

# ON RATIONALLY CONNECTED 3-FOLDS

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ABSTRACT. We study the existence of Hodge metrics of positive total scalar curvature on projective threefolds of negative Kodaira dimension. We give strong indications about the existence of such metrics on rationally connected threefolds. The proofs rely on the theory of minimal models, the weak factorization theorem and on a specialization technique.

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## INTRODUCTION

In this article we address the following question:

**Question 0.1.** *Let  $X$  be a smooth complex  $n$ -fold of Kähler type and negative Kodaira dimension. Does  $X$  admit Kähler metrics of positive total scalar curvature?*

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If we denote by  $s_g$  and  $d\mu_g$  the scalar curvature and the volume form of  $g$ , respectively, this is the same as asking if there is any Kähler metric  $g$  on  $X$  such that  $\int_X s_g d\mu_g > 0$ .

For Kähler metrics, the total scalar curvature has a simpler expression:

$$\int_X s_g d\mu_g = 2\pi n c_1(X) \cup [\omega]^{n-1} \quad (0.1)$$

where  $[\omega]$  is the cohomology class of the Kähler form of  $g$ . The negativity of the Kodaira dimension is a necessary condition [Yau74] because, arguing by contradiction, if for some  $m > 0$ , the  $m^{\text{th}}$  power of the canonical bundle of  $X$  is either trivial or has sections one can immediately see that  $c_1(X) \cup [\omega]^{n-1}$  is negative, which, by (0.1), would imply that the total scalar curvature of  $(X, g)$  was negative.

Question 0.1 has an immediate positive answer in dimension 1. The only smooth complex curve of negative Kodaira dimension is  $\mathbb{P}_1$ , and the Fubini-Study metric satisfies the required inequality. In complex dimension 2, relying on the classification of complex surfaces, a positive answer was given by S.-T. Yau in [Yau74].

Inspired by Yau's approach, we tackle Question 0.1 in the case of projective threefolds, where a satisfactory theory of minimal models exists. As in [Yau74], we can look for Hodge metrics instead, and ask :

**Question 0.2.** *Let  $X$  be a smooth projective 3-fold, with  $\text{Kod}(X) = -\infty$ . Is there any ample line bundle  $H$  on  $X$  such that  $K_X \cdot H^2 < 0$ ?*

A positive answer to this question<sup>1</sup> can be connected to a deep result of S. Mori and Y. Miyaoka [MiyMo86], and regarded as a possible effective characterization of the class of smooth, projective threefolds of negative Kodaira dimension. Also, Question 0.2 can be viewed as extracting some positivity property of the anticanonical bundle. An affirmative answer would yield in dimension three, a weak alternative to the generic semi-positivity theorem of Miyaoka asserting that, for non-uniruled manifolds, the restriction of the cotangent bundle to a general smooth complete intersection curve cut out by elements of  $|mH|$  is semi-positive, for any ample divisor  $H$  and  $m \gg 0$ .

In this article, as an attempt to answer the original Question 0.1, we study Question 0.2 in the case of rationally connected threefolds. Recall that cf. [KMM92], a complex projective manifold  $X$  of dimension  $n \geq 2$  is called *rationally connected* if there is a rational curve passing through any two given points of  $X$ . One reason to restrict our attention to this kind of manifolds comes from the observation that for rationally connected manifolds, answering affirmatively to Question 0.1 is equivalent to answering affirmatively to Question 0.2. This follows from their convenient cohomological properties. Namely, if  $X$  is such a manifold, then  $H^i(X, \mathcal{O}_X) = 0$ , for  $i \geq 1$ , [KMM92]. But in this case, from the Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$$

it follows that  $H^2(X, \mathbb{C}) \simeq H^{1,1}(X)$ . Thus, we can see that any  $(1, 1)$ -form with real coefficients can be approximated<sup>2</sup> by a  $(1, 1)$ -form with rational coefficients. Hence, up to multiplication by positive integers, any Kähler forms can be approximated by first Chern classes of *ample line bundles*. From this the equivalence of our two questions follows easily. However, this is not the only reason to restrict ourselves to the case of rationally connected threefolds, as we will also make use of their wealth of rational curves [ArKo03].

<sup>1</sup>The same question has also been raised in a different context by F. Campana, J. P. Demailly, T. Peternell and M. Schneider, [DPSch96], [CP98]

<sup>2</sup>Here we consider  $H^2(X, \mathbb{R})$  as a finite dimensional real vector space, endowed with any metric topology

To simplify the exposition, for a projective manifold  $X$ , we will say that the property  $\mathcal{P}_X$  holds true if there exists a Cartier divisor  $H$  on  $X$  such that  $K_X \cdot H^2 < 0$ .

Our quest to provide a positive answer to Question 0.2 for rationally connected threefolds falls short, the difficulty being pointed out by the following:

**Conjecture 0.3.** *Let  $X$  be a projective threefold containing a smooth rational curve  $C \subset X$  with  $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ , and let  $X \dashrightarrow X'$  be its flop. If  $X'$  is projective, then*

$$\mathcal{P}_X \text{ holds true} \iff \mathcal{P}_{X'} \text{ holds true.}$$

Our result is:

**Theorem A.** *If Conjecture 0.3 holds true, then for every projective, rationally connected manifold  $X$  of dimension 3 the property  $\mathcal{P}_X$  holds true.*

In Section 3.6 we will formulate a very special case of our Conjecture 0.3 which we actually need in order to prove Theorem A.

