FUNCTOR CATEGORIES AND STABLE HOMOLOGY VIA FUNCTOR HOMOLOGY

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Abstract. This text is a preliminary version of material used for a course at the University of Tokyo, January 18-22, 2016. These lectures concern application of homological algebra in functor categories to compute stable homology of automorphism groups with twisted coefficients. The principal aim of these lectures is to present the method developped, in collaboration with Aurélien Djament, in [DV10] and used in [Dja12],[DV15] and [Dja] to compute the stable homology of families of groups with coefficients given by a polynomial functor.

Keywords: stable homology; polynomial functors; functor homology.

Contents

The aim of these lectures is to explain the method developed in [DV10] to compute stable homology of automorphism groups with twisted coefficients using functor homology.

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1. Categories of functors

A category $\mathcal{C}$ is called "small" if both objects and morphisms are actually sets and not proper classes. In these lectures $\mathcal{C}$ denotes a small category. For c and c' two objects of $\mathcal{C}$, the set of morphisms from c to c' in $\mathcal{C}$ will be denoted by $\mathcal{C}(c, c')$.

The following examples of small categories will be particularly interesting in these lectures.

Example 1.1.
(1) *Let G be a group. We can define a category with a single object and where the endomorphisms of this object is the underlying set of G. The composition of morphisms in this category is given by the binary operation on the group G. The identity morphism is the identity element in G. This category associated to the group G will be denoted by G. Note that any morphism is an isomorphism since each element in G has an inverse.*
(2) *Let **sets** be (the skeleton of) the category of finite sets with objects $\underline{n} = \{1, \dots, n\}$ and morphisms arbitrary functions of finite sets and FI the category of finite sets and morphisms injective maps.*
(3) *Let Γ be (the skeleton of) the category of finite pointed sets with objects $[n] = \{0, 1, \dots, n\}$ with basepoint 0 and morphisms functions of finite sets preserving basepoint (i.e. sending 0 to 0).*
(4) *For R a ring, let R-**mod** be (a skeleton of) the category of finitely generated free left R-modules. The category $\mathbb{Z}$-**mod** of finitely generated free abelian groups is also denoted by **ab**.*
(5) *Let **gr** be (a skeleton of) the category of finitely generated free groups.*

For $\Bbbk$ a commutative ring, we denote by $\Bbbk$-**Mod** the category of modules over $\Bbbk$.

1.1. **Functor categories.** For $\mathcal{C}$ a small category and $\Bbbk$ a commutative ring, we denote by $\mathcal{F}(\mathcal{C}, \Bbbk)$ the category of all functors from $\mathcal{C}$ to $\Bbbk$-**Mod** having natural transformations as morphisms. An object of $\mathcal{F}(\mathcal{C}, \Bbbk)$ is called a *$\mathcal{C}$-module.*

Here are some examples of interesting objects in $\mathcal{F}(\mathcal{C}, \Bbbk)$ for $\mathcal{C}$ one of the categories considered in Example 1.1.

Example 1.2.
(1) *For $\mathcal{C}$ a small category we denote by $\Bbbk$ the functor in $\mathcal{F}(\mathcal{C}, \Bbbk)$ which is constant and equal to $\Bbbk$. We denote also by $\Bbbk$ the functor in $\mathcal{F}(\mathcal{C}^{op}, \Bbbk)$ defined similarly.*
(2) *For $\mathcal{C} = \Bbbk$-**mod**, we denote by $Id : \Bbbk\text{-}\mathbf{mod} \to \Bbbk\text{-}\mathbf{Mod}$ the forgetful functor and by $T^n : \Bbbk\text{-}\mathbf{mod} \to \Bbbk\text{-}\mathbf{Mod}$ the n-th tensor product functor (i.e. $T^n(G) = G^{\otimes n}$). The symmetric group $\mathfrak{S}_n$ acts naturally on T^n by permutation of the factors. We denote by $S^n : \Bbbk\text{-}\mathbf{mod} \to \Bbbk\text{-}\mathbf{Mod}$ the functor obtained taking the coinvariants of T^n by the action of $\mathfrak{S}_n$. This functor is called the n-th symmetric power. We denote by $\Gamma^n : \Bbbk\text{-}\mathbf{mod} \to \Bbbk\text{-}\mathbf{Mod}$ the functor obtained taking the invariants of T^n by the action of $\mathfrak{S}_n$. This functor is called the n-th divided power. The n-th exterior power functor $\Lambda^n : \Bbbk\text{-}\mathbf{mod} \to \Bbbk\text{-}\mathbf{Mod}$ is defined by: for $V \in \Bbbk$-**mod**, $\Lambda^n(V)$ is the quotient of $T^n(V)$ by the relations $v_1 \otimes \dots \otimes v_n = 0$ if there exists i and j such that $v_i = v_j$.*
(3) *For $\mathcal{C} = $ **gr** and $\Bbbk = \mathbb{Z}$ we denote by $\mathfrak{a} : \mathbf{gr} \to \mathbf{Ab}$ the abelianization functor. One can postcompose $\mathfrak{a}$ with any functor given in the previous example (for $\Bbbk = \mathbb{Z}$).*
(4) *For G a group, we denote by $I(G)$ the augmentation ideal of $\mathbb{Z}[G]$ (i.e. the kernel of the map $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ given by $\epsilon(\sum_{g \in G} \alpha_g g) = \sum_{g \in G} \alpha_g$). Let $Q_n : \mathbf{gr} \to \mathbf{Ab}$ be the functor given by $Q_n(G) = I(G)/I^{n+1}(G)$ (this functor is called sometimes the n-th Passi functor). Note that $Q_1 \simeq \mathfrak{a}$.*

We give an example of a morphism in $\mathcal{F}(\mathcal{C}, \Bbbk)$.

Example 1.3. *The norm homomorphism defines a natural transformation $N : S^n \to \Gamma^n$. If $n!$ is invertible in $\Bbbk$ (in particular if $\Bbbk$ is a field of characteristic zero) then N is a natural isomorphism.*

A sequence

$$0 \to F \to G \to H \to 0$$

is an exact sequence in $\mathcal{F}(\mathcal{C}, \Bbbk)$ if, for all $C \in \mathcal{C}$

$$0 \to F(C) \to G(C) \to H(C) \to 0$$

in exact in $\Bbbk$-**Mod**.

Example 1.4.
- *In $\mathcal{F}(\Bbbk\text{-}\mathbf{mod}, \Bbbk)$ we have a short exact sequence*
$$0 \to \Gamma^2 \to T^2 \to \Lambda^2 \to 0.$$
If $char(\Bbbk) = 2$ this sequence does not split. If $char(\Bbbk) \neq 2$ this sequence has a section $s : \Lambda^2 \to T^2$ given by $s_V(x \wedge y) = \frac{1}{2}(x \otimes y - y \otimes x)$ for $V \in \Bbbk\text{-}\mathbf{mod}$ and $x, y \in V$.
- *For $G \in \mathbf{gr}$, the following short exact sequence of abelian groups:*

$$0 \longrightarrow I^nG/I^{n+1}G \longrightarrow IG/I^{n+1}G \longrightarrow IG/I^nG \longrightarrow 0. \tag{1}$$

gives a non split short exact sequence in $\mathcal{F}(\mathbf{gr}, \mathbb{Z})$:

$$0 \longrightarrow T^n \circ \mathfrak{a} \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow 0. \tag{2}$$

Proposition 1.5. *The category $\mathcal{F}(\mathcal{C}, \Bbbk)$ is abelian.*

Proof. The limits and colimits in $\mathcal{F}(\mathcal{C}, \Bbbk)$ are computed pointwise and $\Bbbk$-**Mod** is an abelian category. □

Remark 1.6. A ring R is the same as a preadditive category (i.e. a category which is enriched over the monoidal category of abelian groups) having one object. A covariant (resp. contravariant) additive functor (i.e. an enriched functor over the monoidal category of abelian groups) from the preadditive category R to $\mathbb{Z}$-**Mod** is a left (resp. right) module. Therefore, functor categories theory can be viewed as a generalization to several objects of the theory of modules over a ring.

1.2. **Projective generators.** Recall that a *set of generators* in an abelian category $\mathcal{A}$ is a set E of objects of $\mathcal{A}$ such that for all $A \in \mathcal{A}$ there exists an epimorphism from a direct sum of object in E to A.

For any $C \in \mathcal{C}$, let us define $P^{\mathcal{C}}_C \in \mathcal{F}(\mathcal{C}, \Bbbk)$ by:
$$P^{\mathcal{C}}_C(X) = \Bbbk[\mathcal{C}(C, X)]$$
where $\Bbbk[-] : \mathbf{Set} \to \Bbbk\text{-}\mathbf{Mod}$ is the $\Bbbk$-linearization functor (i.e. the left adjoint to the forgetful functor $\Bbbk\text{-}\mathbf{Mod} \to \mathbf{Set}$).

We will prove that these functors form a set of projective generators in $\mathcal{F}(\mathcal{C}, \Bbbk)$. The proof relies on the Yoneda lemma that we recall.

Lemma 1.7.
- ***Set-theoretic version***
For all $C \in \mathcal{C}$ and $F : \mathcal{C} \to \mathbf{Ens}$ we have a natural bijection
$$Hom_{Func(\mathcal{C}, \mathbf{Ens})}(\mathcal{C}(C, -), F) \simeq F(C).$$
- ***$\Bbbk$-linear version***
For all $C \in \mathcal{C}$ and $F : \mathcal{C} \to \Bbbk\text{-}\mathbf{Mod}$ we have a natural bijection
$$Hom_{\mathcal{F}(\mathcal{C}, \Bbbk)}(P^{\mathcal{C}}_C, F) \simeq F(C).$$

The following corollary shows that $\mathcal{F}(\mathcal{C}, \Bbbk)$ has enough projective objects (i.e. for every functor $F : \mathcal{C} \to \Bbbk\text{-}\mathbf{Mod}$ there is an epimorphism $P \to F$ where P is projective).

Corollary 1.8. *The set $\{P^{\mathcal{C}}_C\}_{C \in \mathcal{C}}$ is a set of projective generators of the category $\mathcal{F}(\mathcal{C}, \Bbbk)$.*

Proof. Let $\alpha : A \to B$ be an epimorphism in $\mathcal{F}(\mathcal{C}, \Bbbk)$ and $f : P^{\mathcal{C}}_C \to B$. Since $Hom_{\mathcal{F}(\mathcal{C}, \Bbbk)}(P^{\mathcal{C}}_C, B) \simeq B(C)$, $Hom_{\mathcal{F}(\mathcal{C}, \Bbbk)}(P^{\mathcal{C}}_C, A) \simeq A(C)$ and $A(C) \to B(C)$ is surjective we deduce that there exists $g : P^{\mathcal{C}}_C \to A$ such that $\alpha \circ g = f$.