Similar to the case of complex surfaces, the idea to prove this theorem is to start with an arbitrary 3-fold  $X$  of negative Kodaira dimension, and running the minimal models program to go to a minimal model  $X_{min}$ . In doing so, we get a birational map  $f : X \dashrightarrow X_{min}$  to a new three dimensional projective variety  $X_{min}$ , with at most  $\mathbb{Q}$ -factorial terminal singularities, which is either a Mori fiber space, or has nef canonical bundle. The hypothesis on the Kodaira dimension exclude [Miy88] the possibility of  $X_{min}$  having nef canonical bundle. Hence,  $X_{min}$  must be a Mori fiber space, i.e.  $X_{min}$  is either a del Pezzo fibration, a conic bundle or a Fano variety. As in [CP98] and [DPSch96], we can see that  $\mathcal{P}_{X_{min}}$  holds true. The difficult part of this program is to show that

$$\mathcal{P}_{X_{min}} \implies \mathcal{P}_X.$$

In this regard, in Section 2 we will show that it is enough to show that  $\mathcal{P}$  is a birational property of *smooth* projective threefolds. We can use now the weak factorization theorem [AKMW02], which says that any birational map between smooth (projective) manifolds can be decomposed into a finite sequence of blow-ups and blow-downs with nonsingular centers of (projective) manifolds. We prove the following:

**Proposition 0.4.** *Let  $p : Y \rightarrow X$  be the blow-up of a smooth, projective 3-fold at a point. Then*

$$\mathcal{P}_X \text{ holds true} \iff \mathcal{P}_Y \text{ holds true.}$$

For the blowing-up along curves the following result is of crucial importance:

**Proposition 0.5.** *Let  $p : Y \rightarrow X$  be the blow-up of a smooth, projective 3-fold along a smooth curve  $C$ .*

- $\mathcal{P}_X$  holds true  $\implies \mathcal{P}_Y$  holds true.
- If  $K_X \cdot C < 0$ , then

$$\mathcal{P}_Y \text{ holds true} \implies \mathcal{P}_X \text{ holds true.}$$

In the last case to verify,  $\mathcal{P}_Y \implies \mathcal{P}_X$ , where  $Y \rightarrow X$  is the blowing-up of smooth projective threefolds along smooth curves with  $K_X \cdot C \geq 0$ , the methods used to prove the previous results do not work anymore. It is only this last hurdle, where we use the rational connectedness assumption.

Our approach is to reduce this last case to the one we have already solved. More precisely, we are going to find a smooth curve  $C' \subset X$  with  $K_X \cdot C' < 0$ , such that  $\mathcal{P}_{Y'}$  holds true, where  $Y' \rightarrow X$  is the blowing-up of  $X$  along  $C'$ , then to use Proposition 0.5.

The condition  $K_X \cdot C < 0$  imposed in the previous proposition can be interpreted, by the Riemann-Roch theorem, as saying that the curve  $C$  "moves". What we do in our construction is "forcing  $C$  to move", by eventually modifying it, while preserving property  $\mathcal{P}$ . Unfortunately, we were unable to complete entirely this task, but merely to reduce it to the validity of Conjecture 0.3. This is done by a lengthy specialization argument, where we strongly rely on the rational connectedness hypothesis. The construction presented in Section 3.2, on which all the computations are performed is based on the work of Graber, Harris, Starr [GHS03] and Kollár [ArKo03]. We devote to this specialization argument the entire Section 3.

In Section 4 we prove Theorem A. An appendix containing some results used intensively throughout this entire chapter is added for the reader's convenience.

**Conventions:** We work over the field of complex numbers and we use the standard notations and terminology of Hartshorne's Algebraic Geometry book [Hart77].

## 1. MINIMAL MODELS

In this section we introduce the objects which appear in the theory of minimal models and we show that our problem has a positive answer for Mori fiber spaces.

Let  $X$  be a variety with  $\dim X > 1$ , such that  $K_X$  is  $\mathbb{Q}$ -Cartier, i.e.  $mK_X$  is Cartier for some positive integer  $m$ . If  $f : Y \rightarrow X$  is a proper birational morphism such that  $K_Y$  is a line bundle (e.g.  $Y$  is a resolution of  $X$ ), then  $mK_Y$  is linearly equivalent to:

$$f^*(mK_X) + \sum m \cdot a(E_i) \cdot E_i,$$

where the  $E_i$ 's are the exceptional divisors. Using numerical equivalence, we can divide by  $m$  and write:

$$K_Y \equiv_{\mathbb{Q}} f^*K_X + \sum a(E_i) \cdot E_i.$$

**Definition 1.1.** *We say that  $X$  has terminal singularities if for any resolution, and for any  $i$ ,  $a(E_i) > 0$ .*

**Definition 1.2.** *We say that a variety  $X$  is  $\mathbb{Q}$ -factorial if for any Weil divisor  $D$  there exist a positive integer  $m$  such that  $mD$  is a Cartier divisor.*

The minimal model program (MMP) studies the structure of varieties via birational morphisms or birational maps of special types to seemingly simpler varieties. The birational morphisms which appear running the MMP are the following:

**Definition 1.3 (divisorial contractions).** *Let  $X$  be a projective variety with at most  $\mathbb{Q}$ -factorial singularities. A birational morphism  $f : X \rightarrow Y$  is called a divisorial contraction if it contracts a divisor,  $-K_X$  is  $f$ -ample and  $\text{rank } NS(X) = \text{rank } NS(Y) + 1$ .*

The main difficulty in the higher dimensional minimal model program is the existence of non-divisorial contractions. When the variety  $X$  is  $\mathbb{Q}$ -factorial with only terminal singularities, one may get contractions, called contractions of flipping type  $f : X \rightarrow Y$ , where the exceptional locus  $E$  has codimension at least 2, but such that  $-K_X$  is  $f$ -ample and  $\text{rank } NS(X) = \text{rank } NS(Y) + 1$ . In this case,  $K_Y$  is no longer  $\mathbb{Q}$ -Cartier. The remedy in dimension 3 is the existence of special birational maps, called flips, which allow to replace  $X$  by another  $\mathbb{Q}$ -factorial variety  $X^+$  with only terminal singularities, but simpler in some sense, [KoMo98]:

**Definition 1.4 (flips).** *Let  $f : X \rightarrow Y$  be a flipping contraction as above. A variety  $X^+$  together with a map  $f^+ : X^+ \rightarrow Y$  is called a flip of  $f$  if  $X^+$  is  $\mathbb{Q}$ -factorial varieties with terminal singularities and  $K_{X^+}$  is  $f^+$ -ample.*

By abuse of terminology, the birational map  $X \dashrightarrow X^+$  will also be called a flip.