Let $F : \mathcal{C} \to \Bbbk\text{-}\mathbf{Mod}$. We consider the natural transformation
$$\bigoplus_{\substack{C \in \mathcal{C} \\ c \in F(C)}} P^{\mathcal{C}}_C \to F$$
where the component $P^{\mathcal{C}}_C \to F$ indexed by $c \in F(C)$ is the morphism corresponding to c by the Yoneda isomorphism. This map is surjective so the set $\{P^{\mathcal{C}}_C\}_{C \in \mathcal{C}}$ is a set of generators in the category $\mathcal{F}(\mathcal{C}, \Bbbk)$. □

In particular, any functor $F : \mathcal{C} \to \Bbbk$-**Mod** admits a resolution by direct sums of projective generators $P^{\mathcal{C}}_C$. So we can do homological algebra in the category $\mathcal{F}(\mathcal{C}, \Bbbk)$ as in the category of $\Bbbk$-**Mod** (see section 3).

1.3. **Tensor products.** The aim of this section is to define three different tensor product functors on functor categories and to give their basic properties.

• **Pointwise tensor product**.

Definition 1.9. *For $F : \mathcal{C} \to \Bbbk$-**Mod** and $G : \mathcal{C} \to \Bbbk$-**Mod**, $F \otimes G : \mathcal{C} \to \Bbbk$-**Mod** is defined by*

$$(F \otimes G)(C) = F(C) \otimes G(C)$$

for all C in $\mathcal{C}$.

Proposition 1.10. *If the category $\mathcal{C}$ admits coproduct denoted by $\sqcup$ then*

$$P^{\mathcal{C}}_C \otimes P^{\mathcal{C}'}_{C'} \simeq P^{\mathcal{C}}_{C \sqcup C'}.$$

Proof. Direct consequence of the definition of coproduct. □

• **Exterior tensor product**. Exterior tensor product of functors is the analogue in the setup of functor categories of exterior product of modules over different rings.

Definition 1.11. *Let $\mathcal{C}$ and $\mathcal{C}'$ be small categories, $F : \mathcal{C} \to \Bbbk$-**Mod** and $F' : \mathcal{C}' \to \Bbbk$-**Mod** be functors, the exterior tensor product of F and F' is the functor $F \boxtimes F' : \mathcal{C} \times \mathcal{C}' \to \Bbbk$-**Mod** such that for $C \in \mathcal{C}$, $C' \in \mathcal{C}'$, we have:*

$$(F \boxtimes F')(C, C') = F(C) \otimes F'(C').$$

Lemma 1.12. *We have an isomorphism*

$$P^{\mathcal{C}}_C \boxtimes P^{\mathcal{C}'}_{C'} \simeq P^{\mathcal{C} \times \mathcal{C}'}_{(C,C')}$$

which is natural in $C \in \mathcal{C}$ and in $C' \in \mathcal{C}'$.

• **Tensor product over a category**.

Definition 1.13. *Let $F : \mathcal{C} \to \Bbbk$-**Mod** and $G : \mathcal{C}^{op} \to \Bbbk$-**Mod**, we define $G \underset{\mathcal{C}}{\otimes} F \in \Bbbk$-**Mod** by*

$$G \underset{\mathcal{C}}{\otimes} F = Coeq\Big(\bigoplus_{f \in \mathcal{C}(c,c')} G(c') \otimes F(c) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \bigoplus_{c \in \mathcal{C}} G(c) \otimes F(c) \Big)$$

where for $f \in \mathcal{C}(c, c')$, $x \in G(c')$ and $y \in F(c)$, $f^*(x \otimes y) = x \otimes F(f)(y) \in G(c') \otimes F(c')$ and $f_*(x \otimes y) = G(f)(x) \otimes y \in G(c) \otimes F(c)$.

Remark 1.14. The previous definition of tensor product over a category is a particular case of coends (see [ML98, IX 6]).

Remark 1.15. The enriched tensor product of such a contravariant and covariant functor is exactly the classical tensor product of a left and a right module over R.

Example 1.16. *Let $\Bbbk$ be the constant functor and $F : \mathcal{C} \to \Bbbk$-**Mod**. We have*

$$\Bbbk \underset{\mathcal{C}}{\otimes} F = \underset{\mathcal{C}}{colim}\, F.$$

(Replacing G by $\Bbbk$ in Definition 1.13, we recover the description of $\underset{\mathcal{C}}{colim}\, F$ in terms of coproduct and coequalizer. See for example [ML98, Chapter V]*).*

Proposition 1.17. *For $F : \mathcal{C} \to \Bbbk$-**Mod** and $G : \mathcal{C}^{op} \to \Bbbk$-**Mod**, we have natural isomorphisms*

$$P^{\mathcal{C}^{op}}_C \underset{\mathcal{C}}{\otimes} F \simeq F(C) \quad \text{and} \quad G \underset{\mathcal{C}}{\otimes} P^{\mathcal{C}}_C \simeq G(C).$$

Another way to characterize the tensor product over a category is by the following adjunction.

Proposition 1.18. *For $F : \mathcal{C} \to \Bbbk\text{-}\mathbf{Mod}$, $G : \mathcal{C}^{op} \to \Bbbk\text{-}\mathbf{Mod}$ and $M \in \Bbbk\text{-}\mathbf{Mod}$ we have the following isomorphism*

$$\Bbbk\text{-}\mathbf{Mod}(G \underset{\mathcal{C}}{\otimes} F, M) \simeq \mathcal{F}(\mathcal{C}, \Bbbk)(F, \mathcal{H}om_{\Bbbk}(G, M))$$

where $\mathcal{H}om_{\Bbbk}(G, M) : \mathcal{C} \to \Bbbk\text{-}\mathbf{Mod}$ is given by $C \mapsto \Bbbk\text{-}\mathbf{Mod}(G(C), M)$.

1.4. **Morita equivalences.** Two rings are called *Morita equivalent* if they have equivalent categories of left modules. As functor categories are generalizations to several objects of the theory of modules over a ring (see Remark 1.6) this definition can be extended in the following form:

Definition 1.19. *Two small categories $\mathcal{C}$ and $\mathcal{C}'$ are Morita equivalent if the functors categories $\mathcal{F}(\mathcal{C}, \Bbbk)$ and $\mathcal{F}(\mathcal{C}', \Bbbk)$ are equivalent.*

Remark 1.20. An equivalence between two functors categories is also called a *Dold-Kan type theorem.* In fact the classical Dold-Kan theorem gives an equivalence between simplicial sets in an abelian category $\mathcal{A}$ (i.e. $\mathcal{F}(\Delta^{op}, \mathcal{A})$) and the category of chain complexes of $\mathcal{A}$. The category of chain complexes can be viewed as the category of functors from an enriched category over Γ preserving nul morphism.

If $\mathcal{C}$ and $\mathcal{C}'$ are equivalent they are obviously Morita equivalent but there are lots of examples of non equivalent categories which are Morita equivalent. The following theorem will be useful in Djament's lecture 2.

Theorem 1.21 (Dold-Kan type theorem of Pirashvili)**.** [Pir00a] *Let Ω be the category of finite sets and surjections. The functor:*

$$cr : \mathcal{F}(\Gamma; \Bbbk) \to \mathcal{F}(\Omega; \Bbbk)$$

given by $cr(F)(\underline{n}) = cr_n F([1], \dots, [1])$ for $F \in \mathcal{F}(\Gamma; \Bbbk)$ is an equivalence of categories (where $cr_n F$ is the n-th cross-effect of F, see Definition 2.2).

Remark 1.22. In [DPV] we extend this result to the PROP associated to a set-operad. The previous theorem corresponding to the PROP associated to the operad $\mathcal{C}om$.

Remark 1.23. In [Sło04] and [LS15] the authors give general conditions in order to obtain equivalences of functor categories. Their methods cover the classical Dold-Kan theorem but also the Dold-Kan type theorem of Pirashvili and its extension to PROP associated to the associative set-operad.

2. Polynomial functors

The definition of cross-effects and polynomial functors comes from the work of Eilenberg and Mac Lane on homology of spaces thereafter linked to their names [EML54]. In this paper the authors take for $\mathcal{C}$ a category of finitely generated free modules over a ring. This definition can easily be extended to a small symmetric monoidal category where the unit 0 is the null object of $\mathcal{C}$. (see below).

2.1. **Definition.** Let $(\mathcal{C}, \oplus, 0)$ be a small symmetric monoidal category where the unit 0 is the null object of $\mathcal{C}$. Several examples in 3.6 are such categories.

Example 2.1. (1) $(\Gamma, \amalg, \{0\})$.
(2) $(\Bbbk\text{-}\mathbf{mod}; \oplus, 0)$.
(3) $(gr, *, 0)$.

Let $F : \mathcal{C} \to \Bbbk\text{-}\mathbf{Mod}$ be a functor. The functor F is said *reduced* if $F(0) = 0$. We have a canonical decomposition $F \simeq F(0) \oplus \bar{F}$ where $F(0)$ is the constant functor equal to $F(0)$ and

$$\bar{F}(C) = ker(F(C) \to F(0)) \simeq coker(F(0) \to F(C)).$$

The functor $\bar{F}$ is reduced and is called the reduced functor associated to F.

Fo $X_1 \in \mathcal{C}$ and $X_2 \in \mathcal{C}$, we denote by p_1 the composition $p_1 : X_1 \oplus X_2 \xrightarrow{1_{X_1} \oplus t} X_1 \oplus 0 \simeq X_1$ where t is the unique element in $\mathcal{C}(X_2, 0)$ (0 is terminal by hypothesis). We define similarly $p_2 : X_1 \oplus X_2 \to X_2$.

Definition 2.2. *The n-th cross-effect of F is a functor $cr_n F : \mathcal{C}^{\times n} \to \Bbbk\text{-}\mathbf{Mod}$ (or a multi-functor) defined inductively by*

$$cr_1 F(X) = ker(F(0) : F(X) \to F(0))$$

$$cr_2 F(X_1, X_2) = ker((F(p_1), F(p_2))^t : F(X_1 \oplus X_2) \to F(X_1) \oplus F(X_2))$$

and, for $n \geq 3$, by

$$cr_n F(X_1, \ldots, X_n) = cr_2(cr_{n-1} F(-, X_3, \ldots, X_n))(X_1, X_2).$$

In other words, to define the n-th cross-effect of F we consider the $(n-1)$-st cross-effect, we fix the $n-2$ last variables and we consider the second cross-effect of this functor.