The minimal model program starts with an arbitrary projective  $\mathbb{Q}$ -factorial threefold with at most terminal singularities  $X$  on which one applies an suitable sequence of divisorial contractions and flips. To describe the outcome of the MMP, we recall the following definition:

**Definition 1.5 (Mori fiber spaces).** *Let  $X$  and  $Y$  be two irreducible  $\mathbb{Q}$ -factorial varieties with terminal singularities,  $\dim X > \dim Y$ , and  $f : X \rightarrow Y$  a morphism. The triplet  $(X, Y, f)$  is called a Mori fiber space if  $-K_X$  is  $f$ -ample, and*

$$\text{rank } NS(X) = \text{rank } NS(Y) + 1.$$

**Theorem 1.6 (Mori).** *Let  $X$  be a projective variety with only  $\mathbb{Q}$ -factorial terminal singularities, and  $\dim X = 3$ . Then there exist a birational map  $f : X \dashrightarrow X_{\min}$ , which is a composition of divisorial contractions and flips, such that either  $K_{X_{\min}}$  is nef or  $X_{\min}$  has a Mori fiber space structure.*

In their approach to Question 0.2, Campana and Peternell proved some important cases. Because of the simplicity, we include their proofs.

The following easy lemma will be used frequently throughout the proofs, sometimes without referring to it.

**Lemma 1.7.** *Let  $X$  be a projective  $\mathbb{Q}$ -factorial variety of dimension 3, with (at most) terminal singularities for which there exists a nef line bundle  $D$  with  $K_X \cdot D^2 < 0$ . Then there exists an ample line bundle  $H$  on  $X$  such that  $K_X \cdot H^2 < 0$ .*

*Proof.* Let  $L$  be any ample on  $X$ . Then for any positive integer  $m$ ,  $H_m := mD + L$  is an ample line bundle with:

$$K_X \cdot H_m^2 = K_X \cdot (mD + L)^2 = mK_X \cdot D^2 + 2K_X \cdot D \cdot L + K_X \cdot L^2 < 0$$

for  $m \gg 0$ . □

**Proposition 1.8 (Mori fiber spaces).** *Let  $(X, Y, f)$  be a Mori fiber space, with  $\dim X = 3$ . Then the property  $\mathcal{P}_X$  holds true.*

*Proof.* Since  $\dim X = 3$  we have 3 cases, according to the dimension of  $Y$  :

**Case 1 (dim  $Y=0$ ):** In this case  $X$  is a  $\mathbb{Q}$ -Fano variety with  $\text{rank } NS(X) = 1$ .

In particular,  $-mK_X$  is an ample line bundle, for some integer  $m > 0$ , and the property  $\mathcal{P}_X$  follows immediately.

**Case 2 (dim  $Y=1$ ):** Take  $L_Y$  be any ample line bundle on  $Y$ . Since  $-K_X$  is  $f$ -ample it follows that  $K_X \cdot (f^*L)^2 < 0$ , and as  $f^*L_Y$  is nef, from Lemma 1.7 we can see that  $\mathcal{P}_X$  holds true.

**Case 3 (dim  $Y=2$ ):** As before we take  $L_Y$  be any ample line bundle on  $Y$  and  $H_X$  be an ample line bundle on  $X$ . Then for any positive integer  $H_m := mf^*L + H_X$ , is an ample line bundle and we have:

$$K_X \cdot H_m^2 = K_X \cdot (mf^*L + H_X)^2 = K_X \cdot H_X^2 + 2mK_X \cdot H_X \cdot f^*L_Y < 0,$$

for  $m \gg 0$ , again because  $-K_X$  is  $f$ -ample. □

**Corollary 1.9.** *Let  $X$  be a  $\mathbb{Q}$ -factorial, projective variety of dimension three with at most terminal singularities. If  $\text{Kod}(X) = -\infty$  then  $\mathcal{P}_{X_{min}}$  holds true.*

*Proof.* From Theorem 1.6 we know that  $X_{min}$  is either a Mori fiber space, for which  $\mathcal{P}_{X_{min}}$  holds true, or  $K_{X_{min}}$  is nef. To exclude the second possibility, we note that from Miyaoka's abundance theorem [MiPe97, page 88], in dimension three this would imply that  $\text{Kod}(X_{min}) \geq 0$ . However, this is impossible since the Kodaira dimension is a birational invariant.  $\square$

## 2. VARIOUS REDUCTIONS

Our first reduction takes care of the singularities. Let  $X$  be a projective  $\mathbb{Q}$ -factorial variety  $X$  of dimension three, with terminal singularities. A nice feature of these singularities is that, in dimension three, they are isolated points. By Hironaka's resolution of singularities we can always find resolution  $p : X' \rightarrow X$  which is an isomorphism outside the singular points of  $X$ . We begin by proving the following:

**Proposition 2.1.**  *$\mathcal{P}_{X'}$  holds true if and only if  $\mathcal{P}_X$  holds true.*

*Proof.* Suppose first that  $\mathcal{P}_X$  holds true. Hence there exists an ample line bundle  $L$  on  $X$  such that  $K_X \cdot L^2 < 0$ , and let  $D' = p^*L$ . Then  $D$  is a nef line bundle on  $X'$  and

$$K_{X'} \cdot D'^2 = (p^*K_X + \sum_i a_i E_i) \cdot p^*L \cdot p^*L = K_X \cdot L^2 < 0.$$

Using Lemma 1.7 it follows that  $\mathcal{P}_{X'}$  holds also true.