Definition 2.3. *A functor $F : \mathcal{C} \to \Bbbk\text{-}\mathbf{Mod}$ is said to be polynomial of degree lower or equal to n if $cr_{n+1} F = 0$.*

We denote by $\mathcal{P}ol_n(\mathcal{C}, \Bbbk)$ the full subcategory of $\mathcal{F}(\mathcal{C}, \Bbbk)$ consisting of polynomial functors of degree lower or equal to n. We have a filtration of categories

$$\ldots \hookrightarrow \mathcal{P}ol_{n-1}(\mathcal{C}, \Bbbk) \hookrightarrow \mathcal{P}ol_n(\mathcal{C}, \Bbbk) \hookrightarrow \ldots \hookrightarrow \mathcal{F}(\mathcal{C}; \Bbbk).$$

In general, it is difficult to describe the categories $\mathcal{P}ol_n(\mathcal{C}, \Bbbk)$ for $n > 1$ (see, for example, [HV11] for the case $n = 2$ and [HPV15] for $\mathcal{C} = \mathbf{gr}$ and general n). However, we will see in section 2.4 that the subquotients of this filtration have easy descriptions.

2.2. **Basic properties and examples.** The following decomposition result is particularly important.

Proposition 2.4. *Let $F : \mathcal{C} \to \Bbbk\text{-}\mathbf{Mod}$ be a reduced functor. Then there is a natural decomposition*

$$F(X_1 \oplus \ldots \oplus X_n) \simeq \bigoplus_{k=1}^{n} \bigoplus_{1 \leq i_1 < \ldots < i_k \leq n} cr_k F(X_{i_1}, \ldots, X_{i_k}).$$

For example this proposition gives the following decompositions:

$$F(X_1 \oplus X_2) \simeq F(X_1) \oplus F(X_2) \oplus cr_2 F(X_1, X_2)$$

$$F(X_1 \oplus X_2 \oplus X_3) \simeq F(X_1) \oplus F(X_2) \oplus F(X_3) \oplus cr_2 F(X_1, X_2) \oplus cr_2 F(X_1, X_3) \oplus cr_2 F(X_2, X_3) \oplus cr_3 F(X_1, X_2, X_3).$$

Example 2.5.
1. *The abelianization functor $\mathfrak{a}$ is reduced ($\mathfrak{a}(0) = 0$) and $\mathfrak{a}(G * H) \simeq \mathfrak{a}(G) \oplus \mathfrak{a}(H)$. By the previous proposition we deduce that $cr_2(\mathfrak{a})(G, H) = 0$, so $\mathfrak{a}$ is polynomial of degree 1.*
2. *The functor $Id : \Bbbk\text{-}\mathbf{mod} \to \Bbbk\text{-}\mathbf{Mod}$ is reduced and $Id(U \oplus V) = Id(U) \oplus Id(V)$.By the previous proposition we deduce that $cr_2(Id)(U, V) = 0$, so Id is polynomial of degree 1.*
3. *The functor $T^2 : \Bbbk\text{-}\mathbf{mod} \to \Bbbk\text{-}\mathbf{Mod}$ is reduced and we have:*

$$T^2(U \oplus V) = (U \oplus V) \otimes (U \oplus V) = T^2(U) \oplus T^2(V) \oplus (U \otimes V \oplus V \otimes U).$$

By the previous proposition we deduce that $cr_2(T^2)(U, V) = U \otimes V \oplus V \otimes U$. Furthermore, we have:

$$T^2(U \oplus V \oplus W) = (U \oplus V \oplus W) \otimes (U \oplus V \oplus W) = T^2(U) \oplus T^2(V) \oplus T^2(W) \oplus cr_2 T^2(U, V) \oplus cr_2 T^2(U, W) \oplus cr_2 T^2(V, W).$$

We deduce from the previous proposition that $cr_3 T^2(U, V, W) = 0$. So T^2 is a polynomial functor of degree 2.

Proposition 2.6. *If $F : \mathcal{C} \to \Bbbk\text{-}\mathbf{Mod}$ and $G : \mathcal{C} \to \Bbbk\text{-}\mathbf{Mod}$ are polynomial functors, then $F \otimes G$ is polynomial and*

$$deg(F \otimes G) = deg(F) + deg(G).$$

Example 2.7. *Using this proposition and Example 2.5(2) we can prove by induction that the functor $T^n : \Bbbk\text{-}\mathbf{mod} \to \Bbbk\text{-}\mathbf{Mod}$ is polynomial of degree n. Using this proposition and the exponential property (see section 2.3) we can prove that Γ^n, S^n, Λ^n from $\Bbbk\text{-}\mathbf{mod}$ to $\Bbbk\text{-}\mathbf{Mod}$ are polynomial of degree n.*

Proposition 2.8. *The functor $cr_n : \mathcal{F}(\mathcal{C}; \Bbbk) \to \mathcal{F}(\mathcal{C}^{\times n}; \Bbbk)$ is exact for all $n \geq 1$.*

Definition 2.9. *A full subcategory $\mathcal{C}'$ of an abelian category $\mathcal{C}$ is thick if it contains* 0 *and is closed under quotients, subobjects and extensions i.e. for an exact sequence $0 \to B \to A \to C \to 0$ in $\mathcal{C}$, if B and C are in $\mathcal{C}'$ then $A \in \mathcal{C}'$.*

Proposition 2.10. *The subcategory $\mathcal{P}ol_n(\mathcal{C}, \Bbbk)$ of $\mathcal{F}(\mathcal{C}; \Bbbk)$ is thick.*

Proof. Immediate consequence of Proposition 2.8. □

Example 2.11. *Using the short exact sequence (2) in Example 1.4, the fact that $\mathfrak{a}$ is polynomial of degree* 1 *and the previous proposition we can prove by induction that $Q_n : \boldsymbol{gr} \to \boldsymbol{Ab}$ is a polynomial functor of degree n.*

Remark 2.12. Polynomial functors can be defined in several other ways which can be more or less useful depending on the context. There are definitions in terms of cokernel instead of kernel, idempotent or the difference functor. We refer the reader to [DV15, Section 2.2] for the equivalence between these definitions when $\mathcal{C}$ has a null object.

Remark 2.13. In a recent preprint [DV] we extend the notion of polynomial functors from a symmetric monoidal category where the unit 0 is the **initial** object.

2.3. **Exponential functors.** One of the most important property of exponential function is that the exponential of a sum is the product of exponentials. In this section we introduce functors satisfying a similar property. We will see in Djament's lecture 1 that functor homology of exponential functors has interesting properties. Let $\mathcal{C}$ be a small category having finite coproducts denoted by $\sqcup$.

Definition 2.14.
(1) *An exponential functor of $\mathcal{F}(\mathcal{C}, \Bbbk)$ is an object E in $\mathcal{F}(\mathcal{C}, \Bbbk)$ equipped with natural isomorphisms*
$$E(C \sqcup C') \simeq E(C) \otimes E(C')$$
for C and C' in $\mathcal{C}$.
(2) *A graded functor $E^\bullet = (E^n)$ where $E^n : \mathcal{C} \to \Bbbk$-**Mod** is exponential if we have a natural isomorphism:*
$$E^n(U \sqcup V) \simeq \bigoplus_{i=0}^{n} E^i(U) \otimes E^{n-i}(V).$$

Example 2.15. *The graded functors $\Lambda^\bullet, S^\bullet, \Gamma^\bullet$ are exponential functors. The graded functor $T^\bullet$ is not exponential but it is not far to be exponential in the sense that we have the following isomorphism*
$$T^n(U \oplus V) \simeq \bigoplus_{i=0}^{n} (T^i(U) \otimes T^{n-i}(V)) \underset{\mathfrak{S}_i \times \mathfrak{S}_{n-i}}{\otimes} \mathbb{Z}[\mathfrak{S}_n].$$

2.4. **Quotient categories.** In order to understand polynomial functors of degree n we would like to describe the category of polynomial functors of degree $\leq n$ from the category of polynomial functors of degree $\leq n-1$ and another category which morally mesures the difference between the functors of degree $\leq n$ and the functors of degree $\leq n-1$. For this we present in this section a general way to study an abelian category $\mathcal{C}$ from "smaller" categories. More precisely, for $\mathcal{C}'$ a subcategory of $\mathcal{C}$ having good properties, we define the quotient category $\mathcal{C}/\mathcal{C}'$. The study of $\mathcal{C}$ can be reduced to the study of $\mathcal{C}'$ and $\mathcal{C}/\mathcal{C}'$.

The original reference for this section is [Gab62, Chapitre III].

Remark 2.16. A quotient category is a particular case of the localisation of Gabriel-Zisman [GZ67] of a category relatively to a set of morphisms (here, we inverse the morphisms whose kernel and cokernel are in the subcategory $\mathcal{C}'$).

Proposition 2.17. *Let $F : \mathcal{C} \to \mathcal{A}$ be a functor between abelian categories. If F is exact then the kernel of F (i.e. the full subcategory of $\mathcal{C}$ of objects which are sent to* 0 *by F) is thick.*

The converse of this proposition is given by the notion of quotient category defined below. More precisely, if $\mathcal{C}'$ is a thick subcategory of $\mathcal{C}$, we define a new abelian category $\mathcal{C}/\mathcal{C}'$ called "quotient category" such that there exists an exact functor $F : \mathcal{C} \to \mathcal{C}/\mathcal{C}'$ whose kernel is $\mathcal{C}'$.

Definition 2.18. *Let $\mathcal{C}$ be an abelian category and $\mathcal{C}'$ a thick subcategory of $\mathcal{C}$. The quotient category $\mathcal{C}/\mathcal{C}'$ has as objects the objects of $\mathcal{C}$ and for X and Y two objects of $\mathcal{C}$*

$$\mathcal{C}/\mathcal{C}'(X,Y) = colim\ \mathcal{C}(X', Y/Y')$$

where the colimit runs through all subobjects $X' \subset X$, $Y' \subset Y$ such that X/X' and Y' are objects in $\mathcal{C}'$.

The composition of morphisms in $\mathcal{C}/\mathcal{C}'$ is given carefully in [Gab62].

We have a quotient functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{C}'$ given by $T(X) = X$ for $X \in \mathcal{C}$ and for $f \in Hom_{\mathcal{C}}(X,Y)$, $T(f)$ is the image of f in colim $Hom_{\mathcal{C}}(X', Y/Y')$.

Proposition 2.19. *The functor $T : \mathcal{C} \to \mathcal{C}/\mathcal{C}'$ is exact and its kernel is $\mathcal{C}'$. Moreover, for $\mathcal{D}$ an abelian category, $F : \mathcal{C} \to \mathcal{D}$ an exact functor which is trivial on $\mathcal{C}'$, there exists a unique functor $G : \mathcal{C}/\mathcal{C}' \to \mathcal{D}$ such that $G \circ T = F$.*

By Proposition 2.10, the category $\mathcal{P}ol_n(\mathcal{C}, \Bbbk)$ is thick. We will describe below the quotient categories $\mathcal{P}ol_n(\mathcal{C}, \Bbbk)/\mathcal{P}ol_{n-1}(\mathcal{C}, \Bbbk)$.