Conversely, suppose now  $\mathcal{P}_{X'}$  holds true, and let  $H'$  be an ample line on  $X'$  such that  $K_{X'} \cdot H'^2 < 0$ . Without loss of generality we can assume  $H'$  very ample and represented by an irreducible divisor, still denoted by  $H'$ . Let  $D := p(H')$  its pushforward in  $X$ . Following [KoMo98, Lemma 3.39] we have

$$p^*D \equiv_{\mathbb{Q}} H' + \sum_i c_i E_i,$$

where the  $E_i$ 's are the exceptional divisors of the resolutions and  $c_i \geq 0$ .

Now, if  $C$  is any curve in  $X$ , let  $C'$  be its strict transform in  $X'$  and so  $p_*C' = C$ . Then,  $D \cdot C = D \cdot p_*C' = p^*D \cdot C' = H' \cdot C' + \sum c_i (E_i \cdot C') > 0$ ,  $C'$  not being contained in any of the exceptional divisors. Thus  $D$  is a strictly nef divisor. The singularities of  $X$  being terminal,  $K_{X'} \equiv_{\mathbb{Q}} p^*K_X + \sum a_i E_i$ , with  $a_i > 0$ . Again, since the singularities of  $X$  are a finite number of isolated points,  $p^*L \cdot E_i = 0$  for any Cartier divisor  $L$  on  $X$ . We immediately obtain:

$$\begin{aligned} K_X \cdot D^2 &= (p^*K_X) \cdot (p^*D)^2 = (p^*K_X) \cdot (H')^2 \\ &= K_{X'} \cdot H'^2 - \sum a_i E_i \cdot H'^2 < 0. \end{aligned}$$

The proposition follows now from Lemma 1.7.  $\square$

Proposition 2.1 allows us to interpret the results we proved in the previous section in the following way. From the minimal models program we obtain a birational map  $f : X \dashrightarrow Y$  from a smooth projective threefold  $X$  to a singular threefold  $Y$  for which  $\mathcal{P}_Y$  holds true. We can replace now  $Y$  by a smooth projective threefold  $X'$  for which  $\mathcal{P}_{X'}$  holds true. Thus the problem we study reduces to the following:

**Question 2.2.** *Is  $\mathcal{P}$  a birational property of the class of projective threefolds of negative Kodaira dimension?*

This is already a major simplification, because we can use now the weak factorization theorem [AKMW02] of Abramovich, Karu, Matsuki and Włodarczyk:

**Theorem 2.3 (Abramovich, Karu, Matsuki, Włodarczyk).** *A birational map between projective nonsingular varieties over an algebraically closed field  $K$  of characteristic zero is a composite of blowings up and blowings down with smooth centers of smooth projective varieties.*

Therefore, what is left to prove is that the property  $\mathcal{P}$  is preserved under blowing-ups and blowing-downs at points and smooth curves, respectively.

### 2.1. Blowing-up at points.

**Proposition 2.4.** *Let  $p : Y \rightarrow X$  be the blow-up of a smooth, projective 3-fold at a point. Then  $\mathcal{P}_X$  holds true if and only if  $\mathcal{P}_Y$  holds true.*

*Proof.* Let  $E$  be the exceptional divisor of  $p$ . Then by [Hart77, Ex. II.8.5],  $\text{Pic}(Y) \cong \text{Pic}(X) \oplus \mathbb{Z}[E]$  and  $K_Y = p^*K_X + 2E$ .

Suppose first that  $\mathcal{P}_X$  holds true and let  $H_X$  be an ample line bundle on  $X$  such that  $K_X \cdot H_X^2 < 0$ . Then  $D_Y \stackrel{\text{def}}{=} p^*H_X$  is a nef line bundle on  $Y$  such that  $K_Y \cdot D_Y^2 = (p^*K_X + 2E) \cdot p^*H_X \cdot p^*H_X = p^*K_X \cdot p^*H_X \cdot p^*H_X = K_X \cdot H_X^2 < 0$ . Using again Lemma 1.7 it follows that  $\mathcal{P}_Y$  holds true.

Conversely, suppose that  $\mathcal{P}_Y$  holds true, and let  $H_Y$  be an ample line bundle on  $Y$  such that  $K_Y \cdot H_Y^2 < 0$ . Then  $H_Y = p^*D_X - aE$ , for some line bundle  $D_X \in \text{Pic}(X)$ , and some positive integer  $a$ . As in the proof of Proposition 2.1, we can show that  $D_X$  is nef and  $K_X \cdot D_X^2 < 0$ . Let  $C$  be any curve in  $X$  and let  $C'$  be its strict transform in  $Y$ . Then  $p_*C' = C$ , and  $D_X \cdot C = D_X \cdot p_*C' = p^*D_X \cdot C' = H_X \cdot C' + aE \cdot C' > 0$ , because  $H_X$  is ample and  $C'$  is not contained in  $E$ . Therefore  $D_X$  is a nef line bundle and

$$K_X \cdot D_X^2 = p^*K_X \cdot p^*D_X \cdot p^*D_X = p^*K_X \cdot H_Y \cdot H_Y = K_Y \cdot H_Y^2 - 2E \cdot H_Y^2 < 0.$$

Applying again Lemma 1.7 we can conclude the proof of the proposition.  $\square$

**2.2. Blowing-up along curves.** In the case of 1-dimensional blowing-up centers, it is easy to prove in one direction:

**Proposition 2.5.** *Let  $p : Y \rightarrow X$  be the blow-up of a smooth, projective 3-fold along a smooth curve  $C$ . If  $\mathcal{P}_X$  holds true then  $\mathcal{P}_Y$  holds true.*

*Proof.* Let  $H_X$  be an ample line bundle on  $X$  satisfying  $K_X \cdot H_X^2 < 0$  and let  $D_Y \stackrel{\text{def}}{=} p^*H_X$ . Then  $D_Y$  is a nef line bundle on  $Y$  and we have

$$K_Y \cdot D_Y^2 = (p^*K_X + E) \cdot p^*H_X \cdot p^*H_X = p^*K_X \cdot p^*H_X \cdot p^*H_X = K_X \cdot H_X^2 < 0.$$

The conclusion follows again from Lemma 1.7.  $\square$

For the converse of Proposition 2.5 the following proposition is the key step in our line of argument. Its proof is rather long, but elementary, based on Proposition C.3.

**Proposition 2.6.** *Let  $p : Y \rightarrow X$  be the blowing-up of a smooth, projective 3-fold along a smooth curve  $C$  such that  $K_X \cdot C < 0$ . If  $\mathcal{P}_Y$  holds true then  $\mathcal{P}_X$  holds true.*

*Proof.* Let  $H_Y$  be an ample line bundle on  $X$  such that  $\mathcal{P}_Y$  holds true. Without loss of generality, we can assume that  $H_X$  is very ample. Since  $p : Y \rightarrow X$  is the blowing-up of  $X$  along  $C \subset X$ , the exceptional divisor  $E = \mathbb{P}_C(N_{C/X}^\vee)$  will be a ruled surface over  $C$ .

Let  $d = \deg_C(N_{C/X})$ , and let  $g$  be the genus of  $C$ . Let  $f$  be a fiber of  $p|_E : E \rightarrow C$ . We denote by  $a$  the intersection number  $(H_Y \cdot f)$  in the Chow ring  $A(Y)$ . We can write:

$$H_Y = p^*L_X - aE \quad (2.1)$$

for some line bundle  $L_X \in \text{Pic}(X)$ . As in the proof of Proposition 2.4, we can check that  $L_X \cdot C' > 0$  for any irreducible curve  $C' \subset X$ , different than  $C$ . Let  $\tilde{C}$  be the strict transform of  $C'$ . Then:

$$L_X \cdot C' = (H_Y + aE) \cdot C' = H_Y \cdot \tilde{C} + aE \cdot \tilde{C} > 0,$$

because  $E \cdot \tilde{C} \geq 0$ , the curve  $\tilde{C}$  is an irreducible curve, obviously not contained in  $E$ . Thus, in order to show that  $L_X$  is nef we only have to check that  $L_X \cdot C \geq 0$ . A straightforward application of the projection formula and of Proposition B.2 gives:

$$\begin{aligned} K_Y \cdot H_Y^2 &= (p^*K_X + E) \cdot (p^*L_X - aE) \cdot (p^*L_X - aE) \\ &= p^*K_X \cdot p^*L_X \cdot p^*E - 2ap^*K_X \cdot p^*L_X \cdot E + a^2p^*K_X \cdot E^2 \\ &\quad + E \cdot p^*L_X \cdot p^*L_X - 2aE^2 \cdot p^*L_X + a^2E^3 \\ &= K_X \cdot L_X^2 - 2aE^2 \cdot p^*L_X + a^2p^*K_X \cdot E^2 + a^2E^3 \\ &= K_X \cdot L_X^2 + 2a(L_X \cdot C) - a^2(K_X \cdot C) - a^2d \\ &= K_X \cdot L_X^2 + 2a(L_X \cdot C) - a^2(2g - 2). \end{aligned}$$

**Observation 2.7.** Since  $K_Y \cdot H_Y^2 < 0$ , to conclude the proof of the Proposition 2.6 it will suffice to prove that

$$2a(L_X \cdot C) - a^2(2g - 2) \geq 0, \quad (2.2)$$

because we would obtain:

- $K_X \cdot L_X^2 < 0$ ,
- $L_X \cdot C \geq 0$ , if  $g \geq 1$ .

If  $g = 0$ , we still have to check that  $L_X \cdot C \geq 0$ .

For a better understanding of (2.2) the following considerations are necessary.

On  $E = \mathbb{P}_C(N_{C/X}^\vee)$ , let  $C_0$  be the section of minimal self-intersection  $C_0^2 = -e$ . We use  $\{C_0, f\}$  as a basis for  $\text{Num}_{\mathbb{Z}}(E)$ . With respect to this basis:

$$\begin{aligned} H_{X|_E} &\equiv aC_0 + bf, \text{ for some } b \in \mathbb{Z}; \\ E|_E &\equiv xC_0 + yf, \end{aligned}$$

where  $x$  and  $y$  can be determined as follows:

$$\begin{aligned} -1 &= E \cdot f = E|_E \cdot_E f = (xC_0 + yf) \cdot_E f = x; \\ -d &= E^3 = E|_E \cdot_E E|_E = (-C_0 + yf)^2 = -e - 2y, \end{aligned}$$

so  $y = \frac{d-e}{2}$ . Here we denoted by " $\cdot_E$ " the intersection product on the exceptional smooth divisor  $E$ .

**Remark 2.8.** Note that the two invariants,  $d$  and  $e$ , of  $E = \mathbb{P}_C(N_{C/X}^\vee)$ , have the same parity.