Proposition 2.20. *The functor $cr_n : \mathcal{P}ol_n(\boldsymbol{ab}, \Bbbk) \to \Bbbk[\mathfrak{S}_n]$-**Mod**, $F \mapsto cr_n F(\mathbb{Z}, \ldots, \mathbb{Z})$ induces an equivalence of categories:*

$$\mathcal{P}ol_n(\boldsymbol{ab}, \Bbbk)/\mathcal{P}ol_{n-1}(\boldsymbol{ab}, \Bbbk) \simeq \Bbbk[\mathfrak{S}_n]\text{-}\mathbf{Mod}.$$

For $n > 1$ the categories $\mathcal{P}ol_n(\mathbf{ab}, \Bbbk)$ and $\mathcal{P}ol_n(\mathbf{gr}, \Bbbk)$ are not equivalent. However we have the following result.

Proposition 2.21. [DV15, Corollaire 3.6] *The abelianization functor $\mathfrak{a} : \boldsymbol{gr} \to \boldsymbol{Ab}$ induces an equivalence of categories:*

$$\mathcal{P}ol_n(\boldsymbol{gr}, \Bbbk)/\mathcal{P}ol_{n-1}(\boldsymbol{gr}, \Bbbk) \simeq \mathcal{P}ol_n(\boldsymbol{ab}, \Bbbk)/\mathcal{P}ol_{n-1}(\boldsymbol{ab}, \Bbbk)$$

Combinig the last two propositions we obtain the equivalence of categories

$$\mathcal{P}ol_n(\mathbf{gr}, \Bbbk)/\mathcal{P}ol_{n-1}(\mathbf{gr}, \Bbbk) \simeq \Bbbk[\mathfrak{S}_n]\text{-}\mathbf{Mod}.$$

Remark 2.22. In [DV15], we give more generally the description of $\mathcal{P}ol_n(\langle E\rangle_{\mathcal{C}}, \Bbbk)/\mathcal{P}ol_{n-1}(\langle E\rangle_{\mathcal{C}}, \Bbbk)$ where $\mathcal{C}$ is a small pointed category with finite coproduct, E is a fixed object in $\mathcal{C}$ and $\langle E\rangle_{\mathcal{C}}$ is the full subcategory of $\mathcal{C}$ having as objects finite coproducts of E. More precisely, we give the *recollement diagram* between the categories $\mathcal{P}ol_n(\langle E\rangle_{\mathcal{C}}, \Bbbk)$, $\mathcal{P}ol_{n-1}(\langle E\rangle_{\mathcal{C}}, \Bbbk)$ and $\Bbbk[\mathfrak{S}_n]$-**mod**. In [HV11], we give a complete description of the category $\mathcal{P}ol_2(\langle E\rangle_{\mathcal{C}}, \Bbbk)$.

3. Homology of functors I

The terms "functor homology" denote homological algebra in functor categories. In this lecture we will explain in more details what does it mean, give some basic properties and several results concerning functor homology over **gr**.

3.1. Tor and Ext.

• **Tor groups**. The functors $- \underset{\mathcal{C}}{\otimes} F$ and $G \underset{\mathcal{C}}{\otimes} -$ are right exact. They commute with colimits in each variables. We can derive these functors on the left.

Definition 3.1. *For $F \in \mathcal{F}(\mathcal{C}, \Bbbk)$ and $G \in \mathcal{F}(\mathcal{C}^{op}, \Bbbk)$ we define:*

$$Tor_i^{\mathcal{C}}(G, F) = H_i(G \underset{\mathcal{C}}{\otimes} P_\bullet)$$

where $P_\bullet$ is a projective resolution of $F \in \mathcal{F}(\mathcal{C}, \Bbbk)$.

Remark 3.2. We could equally well resolve G.

Homology of a category

Definition 3.3. *For $F \in \mathcal{F}(\mathcal{C}, \Bbbk)$ we define the graded $\Bbbk$-module*

$$H_*(\mathcal{C}, F) = Tor_*^{\mathcal{C}}(\Bbbk, F).$$

Remark 3.4. We have $H_0(\mathcal{C},F)=\Bbbk\underset{\mathcal{C}}{\otimes}F=\mathrm{colim}(F)$ by Example 1.16.

Example 3.5. *If G is the category associated to a group G (see example 1.1), a functor $F:G\to\Bbbk$-**Mod** is a $\Bbbk[G]$-module and the homology of the category G, $H_*(G,F)$, corresponds to the usual notion of homology of the group G with coefficients in a $\Bbbk[G]$-module.*

We give two simple examples of computation of homologies of categories which will be very useful in the sequel.

Example 3.6. *If $\mathcal{C}$ has an initial object I then $H_0(\mathcal{C},\Bbbk)=\Bbbk$ and $H_*(\mathcal{C},\Bbbk)=0$ for $*>0$.*
In fact, $\Bbbk=P_I^{\mathcal{C}}(-)=\Bbbk[\mathcal{C}(I,-)]$ so $\Bbbk$ is a projective object in $\mathcal{F}(\mathcal{C},\Bbbk)$.

Example 3.7. *If $\mathcal{C}$ has a terminal object T then $H_0(\mathcal{C};F)=F(T)$ and $H_i(\mathcal{C};F)=0$ for $*>0$.*
In fact, $\Bbbk=P_T^{\mathcal{C}^{op}}(-)=\Bbbk[\mathcal{C}(-,T)]$ so $\Bbbk$ is a projective object in $\mathcal{F}(\mathcal{C}^{op},\Bbbk)$.

• **Ext groups**.

Definition 3.8. *For $F\in\mathcal{F}(\mathcal{C},\Bbbk)$ and $G\in\mathcal{F}(\mathcal{C},\Bbbk)$ we define:*

$$Ext^i_{\mathcal{F}(\mathcal{C}\Bbbk)}(G,F)=H_i(Hom_{\mathcal{F}(\mathcal{C},\Bbbk)}(P_\bullet,F))$$

where $P_\bullet$ is a projective resolution of $G\in\mathcal{F}(\mathcal{C},\Bbbk)$.

3.2. **First properties.**
• **Relation between Ext and Tor**.

Proposition 3.9. *Let $G:\mathcal{C}^{op}\to\Bbbk$-**Mod**, $F:\mathcal{C}\to\Bbbk$-**Mod** and I an injective $\Bbbk$-module then we have a natural graded isomorphism*

$$Tor^{\mathcal{C}}_\bullet(G,F)^\vee\simeq Ext^\bullet_{\mathcal{F}(\mathcal{C})}(F,\mathcal{H}om(G,I))$$

where $V^\vee=Hom_\Bbbk(V,I)$ for a $\Bbbk$-module V and $\mathcal{H}om(G,I):\mathcal{C}\to\Bbbk$-Mod, $C\mapsto Hom_{\Bbbk\text{-}\mathbf{Mod}}(G(C),I)$.

Proof. We use the characterization of Ext groups given in [ML63, Chapter III, Theorem 10.1] for $Ex^n(F)=Tor^{\mathcal{C}}_n(G,F)^\vee$.

We have

$$Ex^0(F)=Tor^{\mathcal{C}}_0(G,F)^\vee=(G\underset{\mathcal{C}}{\otimes}F)^\vee=Hom_\Bbbk(G\underset{\mathcal{C}}{\otimes}F,I)\simeq Hom_{\mathcal{C}}(F,\mathcal{H}om_\Bbbk(G,I))$$

by Proposition 1.18.

For a projective functor P we have

$$Ex^n(P)=Tor^{\mathcal{C}}_n(G,P)^\vee=0$$

Since I is an injective $\Bbbk$-module, $Hom_\Bbbk(-,I)$ is exact and Ex^n send short exact sequence to long exact sequence.

We deduce that $Ex^n\simeq Ext^n(-,G^*)$. □

• **Functoriality**.

Lemma 3.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small category, $\varphi:\mathcal{C}\to\mathcal{D}$, $F:\mathcal{D}\to\Bbbk$-Mod and $G:\mathcal{D}^{op}\to\Bbbk$-Mod be functors then φ induces a natural morphism*

$$\varphi_\bullet:Tor^{\mathcal{C}}_\bullet(\varphi^*(G),\varphi^*(F))\to Tor^{\mathcal{D}}_\bullet(G,F).$$

Proof. To construct the maps $\varphi_\bullet$ we use the notion of universal δ-functors (see [Wei94, Section 2.1] for the definition) and the following result (see [Wei94, Theorem 2.4.7 p47]) *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. If $\mathcal{A}$ has enough projective objects, then for any right exact functor $F:\mathcal{A}\to\mathcal{B}$, the derived functors L_nF form a universal δ-functor.*

The categories $\mathcal{F}(\mathcal{C},\Bbbk)$ and $\mathcal{F}(\mathcal{D},\Bbbk)$ are abelian by Proposition 1.5 and have enough projective objects by Corollary 1.8. For $G:\mathcal{D}^{op}\to\Bbbk$-**Mod**, the functor $G\underset{\mathcal{D}}{\otimes}-:\mathcal{F}(\mathcal{D},\Bbbk)\to\Bbbk$-**Mod** is right exact. Since the functor φ^* is exact, the functor $\varphi^*(G)\underset{\mathcal{C}}{\otimes}-:\mathcal{F}(\mathcal{C},\Bbbk)\to\Bbbk$-**Mod** is also right exact. We deduce that the functors $Tor_n^{\mathcal{D}}(G,-)$ and $Tor_n^{\mathcal{C}}(\varphi^*(G),-)$ form universal δ-functors. By the universal property of the δ-functor $Tor_n^{\mathcal{C}}(\varphi^*(G),-)$ the natural transformation

$$\varphi_0 : \varphi^*(G) \underset{\mathcal{C}}{\otimes} \varphi^*(-) \to G \underset{\mathcal{D}}{\otimes} -$$

gives morphisms

$$\varphi_n : Tor_n^{\mathcal{C}}(\varphi^*(G), \varphi^*(F)) \to Tor_n^{\mathcal{D}}(G,F)$$

extending φ_0. Moreover these morphisms are unique if we require to have a morphism of δ-functor. □

Corollary 3.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small category, $\varphi : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{D} \to \mathbb{k}$-Mod be functors then φ induces a natural morphism*

$$\varphi_\bullet : H_\bullet(\mathcal{C}, \varphi^*(F)) \to H_\bullet(\mathcal{D}, F).$$

Proof. Apply the previous lemma for G the constant functor equal to $\mathbb{k}$. □

One of the main ingredient of the method developed in [DV10] consists to compare functor homologies of various categories. More precisely, we establish a criterion implying that the natural map $\varphi_* : H_*(\mathcal{C}, \varphi^*(F)) \to H_*(\mathcal{D}, F)$ obtained in the previous corollary is an isomorphism. We will see in Djament's lecture 1 this criterion (see also Proposition 4.9) and Djament's lecture 3 and section 4 that the most difficult step in our method is to prove the hypothesis in this criterion.