**Lemma 2.9.** *In the above notations, we have:*

$$2a(L_X \cdot C) - a^2(2g - 2) = 2ab - a^2e - a^2(K_X \cdot C). \quad (2.3)$$

*Proof.* Computing  $H_Y \cdot E \cdot p^*L_X$  in two ways, we obtain:

$$\begin{aligned} H_Y \cdot E \cdot p^*L_X &= (L_X \cdot C)L_X \cdot f = a(L_X \cdot C); \\ H_Y \cdot E \cdot p^*L_X &= H_Y \cdot E \cdot (H_Y + aE) = H_Y^2 \cdot E + aH_Y \cdot E^2. \end{aligned}$$

Thus

$$\begin{aligned} 2a(L_X \cdot C) - a^2(2g - 2) &= 2(H_Y^2 \cdot E + aH_Y \cdot E^2) - a^2(2g - 2) \\ &= 2(H_Y^2 \cdot E + aH_Y \cdot E^2) - a^2d - a^2(K_X \cdot C). \end{aligned}$$

Furthermore:

$$\begin{aligned} 2(H_Y^2 \cdot E + aH_Y \cdot E^2) &= 2[(H_{X|_E} \cdot H_{X|_E}) + a(H_{X|_E} \cdot E|_E)] \\ &= 2(aC_0 + bf)^2 + a(aC_0 + bf) \cdot \frac{[-2C_0 + (d - e)f]}{E} \\ &= 2a^2C_0^2 + 4ab - 2a^2C_0^2 - 2ab + a^2(d - e) \\ &= 2ab + a^2(d - e). \end{aligned}$$

Therefore  $2a(L_X \cdot C) - a^2(2g - 2) = 2ab - a^2e - a^2(K_X \cdot C)$ .  $\square$

We can finish now the proof of Proposition 2.6:

- If  $g \geq 0$ , by Proposition C.3, we have two subcases:
  - i) **Case  $e \geq 0$**  : Since  $H_Y$  is ample,  $H_{Y|_E}$  is ample, and so, by Proposition C.3,  $a > 0$  and  $b > ae$ . Remembering that  $K_X \cdot C < 0$ , from (2.3) we can see that:

$$\begin{aligned} 2a(L_X \cdot C) - a^2(2g - 2) &= 2ab - a^2e - a^2(K_X \cdot C) \\ &> a^2e - a^2(K_X \cdot C) > 0. \end{aligned}$$

By the crucial Observation 2.7 and by Proposition 1.7 we are done.

- ii) **Case  $e < 0$**  : Similarly, since  $H_Y$  is ample,  $H_{Y|_E}$  is ample, too. Thus, by Proposition C.3,  $a > 0$  and  $b > \frac{1}{2}ae$ . Then:

$$\begin{aligned} 2a(L_X \cdot C) - a^2(2g - 2) &= 2ab - a^2e - a^2(K_X \cdot C) \\ &> -a^2(K_X \cdot C) > 0, \end{aligned}$$

and we are again done.

- If  $g = 0$  then  $e \geq 0$ , and in this case it suffices to show that  $2ab + a^2(d - e) \geq 0$ . Since  $H_{Y|_E}$  is ample,  $a > 0$  and  $b > ae$ . We have:

$$2ab + a^2(d - e) > a^2e + a^2d = a^2(e - 2 - K_X \cdot C).$$

So, if  $K_X \cdot C \leq -2$  it follows immediately that  $L_X \cdot C > 0$ , and with the help of Observation 2.7 and Lemma 1.7 we are done again. If  $K_X \cdot C = -1$ , then  $d = -1$  and since  $d$  and  $e$  have the same parity,  $e \geq 1$  and we obtain again  $L_X \cdot C > 0$ , and we can conclude as above.

With this Proposition 2.6 is completely proved.  $\square$

**Remark 2.10.** The proof of Proposition 2.6 also works when  $K_X \cdot C = 0$  and  $g > 0$ . However, when  $C$  is a rational curve  $d = \deg N_{C/X} = -2$ , and the above arguments show that a possible exception occurs only when  $e = 0$ , and  $0 < b < a$ . In this case,  $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ , and what fails is only the nefness of  $L_X$ .

When  $K_X \cdot C > 0$ , nothing can be said with the above approach.

This remark is the one that inspires Conjecture 0.3.

## 3. SPECIALIZATION ARGUMENT

From what we proved so far, to show that  $\mathcal{P}$  is a birational property in the class of smooth, projective threefolds it would be enough to answer affirmatively to the following question:

**Question 3.1.** *Let  $p : X_C \rightarrow X$  be the blowing up of smooth projective threefold  $X$  along a smooth curve  $C \subset X$  with  $K_X \cdot C \geq 0$ . Suppose that  $\mathcal{P}_{X_C}$  holds true. Does  $\mathcal{P}_X$  also hold true?*

Proposition 2.6 is inspirational, suggesting that a positive answer is possible if we can replace the blowing-up  $p : X_C \rightarrow X$  of  $X$  along the curve  $C$  by the blowing-up  $p' : X_{C'} \rightarrow X$  of  $X$  along a smooth curve  $C' \subset X$ , but such that  $K_X \cdot C' < 0$ , as long as we are able to show that  $\mathcal{P}_{X_{C'}}$  also holds true. We will show that such an approach works in the case of *rationally connected* projective threefolds, if we assume the validity of Conjecture 0.3.

A more precise description of our strategy to answer Question 3.1, and the outline of the structure of this section is the following. In the next subsection we introduce the results from the theory of rationally connected manifolds we need. Then using the outcome of Theorem 3.5, we can construct a smooth family over the unit disk  $\mathcal{X} \rightarrow \Delta$ , whose general fiber  $X_{C_t}$  is the blowing-up of  $X$  along a smooth curve  $C_t$  with  $K_X \cdot C_t < 0$ . The central fiber of this family will be a normal crossing divisor whose irreducible components are smooth rationally connected threefolds. In subsection 3.3 we show that any line bundle on the central fiber of  $\mathcal{X} \rightarrow \Delta$  extends to  $\mathcal{X}$ . Moreover, if the line bundle on the central fiber is chosen to be ample, by eventually shrinking  $\Delta$ , its extension restricted to  $X_{C_t}$  will also be ample. We will next show how a positive answer to Conjecture 0.3 allows us to construct an ample line bundles on the whole central fiber of  $\mathcal{X} \rightarrow \Delta$ . Finally, after setting-up the intersection theory of the central fiber, we will show that on the central fiber of  $\mathcal{X} \rightarrow \Delta$  satisfies property  $\mathcal{P}$  holds true, which will imply that  $\mathcal{P}_{X_{C_t}}$  holds true, too.

**3.1. Rationally connected manifolds.** In this section we collect the necessary information from the theory of rationally connected manifolds. For the definitions and the main results presented we refer the interested reader to [KMM92], [Ko96] and especially to [ArKo03].

Let  $X$  denote a complex projective manifold with  $\dim X \geq 2$ .

**Definition 3.2.** *A nonsingular, complex, projective variety  $X$  will be called rationally connected if any pair of points in  $X$  can be connected by a rational curve.*

The main properties and characterizations of rationally connected manifolds are summarized in the following:

**Theorem 3.3.** *1) Rationally connectedness is a birational property and is invariant under smooth deformations.*

*2) Rationally connected manifolds are simply connected and satisfy*

$$H^0(X, \Omega_X^{\otimes m}) = 0 \text{ for } m > 0 \text{ and } H^i(X, \mathcal{O}_X) = 0 \text{ for } i > 0.$$

*3)  $X$  is rationally connected if and only if for any point  $x \in X$  there exists a smooth rational curve  $L \subset X$  passing through  $x$ , with arbitrarily prescribed tangent direction and such that its normal bundle  $N_{L|X}$  is ample.*

**Definition 3.4.** *A comb with  $n$  teeth is a projective curve with  $n+1$  irreducible components  $C, L_1, \dots, L_n$  such that:*

- The curves  $L_1, \dots, L_n$  are mutually disjoint, smooth rational curves.
- Each  $L_i$ ,  $i \neq 0$  meets  $C$  transversely in a single smooth point of  $C$ .

The curve  $C$  is called the handle of the comb, and  $L_1, \dots, L_n$  are called the teeth.

The key result we use is the following theorem of Graber, Harris and Starr [GHS03], which we present in the shape given by J. Kollár [ArKo03]:

**Theorem 3.5.** *Let  $X$  be a smooth, complex, projective variety of dimension at least 3. Let  $C \subset X$  be a smooth irreducible curve. Let  $L \subset X$  be a rational curve with ample normal bundle intersecting  $C$  and let  $\mathcal{L}$  be a family of rational curves on  $X$  parametrized by a neighborhood of  $[L]$  in  $\text{Hilb}(X)$ . Then there are curves  $L_1, \dots, L_n \in \mathcal{L}$  such that  $C_0 = C \cup L_1 \cup \dots \cup L_n$  is a comb and satisfies the following conditions:*

- 1) The sheaf  $N_{C_0/X}$  is generated by the global sections.
- 2)  $H^1(C_0, N_{C_0/X}) = 0$ .

For a better understanding of this theorem the following corollary [ArKo03] is very useful. Since we consider that its proof gives some useful information about our construction, we include for convenience Kollár's proof.

**Corollary 3.6.**  *$\text{Hilb}(X)$  has a unique irreducible component containing  $[C_0]$ . This component is smooth at  $[C_0]$  and a non-empty subset of it parametrizes smooth, irreducible curves in  $X$ .*

*Proof.* Since the curve  $C_0$  is locally complete intersection, its normal sheaf  $N_{C_0/X}$  is locally free. We have an exact sequence

$$0 \rightarrow N_{C/X} \rightarrow N_{C_0/X}|_C \rightarrow Q \rightarrow 0$$

where  $Q$  is a torsion sheaf supported at the points  $P_i = C \cap L_i$ , for  $i = 1, \dots, n$ . Since  $N_{C_0/X}$  is globally generated, we can find a global section  $s \in H^0(C_0, N_{C_0/X})$  such that, for each  $i$ , the restriction of  $s$  to a neighborhood of  $P_i$  is not in the image of  $N_{C/X}$ . This means that  $s$  corresponds to a first-order deformation of  $C_0$  that smoothes the nodes  $P_i$  of  $C_0$ . From the vanishing of  $H^1(C_0, N_{C_0/X})$  we see that there are no obstructions finding a global deformation of  $C_0$  that smoothes its nodes  $P_i$ .

To be more explicit, we choose local holomorphic coordinates, so that near one of its nodes  $P$ ,  $C_0$  is given by:

$$z_1 z_2 = z_3 = \dots = z_n = 0.$$

Consider now a general 1-parameter deformation corresponding to a section of  $N_{C_0/X}$  which does not belong to the subspace of  $N_{C_0/X,P}$  generated by  $z_3, \dots, z_n$ . This deformation will be given by the equations:

$$z_1 z_2 + t f(t, \mathbf{z}) = z_3 + t f_3(t, \mathbf{z}) = \dots = z_n + t f_n(t, \mathbf{z}),$$

and  $f(t, \mathbf{z}) \neq 0$ , by assumption. We can change new coordinates  $z_1' := z_1$ ,  $z_2' := z_2$  and  $z_i' := z_i + t f_i(t, \mathbf{z})$  for  $i = 3, \dots, n$ , to get new, simpler equations:

$$z_1' z_2' + t(a + F(t, \mathbf{z})) = z_3' = \dots = z_n' = 0, \quad (3.1)$$

where  $a \neq 0$  and  $F(0, 0) = 0$ . The singular points are given by the equations:

$$z_1' + t \frac{\partial F}{\partial z_2'} = z_2' + t \frac{\partial F}{\partial z_1'} = \dots = z_n' = 0.$$

Substituting back these equations into  $z_1'z_2' + t(a + F(t, \mathbf{z})) = 0$  we get a new equation for the supposed singular point:

$$ta = -tF(t, \mathbf{z}) - t^2 \frac{\partial F}{\partial z_1'} \frac{\partial F}{\partial z_2'}$$

The latter has no solution for  $t \neq 0$  and  $t, z_1', z_2', \dots, z_n'$  small since  $a \neq 0$  and  $F(0, 0) = 0$ .  $\square$

**Remark 3.7.** Using the implicit function theorem we can change one more time the coordinates in (3.1) such that near the node  $P_i$ ,  $C_0$  is given by:

$$z_1z_2 + t = z_3 = \dots = z_n = 0. \quad (3.2)$$

This change of coordinates is given by  $z_i := z_i'$  for  $i = 1, \dots, n$ , and  $t := t(a + F(t, \mathbf{z}))$ .