The following lemma will be also useful.

Lemma 3.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small category, $\varphi, \psi : \mathcal{C} \to \mathcal{D}$ be functors, $u : \varphi \to \psi$ a natural transformation and $F : \mathcal{D} \to \mathbb{k}$-**Mod** and $G : \mathcal{D} \to \mathbb{k}$-**Mod** be functors, then the following diagram is commutative*

$$\begin{array}{ccc} Tor_\bullet^{\mathcal{C}}(\psi^* G, \varphi^* F) & \xrightarrow{Tor_\bullet^{\mathcal{C}}(\psi^* F, F \circ u)} & Tor_\bullet^{\mathcal{C}}(\varphi^* G, \varphi^* F) \\ {\scriptstyle Tor_\bullet^{\mathcal{C}}(G\circ u, \varphi^* F)} \downarrow & & \downarrow {\scriptstyle \psi_\bullet} \\ Tor_\bullet^{\mathcal{C}}(\varphi^* G, \varphi^* F) & \xrightarrow[\varphi_\bullet]{} & Tor_\bullet^{\mathcal{D}}(G, F). \end{array}$$

3.3. **Functor homology over gr.** In functor homology as in group homology it is quite rare to have access to an explicit projective resolution of the functors. However we can still calculate some *Tor* groups and *Ext* groups using other methods as we will see in Djament's lecture 1.

The aim of this section is to give several results concerning functor homology over **gr** and, in particular, concrete computations of functor homology over **gr** obtained in [Ves] and useful to compute stable homology of automorphisms groups of free groups with coefficients twisted by a contravariant functor (see section 4). In fact, for $\mathcal{C} = \mathbf{gr}$ computations of functor homology are accessible since we have an explicit projective resolution of the abelianization functor given in the following proposition where we denote by P_i the projective object in $\mathcal{F}(\mathbf{gr}, \mathbb{Z})$, $P^{\mathbf{gr}}_{\mathbb{Z}^{*i}}$ defined in section 1.2.

Proposition 3.13. *(Cf.* [JP91, Proposition 5.1]*) The exact sequence in $\mathcal{F}(\boldsymbol{gr})$:*

$$\ldots P_{n+1} \xrightarrow{d_n} P_n \to \ldots \to P_2 \xrightarrow{d_1} P_1$$

is a projective resolution of the abelianization functor $\mathfrak{a} : \boldsymbol{gr} \to \boldsymbol{Ab}$. The natural transformation $d_n : P_n \to P_{n-1}$ is given on a group G by the linear map $\mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ such that:

$$d_n([g_1, \ldots, g_{n+1}]) = [g_2, \ldots, g_{n+1}]$$

$$+\sum_{i=1}^{n} (-1)^i [g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}] + (-1)^{n+1}[g_1, \ldots, g_n]$$

for all $(g_1, \ldots, g_{n+1}) \in G^{n+1}$.

Remark 3.14. This projective resolution plays an important role in [DV15] (see section 4).

Proof. Let G be a group, applying the functor $\mathbb{Z} \underset{\mathbb{Z}G}{\otimes} -$ to the unormalized bar resolution we obtain the following complex

$$\ldots (B^u_2)_G \to (B^u_1)_G \to (B^u_0)_G = \mathbb{Z}$$

computing $H_*(G, \mathbb{Z})$. For $G = F_n$, since

$$H_*(F_n, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}^{\oplus n} & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}$$

we obtain the exact complex in the first line of the following commutative diagram:

$$\begin{array}{ccccccc} \ldots \longrightarrow & (B^u_2)_{F_n} & \longrightarrow & (B^u_1)_{F_n} & \longrightarrow & \mathbb{Z}^{\oplus n} \\ & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ \ldots \longrightarrow & P_2(F_n) & \longrightarrow & P_1(F_n) & \xrightarrow{\epsilon'} & \mathfrak{a}(F_n). \end{array}$$

The projective resolution of $\mathfrak{a}$ is given by the second line of the diagram. □

Using this projective resolution we can prove the following proposition.

Proposition 3.15. [Ves] *Let $m \geq 1$ be a natural integer, we have an isomorphism:*

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}(\mathfrak{a}, T^m \circ \mathfrak{a}) \simeq \begin{cases} \mathbb{Z} & \text{if } * = m-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We prove the result by induction on m. To start the induction, $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a})$ is the homology of the complex:

$$\ldots \longleftarrow \mathbb{Z}^{n+1} \xleftarrow{\delta_n} \mathbb{Z}^n \longleftarrow \ldots \longleftarrow \mathbb{Z}^2 \xleftarrow{\delta_1} \mathbb{Z}$$

which is trivial for $* \geq 1$ and is isomorphic to $\mathbb{Z}$ for $* = 0$. Assume that the statement is true for m. By Künneth formula ($Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a})$ is torsion free by the previous computation), we have:

$$Ext^n_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, T^{m+1} \circ \mathfrak{a}) \simeq \bigoplus_{i+j=n+1} Ext^{i-1}_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, T^m \circ \mathfrak{a}) \otimes Ext^{j-1}_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a})$$

and we obtain the result by the inductive step and the computation of $Ext^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a})$.

□

Theorem 3.16. [Ves] *Let n and m be integers, we have an isomorphism:*

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \simeq \begin{cases} \mathbb{Z}[Surj(m,n)] & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

where $Surj(m,n)$ is the set of surjections from m to n.

The proof of this theorem uses exponential type property of the graded functor $T^\bullet$ (see section 2.3), the Künneth formula and the sum-diagonal adjunction (see Djament's lecture 1). By Proposition 3.9 giving the relation between Tor and Ext we obtain:

Corollary 3.17. [Ves] *Let $\Bbbk$ be a field, n and m be integers, we have an isomorphism:*

$$Tor^{\boldsymbol{gr}}_*(\mathcal{H}om(T^m \circ \mathfrak{a}_\Bbbk, \Bbbk), T^n \circ \mathfrak{a}_\Bbbk) \simeq \begin{cases} \Bbbk[Surj(m,n)] & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

*where $Surj(m,n)$ is the set of surjections from m to n and $\mathfrak{a}_\Bbbk$ is the functor $\boldsymbol{gr} \to \Bbbk$-**Mod**; $G \mapsto \mathfrak{a}(G) \otimes \Bbbk$.*

Proof. Since $\Bbbk$ is a field, we can apply Proposition 3.9 to the functor $I = \Bbbk$ (the constant functor):

$$Hom(Tor^{\mathbf{gr}}_*(\mathcal{H}om(T^m \circ \mathfrak{a}_\Bbbk, \Bbbk), T^n \circ \mathfrak{a}_\Bbbk), \Bbbk) \simeq Ext^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}((T^n \circ \mathfrak{a}_\Bbbk) \otimes \Bbbk, \mathcal{H}om(\mathcal{H}om((T^m \circ \mathfrak{a}_\Bbbk) \otimes \Bbbk, \Bbbk), \Bbbk)).$$

Since the values of the functor $T^m \circ \mathfrak{a}_\Bbbk$ are finite dimensional $\Bbbk$-modules we have

$$\mathcal{H}om(\mathcal{H}om(T^m \circ \mathfrak{a}_\Bbbk, \Bbbk), \Bbbk) \simeq T^m \circ \mathfrak{a}_\Bbbk$$

so

$$Hom(Tor_*^{\mathbf{gr}}(\mathcal{H}om(T^m \circ \mathfrak{a}_{\Bbbk}, \Bbbk), T^n \circ \mathfrak{a}_{\Bbbk}), \Bbbk) \simeq Ext^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(T^n \circ \mathfrak{a}_{\Bbbk}, T^m \circ \mathfrak{a}_{\Bbbk}) \simeq \begin{cases} \Bbbk[Surj(m,n)] & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

Since these $\Bbbk$-modules are finite we deduce the statement by a last duality argument.

□

In the following proposition we show that the actions of symmetric groups on $Ext^*_{\mathcal{F}(\mathbf{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a})$ are induced by the composition of surjections via the isomorphism in Theorem 3.16 up to a sign that we make explicit.

Proposition 3.18. [Ves] *The symmetric groups $\mathfrak{S}_m$ and $\mathfrak{S}_n$ act on $Ext^{m-n}_{\mathcal{F}(\boldsymbol{gr})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \simeq \mathbb{Z}[Surj(m,n)]$ by the following way: for $\sigma \in \mathfrak{S}_m$, $\tau_{k,l} \in \mathfrak{S}_n$ the transposition of k and l where $k, l \in \{1, \dots, n\}$ and $f \in Surj(m,n)$*

$$[f].\sigma = \prod_{1 \leq i \leq n} \epsilon(\overline{\sigma_{|(f \circ \sigma)^{-1}(i)}})[f \circ \sigma]$$

where $\sigma_{|(f\circ\sigma)^{-1}(i)} : (f \circ \sigma)^{-1}(i) \to \sigma((f \circ \sigma)^{-1}(i))$

$$\tau_{k,l}.[f] = (-1)^{(|f^{-1}(k)|-1)(|f^{-1}(l)|-1)}[\tau_{k,l} \circ f].$$

Using this explicit description of the actions of symmetric groups we can obtain other computations of Ext-groups between functors from $\mathbf{gr}$.

Proposition 3.19. [Ves] *Let n and m be natural integers, we have isomorphisms:*

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((\Lambda^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q}^{\rho(m,n)} & \textit{if } * = m-n \\ 0 & \textit{otherwise} \end{cases}$$

where $\rho(m,n)$ denotes the number of partitions of m into n parts.

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((S^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (S^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \textit{if } n = m \textit{ and } * = 0 \\ 0 & \textit{otherwise} \end{cases}$$

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((\Lambda^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (S^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \textit{if } n = m = 0 \textit{ and } * = 0 \\ & \textit{or } n = m = 1 \textit{ and } * = 0 \\ 0 & \textit{otherwise} \end{cases}$$

$$Ext^*_{\mathcal{F}(\boldsymbol{gr})}((S^n \circ \mathfrak{a}) \otimes \mathbb{Q}, (\Lambda^m \circ \mathfrak{a}) \otimes \mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & \textit{if } n = m = 0 \textit{ and } * = 0 \\ & \textit{or } n = 1 \textit{ and } * = m-1 \\ 0 & \textit{otherwise} \end{cases}$$

This proposition will be used in section 4 to give explicit computations of stable homology of automorphisms groups of free groups with coefficients twisted by a contravariant functor.

In [DPV] we obtain the following comparison result.

Theorem 3.20. [DPV] *Let k be a ring, $d \in \mathbb{N}$ and $F : \boldsymbol{gr} \to k$-**Mod**, $G : \boldsymbol{gr} \to k$-**Mod** polynomial functors of degree $\leq d$. The natural graded linear map induced by the inclusion $\mathcal{P}ol_d(\boldsymbol{gr}, k) \to \mathcal{F}(\boldsymbol{gr}, k)$*

$$\mathrm{Ext}^*_{\mathcal{P}ol_d(\boldsymbol{gr},k)}(F, G) \to \mathrm{Ext}^*_{\mathcal{F}(\boldsymbol{gr},k)}(F, G)$$

is an isomorphism.

The analogous result replacing $\mathbf{gr}$ by $\mathbf{ab}$ is not true. For example, for $Id : \mathbf{ab} \to \mathbf{Ab}$ we have:

$$Ext^4_{\mathcal{P}ol_1(\mathbf{ab},\mathbb{Z})}(Id, Id) \simeq Ext^4_{\mathbf{Ab}}(\mathbb{Z}, \mathbb{Z}) = 0$$

since $\mathcal{P}ol_1(\mathbf{ab}, \mathbb{Z}) \simeq \mathbf{Ab}$ and

$$Ext^4_{\mathcal{F}(\mathbf{ab},\mathbb{Z})}(Id, Id) \simeq \mathbb{Z}/2\mathbb{Z}$$

by a result of Bökstedt, recovered by Franjou and Pirashvili in [FP98] using functor homology methods.

Theorem 3.20 and concrete computations obtained in Theorem 3.16 and Proposition 3.19 illustrate the fact that functor homology over $\mathbf{gr}$ is simpler that over $\mathbf{ab}$.

4. An example: automorphisms of free groups

The aim of this lecture is to give the known results for homological stability and stable homology for the family of automorphisms of free groups. Let $\mathbb{Z}^{*n}$ be a free group on n generators and $Aut(\mathbb{Z}^{*n})$ its group of automorphisms. Recall that we have sequences of group morphisms

$$\ldots \to H_*(Aut(\mathbb{Z}^{*n}); \Bbbk) \to H_*(Aut(\mathbb{Z}^{*n+1}); \Bbbk) \to \ldots \tag{3}$$

and

$$\ldots \to H_*(Aut(\mathbb{Z}^{*n}); M_n) \to H_*(Aut(\mathbb{Z}^{*n+1}); M_{n+1}) \to \ldots \tag{4}$$

for a sequence of compatible representations.

4.1. Homological stability.

Theorem 4.1. [HV98a] *The group morphism*

$$H_i(Aut(\mathbb{Z}^{*n}); \Bbbk) \to H_i(Aut(\mathbb{Z}^{*n+1}); \Bbbk)$$

is an isomorphism for $n \geq 2i + 3$.

The stability was proved previously by Hatcher in [Hat95] with a quadratic bound ($n > i^2/4+2i-1$). The previous theorem improves significantly the stability range.

The homological stability of automorphisms of free groups with twisted coefficients is given in a recent work of Randal-Williams and Wahl where they develop a general framework to study homological stability.

Theorem 4.2. [RWW] *Let* $F : \boldsymbol{gr} \to \mathbb{Z}\text{-}\mathbf{Mod}$ *be a polynomial functor of degree* r. *The group morphism*

$$H_i(Aut(F_n); F(F_n)) \to H_i(Aut(F_{n+1}); F(F_{n+1}))$$

is an isomorphism for $n \geq 2i + r + 3$.

4.2. Stable homology with constant coefficients.

Theorem 4.3 (Galatius [Gal11])**.** *Let* $\mathbb{Z}^{*n}$ *be the free group freely generated on* $\{x_1, \ldots, x_n\}$ *and* $\phi_n : \mathfrak{S}_n \to Aut(\mathbb{Z}^{*n})$ *the homomorphism given by* $\phi_n(\sigma) : x_i \mapsto x_{\sigma(i)}$ *for* $\sigma \in \mathfrak{S}_n$. *The induced homomorphism:*

$$(\phi_n)_* : H_k(\mathfrak{S}_n; \mathbb{Z}) \to H_k(Aut(\mathbb{Z}^{*n}); \mathbb{Z})$$

is an isomorphism for $2k + 1 \leq n$.

By the computations of Nakaoka for the symmetric groups, we can deduce the computation of the stable homology of automorphism groups of free groups $Aut(\mathbb{Z}^{*n})$ with integral coefficients. For k a field, using the universal coefficient theorem we deduce the isomorphism:

$$(\phi_n)_* : H_k(\mathfrak{S}_n; k) \simeq H_k(Aut(\mathbb{Z}^{*n}); k)$$

for $2k + 1 \leq n$. In particular, since $\mathfrak{S}_n$ is a finite group, the homology groups of $\mathfrak{S}_n$ with rational coefficients vanish so the previous theorem has the following corollary.

Corollary 4.4 (Galatius [Gal11])**.** *Let* $\mathbb{Z}^{*n}$ *be the free group, then, for* $k \geq 1$*:*

$$H_k(Aut(\mathbb{Z}^{*n}); \mathbb{Q}) = 0 \quad \text{for } 2k + 1 \leq n.$$

This result had been conjectured by Hatcher and Vogtmann in [HV98b].

4.3. **Stable homology with coefficients twisted by a covariant functor.**

Theorem 4.5. [DV15] *Let $F : \boldsymbol{gr} \to \mathbb{k}$-**Mod** be a reduced polynomial functor where $\mathbb{k}$ is $\mathbb{Z}$ or a prime field (i.e. $\mathbb{Q}$ or $\mathbb{F}_p$). Then*

$$\underset{n\in\mathbb{N}}{colim}\ H_*\Big(\mathrm{Aut}\left(\mathbb{Z}^{*n}\right); F\left(\mathbb{Z}^{*n}\right)\Big) = 0.$$

Remark 4.6. If F is the abelianization functor $\mathfrak{a} : \mathbf{gr} \to \mathbf{Ab}$, Theorem 4.5 was known. In fact, Hatcher and Wahl proved that the stable homology groups of $Aut(\mathbb{Z}^{*n})$ with coefficients in $\mathbb{Z}^n$ are trivial using the stability of the homology groups of the mapping class groups of certain 3-manifolds (see [HW05], [HW] and [HW10]). Furthermore, by an observation of Randal-Williams in [RW], for $F = T^n \circ \mathfrak{a}$ Theorem 4.5 can be deduced from the paper of Hatcher and Vogtmann [HV04] (see also its erratum [HVW] with Wahl). By completely different and purely algebraic methods, Satoh computes completely (i.e. not only the stable value) in [Sat06] and [Sat07] the first and second homology groups of the automorphism group of a free group with coefficients given by the abelianization functor.

Remark 4.7. Theorem 4.5 should be compared with Theorem 4.3 due to Galatius. In the stable range, by Galatius' Theorem, the stable integral homology groups of Aut $\left(\mathbb{Z}^{*n}\right)$ are isomorphic to those of the symmetric group $\mathfrak{S}_n$. Theorem 4.5 tells us that it is no more the case for twisted coefficients. In fact, by a theorem due to Betley [Bet02], stable homology of symmetric groups with twisted coefficients is highly non trivial.

Sketch of the proof of the theorem. Recall that $\mathcal{G}$ is the symmetric monoidal category having as objects finitely generated free groups with the free product. An arrow from G to H is a pair (u, T) where $u : G \to H$ is a group monomorphism and T a subgroup of H such that $H = T * u(G)$.

We saw in lecture 2 of Djament that this category is an homogeneous category. So, by our general theorem we have

Proposition 4.8. *For $F : \boldsymbol{gr} \to \mathbb{k}$-**Mod***

$$\underset{n\in\mathbb{N}}{colim}\ H_*(Aut(\mathbb{Z}^{*n}), F(\mathbb{Z}^{*n})) \simeq H_*(\mathcal{G} \times \underset{n\in\mathbb{N}}{colim}\ (Aut(\mathbb{Z}^{*n})), \Pi^* F)$$

where $\Pi : \mathcal{G} \times \underset{n\in\mathbb{N}}{colim}\ (Aut(\mathbb{Z}^{*n})) \to \mathcal{G}$ is the projection functor.

If $\mathbb{k}$ is a field we have:

$$\underset{n\in\mathbb{N}}{colim}\ H_*(Aut(\mathbb{Z}^{*n}), F(\mathbb{Z}^{*n})) \simeq H_*(\mathcal{G}, F) \otimes \underset{n\in\mathbb{N}}{colim}\ H_*(Aut(\mathbb{Z}^{*n}), \mathbb{k})$$

Note that in this proposition there is no condition on F.

The second step consists to compute $H_*(\mathcal{G}, F)$. Recall that, by Corollary 3.11, the forgetful functor $\iota : \mathcal{G} \to \mathbf{gr}$ induces a morphism

$$\iota_* : H_*(\mathcal{G}, \iota^*(F)) \to H_*(\mathbf{gr}, F)$$

for $F : \mathbf{gr} \to \mathbb{k}$-**Mod**.

Recall the comparison criterion presented in Djament's lecture 1:

Proposition 4.9. *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor between small categories and $F : \mathcal{D} \to \mathbb{k}$-**Mod**. For $i \in \mathbb{N}$, let us denote by $L_i : \mathcal{D}^{op} \to \mathbb{k}$-**Mod** the functor defined by: $L_i(d) = \tilde{H}_i(\varphi \setminus d; \mathbb{k})$ (where $\tilde{H}$ is the reduced homology and $\varphi \setminus d$ is the comma category associated to the functor $\mathcal{D}(d, -) \circ \varphi : \mathcal{C} \to$ **Set**). If $Tor_*^{\mathcal{D}}(L_i; F) = 0$ for each i, then the canonical map (see Corollary 3.11)*

$$H_*(\mathcal{C}; \varphi^* F) \to H_*(\mathcal{D}; F)$$

is an isomorphism.

Applying this criterion to $\varphi = \iota : \mathcal{G} \to \mathbf{gr}$ we obtain:

Proposition 4.10. *If $F : \boldsymbol{gr} \to \mathbb{k}$-**Mod** is a reduced functor satisfying the condition $Tor_p^{\boldsymbol{gr}}(L_i, F) = 0$ then*

$$H_*(\mathcal{G}, \iota^*(F)) \xrightarrow{\simeq} H_*(\boldsymbol{gr}, F) = 0.$$

The last equality is a consequence of Example 3.7 using the fact that F is reduced.

Combining Propositions 4.8 and 4.10 we obtain that Theorem 4.5 is true for $F : \mathbf{gr} \to \Bbbk\text{-}\mathbf{Mod}$ a reduced functor such that $Tor_p^{\mathbf{gr}}(L_i, F) = 0$.

It remains to show the following lemma which is the difficult part of the proof of Theorem 4.5.

Lemma 4.11. ***Cancellation result***
For $F : \mathbf{gr} \to \Bbbk\text{-}\mathbf{Mod}$ a reduced polynomial functor, we have

$$Tor_p^{\mathbf{gr}}(L_i, F) = 0.$$

Proof. Let $X : \mathbf{gr}^{op} \to \Bbbk\text{-}\mathbf{Mod}$. In this proof we denote by P_n the projective functor $P^{\mathbf{gr}}_{\mathbb{Z}^{*n}} : \mathbf{gr} \to \mathbf{Ab}$.

(1) **A complex computing** $Tor_*^{\mathbf{gr}}(X, \mathfrak{a} \otimes P_n)$
By Proposition 3.13

$$\dots P_{n+1} \xrightarrow{d_n} P_n \to \dots \to P_2 \xrightarrow{d_1} P_1$$

is a projective resolution of the abelianization functor $\mathfrak{a} : \mathbf{gr} \to \mathbf{Ab}$. As P_r is projective, $- \otimes P_r$ is exact and by Proposition 1.10 $P_i \otimes P_j \simeq P_{i+j}$, so we obtain the following projective resolution of $\mathfrak{a} \otimes P^{\mathbf{gr}}_{\mathbb{Z}^n}$

$$\dots P_{n+1+r} \xrightarrow{d_n} P_{n+r} \to \dots \to P_{2+r} \xrightarrow{d_1} P_{1+r}$$

Applying $X \underset{\mathbf{gr}}{\otimes} -$ to this resolution and using Proposition 1.17 we obtain the complex

$$\dots X(\mathbb{Z}^{n+r+1}) \xrightarrow{d_n} X(\mathbb{Z}^{n+r}) \to \dots \to X(\mathbb{Z}^{r+2}) \xrightarrow{d_1} X(\mathbb{Z}^{r+1}) \tag{5}$$

whose homology is $Tor_*^{\mathbf{gr}}(X, \mathfrak{a} \otimes P_n)$. (In [DV15] the maps of this complex are described carefully).

(2) **A vanishing criterion**
In [DV15, Proposition 5.4] we give the following vanishing criterion:

Proposition 4.12. *Let $X : \mathbf{gr}^{op} \to \Bbbk\text{-}\mathbf{Mod}$ be a functor such that there exists, for all objects A and T in $\mathbf{gr}$, a linear map $\xi(A,T) : X(A) \to X(T * A)$ satisfying the following properties:*

(a) *for all $\varphi : A \to B$ and T in $\mathbf{gr}$, the compositions*

$$X(B) \xrightarrow{X(\varphi)} X(A) \xrightarrow{\xi(A,T)} X(T * A)$$

and

$$X(B) \xrightarrow{\xi(B,T)} X(T * B) \xrightarrow{X(T*\varphi)} X(T * A)$$

coincide;

(b) *the composition*

$$X(A) \xrightarrow{\xi(A,T)} X(T * A) \xrightarrow{X(u(A,T))} X(A)$$

is the identity, where $u(A,T) : A \to T * A$ is the canonical inclusion;

(c) *let $\varphi : A \to B$, T and $\tau : T \to T * B$ in $\mathbf{gr}$. Assume that the morphism $\theta : T * B \to T * B$ which is equal to the identity on B and to τ on T is an isomorphism. For $\psi : T * A \to T * B$ the morphism with components $A \xrightarrow{\varphi} B \xrightarrow{u(B,T)} T * B$ and $T \xrightarrow{\tau} T * B$, then the compositions*

$$X(B) \xrightarrow{X(\varphi)} X(A) \xrightarrow{\xi(A,T)} X(T * A)$$

and

$$X(B) \xrightarrow{\xi(B,T)} X(T * B) \xrightarrow{X(\psi)} X(T * A)$$

coincide.

Then $\mathrm{Tor}_*^{\mathbf{gr}}(X, \mathfrak{a} \otimes P_r) = 0$ *for all r.*

To prove this proposition we show that $h_n = \xi(\mathbb{Z}^{*n+r}, \mathbb{Z}) : X(n+r) \to X(n+r+1)$ is an homotopy of the complex (5).

(3) L_i **satisfies the conditions of the criterion.** See [DV15, Lemme 6.2].
We conclude by Proposition 4.12 that $\mathrm{Tor}_*^{\mathbf{gr}}(L_i, \mathfrak{a} \otimes P_r) = 0$ for all r.

(4) **For $F : \mathbf{gr} \to \Bbbk$-Mod reduced polynomial and $G : \mathbf{gr} \to \Bbbk$-Mod, if $Tor^{\mathbf{gr}}_*(X, \mathfrak{a} \otimes P_r) = 0$ then $Tor^{\mathbf{gr}}_*(X, F \otimes G) = 0$.** See [DV15, Proposition 6.3].
In this step we use a refined version of Proposition 2.21 (we use the recollement diagram between the three categories).

Combining (2), (3) and (4) applied to the constant functor $G = \Bbbk$ we obtain the result. □

4.4. **Stable homology with coefficients twisted by a contravariant functor.** Contrary to the case of general linear groups, the stable homology of automorphism groups of free groups with coefficients given by a contravariant functor can not be obtained from the one with coefficients given by of a covariant functor. Recently, Djament relates this stable homology to functor homology. More precisely, he obtains the following theorem. Recall that $\mathfrak{a}_{\mathbb{Q}} : \mathbf{gr} \to \mathbb{Q}$-**Mod**; $G \mapsto \mathfrak{a}(G) \otimes \mathbb{Q}$.

Theorem 4.13. [Dja] *Let $N : \mathbf{gr}^{op} \to \mathbb{Q}$-**Mod** be a polynomial functor, then there is a natural isomorphism*

$$\underset{n\in\mathbb{N}}{colim}\ H_r(Aut(F_n); N(F_n)) \simeq \bigoplus_{i+j=r} Tor^{\mathbf{gr}}_i(N, \Lambda^j \circ \mathfrak{a}_{\mathbb{Q}}).$$

This theorem is equivalent to the following statement in cohomology:
let $M : \mathbf{gr} \to \mathbb{Q}$-**Mod** be a polynomial functor, then there is a natural isomorphism

$$\lim_{n\in\mathbb{N}} H^r(Aut(F_n); M(F_n)) \simeq \bigoplus_{i+j=r} Ext^i_{\mathcal{F}(\mathbf{gr},\mathbb{Q})}(\Lambda^j \circ \mathfrak{a}_{\mathbb{Q}}, M).$$

The proof of Theorem 4.13 uses Galatius's theorem 4.3, the stability of the homology of automorphisms groups of free groups with twisted coefficients proved by Randal-Williams and Wahl in [RWW] (see Theorem 4.2) and, above all, functor homology arguments. In fact, to prove his theorem, Djament compares the homology of several categories having as objects free groups.

Remark 4.14. In Theorem 4.13, N is not supposed to be reduced. In fact, if $N = \mathbb{Q}$ this statement gives

$$\underset{n\in\mathbb{N}}{\operatorname{colim}}\ H_r(Aut(F_n); \mathbb{Q}) \simeq \bigoplus_{i+j=r} Tor^{\mathbf{gr}}_i(\mathbb{Q}, \Lambda^j \circ \mathfrak{a}_{\mathbb{Q}}) = \bigoplus_{i+j=r} H_i(\mathbf{gr}, \Lambda^j \circ \mathfrak{a}_{\mathbb{Q}}) = \Lambda^r \circ \mathfrak{a}_{\mathbb{Q}}(0)$$

where the last equality comes from Example 3.7 (**gr** has 0 as terminal object). For $r = 0$ we obtain $\mathbb{Q}$ and for $r > 0$ we obtain 0, so this case corresponds to the corollary 4.4 of the theorem of Galatius.

Remark 4.15. Taking $r = 1$ in Theorem 4.13 we obtain:

$$\underset{n\in\mathbb{N}}{\operatorname{colim}}\ H_1(Aut(F_n); N(F_n)) \simeq \bigoplus_{i+j=1} Tor^{\mathbf{gr}}_i(N, \Lambda^j \circ \mathfrak{a}_{\mathbb{Q}}) = N \underset{\mathbf{gr}}{\otimes} \Lambda^1 \circ \mathfrak{a}_{\mathbb{Q}} \oplus Tor^{\mathbf{gr}}_1(N, \mathbb{Q}).$$

By Example 3.7, $Tor^{\mathbf{gr}}_1(G, \mathbb{Q}) = 0$. So

$$\underset{n\in\mathbb{N}}{\operatorname{colim}}\ H_1(Aut(F_n); N(F_n)) \simeq N \underset{\mathbf{gr}}{\otimes} \mathfrak{a}_{\mathbb{Q}}.$$

This result can be found at the end of section 6 in [Kaw06] and is reproved in [DV15, Proposition 7.2] by functor homology methods and with target category **Ab**.

Combining this result with the concrete computations presented in section 3 we obtain the following concrete computations of stable homology:

Proposition 4.16. *Let $H_n = Hom_{\mathbf{gr}}(F_n, \mathbb{Q})$. We have*

$$\underset{n\in\mathbb{N}}{colim}\ H_*(Aut(F_n), \Lambda^d \circ \mathfrak{a}_{\mathbb{Q}}(H_n)) \simeq \begin{cases} \mathbb{Q}^{\rho(d)} & if * = d \\ 0 & otherwise \end{cases}$$

where $\rho(d)$ denotes the number of partitions of d, and

$$\underset{n\in\mathbb{N}}{colim}\ H_*(Aut(F_n), S^d \circ \mathfrak{a}_{\mathbb{Q}}(H_n)) \simeq \begin{cases} \mathbb{Q} & if * = d = 1 \ or * = d = 0 \\ 0 & otherwise \end{cases}$$

Remark 4.17. Two particular cases of Proposition 4.16 were known before. First Kawazumi obtained at the end of section 6 in [Kaw06] that $\operatorname{colim}_{n\in\mathbb{N}} H_1(Aut(F_n), \mathfrak{a}_{\mathbb{Q}}(H_n)) = \mathbb{Q}$ and as a consequence of results of Satoh in [Sat07] we have $\operatorname{colim}_{n\in\mathbb{N}} H_2(Aut(F_n), \mathfrak{a}_{\mathbb{Q}}(H_n)) = 0$.

Proof. We have

$$\begin{aligned}\operatorname*{colim}_{n\in\mathbb{N}} H_r(Aut(F_n); \Lambda^d \circ \mathfrak{a}_{\mathbb{Q}}(H_n)) &\simeq \bigoplus_{i+j=r} Tor_i^{\mathbf{gr}}(\mathcal{H}om(\Lambda^d \circ \mathfrak{a}_{\mathbb{Q}}, \mathbb{Q}), \Lambda^j \circ \mathfrak{a}_{\mathbb{Q}})\\ &\simeq \bigoplus_{i+j=r} Ext^i_{\mathcal{F}(\mathbf{gr},\mathbb{Q})}(\Lambda^j \circ \mathfrak{a}_{\mathbb{Q}}, \Lambda^d \circ \mathfrak{a}_{\mathbb{Q}})\end{aligned}$$

where the first isomorphism is given by Theorem 4.13 and the second by duality (see the argument of Corollary 3.17 for a similar proof). By Proposition 3.19 we obtain:

$$\operatorname*{colim}_{n\in\mathbb{N}} H_r(Aut(F_n); \Lambda^d \circ \mathfrak{a}_{\mathbb{Q}}(H_n)) \simeq \begin{cases} \bigoplus_{i+j=d} \mathbb{Q}^{\rho(d,j)} = \mathbb{Q}^{\rho(d)} & \text{if } r = d\\ 0 & \text{otherwise}\end{cases}$$

where $\rho(d, j)$ denotes the number of partitions of d into j parts.

Similarly, we have

$$\begin{aligned}\operatorname*{colim}_{n\in\mathbb{N}} H_r(Aut(F_n); S^d \circ \mathfrak{a}_{\mathbb{Q}}(H_n)) &\simeq \bigoplus_{i+j=r} Tor_i^{\mathbf{gr}}(\mathcal{H}om(S^d \circ \mathfrak{a}_{\mathbb{Q}}, \mathbb{Q}), \Lambda^j \circ \mathfrak{a}_{\mathbb{Q}})\\ &\simeq \bigoplus_{i+j=r} Ext^i_{\mathcal{F}(\mathbf{gr},\mathbb{Q})}(\Lambda^j \circ \mathfrak{a}_{\mathbb{Q}}, S^d \circ \mathfrak{a}_{\mathbb{Q}}).\end{aligned}$$

By Proposition 3.19 we obtain:

$$\operatorname*{colim}_{n\in\mathbb{N}} H_r(Aut(F_n); S^d \circ \mathfrak{a}_{\mathbb{Q}}(H_n)) \simeq \begin{cases} \mathbb{Q} & \text{if } r = d = 0 \text{ or } d = r = 1\\ 0 & \text{otherwise}\end{cases}$$

where $\rho(d, j)$ denotes the number of partitions of d into j parts.

□

Theorem 4.5 and Proposition 4.16 answer partially to Problem 17 asked by Morita in [Mor06]: Compute the cohomology of $AutF_n$ and $OutF_n$ with coefficients in various $GL(n, Q)$-modules.

Note that the results of Proposition 4.16 were conjectured by Randal-Williams in [RW]. His conjectures are based on the collapsing at the E_2-page of a topological spectral sequence. In the proof of his theorem 4.13, Djament proves the collapsing at the E_2-page of a purely algebraic spectral sequence.

5. Functor homology and homology of algebras

The aim of this section is to give an overview of several results giving an interpretation of classical homology theories in terms of functor homology and to explain why is this point of view interesting.

In this section $\mathbb{k}$ is a field.

5.1. **Functor homology of Γ-modules and higher order Hochschild homology.**

- Hochschild homology:

For A a commutative, associative and unital $\mathbb{k}$-algebra and M a symmetric A-bimodule. Assume that A and M are $\mathbb{k}$-projective.

Definition 5.1. *The i-th Hochschild homology group of A with coefficients in M, $HH_i(A; M)$ is the i-th homology group of the complex*

$$\ldots \xrightarrow{d} M \otimes A^{\otimes n} \xrightarrow{d} M \otimes A^{\otimes n-1} \xrightarrow{d} \ldots$$

where $d = \sum_{i=0}^{n} (-1)^i d_i$ *with*

$$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \text{ for } i < n$$

and

$$d_n(a_0 \otimes \ldots \otimes a_n) = a_n a_0 \otimes \ldots \otimes a_{n-1}$$

Theorem 5.2. ***Classical Hodge decomposition***
If $\mathbb{k}$ is a field of characteristic zero, then

$$HH_n(A;M) \simeq \bigoplus_{i=0}^{n} HH_n^{(i)}(A;M) \tag{6}$$

for $n \geq 1$ where $HH_n^{(1)}(A,M)$ is the Harrison homology and $HH_n^{(n)}(A,M)$ can be described in terms of differential forms.

This theorem was first proved by Quillen by a spectral sequence argument. Later it was reproved independently by Hain, Gerstenhaber-Schack [GS87] and Loday [Lod89] by combinatorially arguments.

We will see below how Pirashvili recovers and extends this theorem to what is called higher order Hochschild homology.

- Γ-modules and homotopy theory

Let $\mathbf{Set}_*$ be the category of all pointed sets. A functor $F : \Gamma \to \mathbb{k}\text{-}\mathbf{Mod}$ can be extended in a functor $s\mathbf{Set}_* \to s\mathbb{k}\text{-}\mathbf{Mod}$ where $s\mathbf{Set}_*$ is the category of simplicial pointed sets and $s\mathbb{k}\text{-}\mathbf{Mod}$ the category of simplicial vector spaces. First, one can extend F by direct limits to a functor $\mathbf{Set}_* \to \mathbb{k}\text{-}\mathbf{Mod}$, then by the degreewise action one obtains a functor $s\mathbf{Set}_* \to s\mathbb{k}\text{-}\mathbf{Mod}$. By abuse of notation we will still denote this functor by F.

Let Δ be the category of finite, non-empty totally ordered sets and order-preserving functions between them. Recall that the sphere S^1 has the following simplicial model (i.e. S^1 can be interpreted as a functor $S^1 : \Delta^{op} \to \Gamma$): $S^1_n = [n]$ and the face and degeneracy maps d_i, s_i are given by:

- $s_i : [n] \to [n+1]$ is the unique monotone injection that does not contain $i+1$;
- $d_i : [n] \to [n-1]$

$$d_i(j) = \begin{cases} j & \text{if} \quad j < i \\ i & \text{if} \quad j = i < n \\ 0 & \text{if} \quad j = i = n \\ j-1 & \text{if} \quad j > i. \end{cases}$$

The aim of this section is to give results concerning $\pi_* F(L)$ for $L \in s\mathbf{Set}_*$. The case $L = S^n$ (the n-sphere) is particulary interesting.

- The functors $t : \Gamma^{op} \to \mathbb{k}\text{-}\mathbf{Mod}$, $\theta^n : \Gamma^{op} \to \mathbb{k}\text{-}\mathbf{Mod}$ and the Loday functor $\mathcal{L}(A,M) : \Gamma \to \mathbb{k}\text{-}\mathbf{Mod}$

Let $t : \Gamma^{op} \to \mathbb{k}\text{-}\mathbf{Mod}$ given by

$$t([n]) := \mathbf{Set}_*([n], \mathbb{k})$$

where the field $\mathbb{k}$ is considered as a pointed set with basepoint 0. Note that we have an exact sequence

$$\mathbb{k}[\Gamma(-,[2])] \to \mathbb{k}[\Gamma(-,[1])] \to t \to 0.$$

For A a commutative, associative and unital $\mathbb{k}$-algebra and M a symmetric A-bimodule. The Loday functor $\mathcal{L}(A,M) : \Gamma \to \mathbb{k}\text{-}\mathbf{Mod}$ is defined by $\mathcal{L}(A,M)([n]) = M \otimes A^{\otimes n}$ and for f an element in $\Gamma([n],[m])$ the induced map $f_* : \mathcal{L}(A,M)([n]) \to \mathcal{L}(A,M)([m])$ is given by

$$f_*(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = b_0 \otimes \ldots \otimes b_m$$

where $b_i = \prod\limits_{j \in f^{-1}(i)} a_j$ or $b_i = 1$ if $f^{-1}(i) = \emptyset$.

We have:

$$HH_*(A;M) \simeq \pi_*(\mathcal{L}(A,M))(S^1).$$

- Higher order Hochschild homology and its Hodge decomposition:

Theorem 5.3. *Let $\mathbb{k}$ be a field of characteristic zero. For $F : \Gamma \to \mathbb{k}\text{-}\mathbf{Mod}$ we have:*

$$\pi_n F(S^1) \simeq \bigoplus_{i=0}^{n} HH_n^{(i)}(F)$$

where $HH_n^{(i)}(F) \simeq Tor_{n-i}^{\Gamma}(\Lambda^i \circ t, F)$.

Remark 5.4. Djament's theorem 4.13 looks like very similar to this decomposition...

In the classical Hodge decompositon 5.2 the pieces of the decomposition are given by functor homology:

$$HH_n^{(i)}(A;M) \simeq Tor_{n-i}^{\Gamma}(\Lambda^i \circ t, \mathcal{L}(A,M)).$$

In [Pir00b] Pirashvili extends this result to what he called "higher Hochschild homology".

Definition 5.5. [Pir00b] *The Hochschild homology of order n, denoted by* $HH_*^{[n]}(A;M)$*, is*

$$HH_*^{[n]}(A;M) := \pi_*(\mathcal{L}(A;M))(S^n).$$

Higher Hochschild homology $HH_*^{[n]}(A;M)$ has a decomposition generalizing Theorem 5.2:

Theorem 5.6. [Pir00b, Proposition 5.2] *Let* $\Bbbk$ *be a field of characteristic zero.*

- *If d is odd,*

$$HH_n^{[d]}(A,M) \simeq \bigoplus_{i+dj=n} Tor_i^{\Gamma}(\Lambda^j \circ t, \mathcal{L}(A;M)) \simeq \bigoplus_{i+dj=n} HH_{i+j}^{(j)}(A;M)$$

- *If d is even*

$$HH_n^{[d]}(A,M) \simeq \bigoplus_{i+dj=n} Tor_i^{\Gamma}(\theta^j, \mathcal{L}(A;M)).$$

Remark 5.7. For $1 \leq n \leq \infty$, an E_n-algebra is an algebra over an operad in chain complexes which is weakly equivalent to the chain complex of the little n-cubes operad. The E_n-homology of an E_n-algebra are homological invariants specifically suited to E_n-algebras. By neglect of structure, any commutative algebra can be viewed as an E_n-algebra. Therefore one can consider E_n-homology of commutative algebras. Higher order Hochschild homology defined previously corresponds to this homology (see [GTZ14] and [Zie]).

5.2. Functor homology and other homology theories.

We saw in the previous section that pieces of usual Hochschild homology of commutative algebra are given by functor homology over the category Γ. We will see in this section that there are many notions of homology which can be expressed as functor homology over a suitable category.

In the 90s, Robinson and Whitehouse defined and study in [RW02] Γ-homology of commutative algebras, a homology theory suited in the context of differential graded modules over a field of positive characteristic. In [PR00] Pirashvili and Richter proved that this homology can be interpreted as functor homology over Γ.

For associative algebras, similar results are obtained by Pirashvili and Richter. In [PR02], they interpret Hochschild and cyclic homology of associative algebras as functor homology. In this setting, the category Γ is replaced by a category associated with the associative set operad.

In [HV15], we obtain a description of Leibniz homology of Lie algebras as functor homology. In this setting, the category Γ is replaced by a category associated with the Lie operad satisfying a shuffle condition.

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