**3.2. Construction of the specialization.** We start with our blowing-up  $p : X_C \rightarrow X$  of a projective, rationally connected threefold  $X$  along a smooth curve  $C \subset X$ , with  $E$  the exceptional divisor.

Since  $X$  is rationally connected, we can always attach [ArKo03] to the curve  $C \subset X$  a finite number of disjoint, smooth rational curves  $L_1, \dots, L_n \subset X$ , with ample normal bundle, each meeting  $C$  transversely at exactly one point  $P_i = C \cap L_i$ ,  $i = 1, \dots, n$ . Using Theorem 3.5 and Corollary 3.6, the comb  $C_0 = C \cup L_1 \cup \dots \cup L_n$  is smoothable for  $n \gg 0$ . As in the proof of Corollary 3.6, this means that we can find a small deformation of  $C_0$  parametrized by a one dimensional disk  $\Delta \subset \text{Hilb}(X)$  centered in  $[C_0]$ . That is there exists a smooth submanifold  $\mathcal{C} \subset X \times \Delta$ , such that its projection  $\pi : \mathcal{C} \rightarrow \Delta$  is flat, and

$$\pi^{-1}(t) = \begin{cases} C_0, & \text{if } t = 0 \\ C_t, & \text{if } t \neq 0, \end{cases}$$

where  $C_t$  is a smooth irreducible curve. From Corollary 3.6 and Remark 3.7, in local coordinates chosen w.r.t. a neighborhood of the node  $P_i$ ,  $\pi$  is the projection

$$(z_1, z_2, z_3, t) \mapsto t$$

and  $\mathcal{C}$  is given by  $z_1z_2 + t = z_3 = 0$ . In these local coordinates,  $C \subset \mathcal{C}$  is given by  $z_1 = z_3 = t = 0$ , and  $L_i$  by  $z_2 = z_3 = t = 0$ .

Let  $\varpi : \mathcal{X}_{\mathcal{C}} \rightarrow X \times \Delta$  be the blow-up of  $X \times \Delta$  along  $\mathcal{C}$ , and let

$$\Pi : \mathcal{X}_{\mathcal{C}} \rightarrow \Delta$$

be the projection onto  $\Delta$ .

**Lemma 3.8. (Structure of  $\Pi : \mathcal{X}_{\mathcal{C}} \rightarrow \Delta$ )**

- i)  $\mathcal{X}_{\mathcal{C}}$  is a smooth variety, and  $\Pi : \mathcal{X}_{\mathcal{C}} \rightarrow \Delta$  is a flat, proper family of projective varieties.
- ii) For  $t \neq 0$ ,  $X_{\mathcal{C},t} = \Pi^{-1}(t)$  is the blowing-up of  $X$  along  $C_t$ , while  $X_{\mathcal{C},0} = \Pi^{-1}(0)$  is the blowing-up of  $X$  along the ideal sheaf of  $C_0 \subset X$ .

*Proof.* i) This are standard facts about blowing-up, see sections II. 7 and II. 8 of [Hart77].

ii) For the proof we can either quote the universal property of the blowing-up, Corollary II.7.15 of [Hart77] or use local equations as in the proof of Corollary 3.6. We adopt the latter. The results we want to prove here are of local nature. In a neighborhood of a node of  $C_0 \subset U \subset X \times \Delta$ ,  $\mathcal{C}$  is given by the equations:

$$z_1z_2 + t = z_3 = 0.$$

$\mathcal{X}_{C|U} \subset U \times \mathbb{P}_1$  will therefore be given by the equations:

$$(z_1 z_2 + t)v = z_3 u, \quad (3.3)$$

where  $[u : v]$  are the homogeneous coordinates on  $\mathbb{P}_1$ , and the conclusion follows now immediately.  $\square$

Let  $\Pi_0 : X_{C,0} \rightarrow X$  denote the blowing-up map of  $X$  along the ideal sheaf of  $C_0$ .

**Lemma 3.9. (Structure of the central fiber  $X_{C,0}$ )**

- i)  $X_{C,0}$  has exactly  $n$  distinct ordinary double points as singularities.
- ii) The exceptional divisor of  $\Pi_0$ , denoted by  $E^*$  is a union of smooth Weil divisors  $E_C^*, E_1^*, \dots, E_n^*$ .

*Proof.* i) From the arguments used in the above Lemma we can see that the singular points of  $X_{C,0}$  can only occur over the singular points of  $C_0$ . For every node of  $C_0$ , in the above coordinates, there is exactly one singular point of  $X_{C,0}$ , which appears in the chart where  $v \neq 0$  and is given by the local equation

$$z_1 z_2 = z_3 u', \quad (3.4)$$

where  $u' = \frac{u}{v}$ .

ii) Since the center of the blowing-up has exactly  $n + 1$  components, it follows the exceptional divisor of  $X_{C,0}$  has  $n + 1$  components too, one over each of the components of  $C_0$ . Using (3.4) the other claims easily follow.  $\square$

Let  $Q_i \in \mathcal{X}_C$ ,  $1 = 1, \dots, n$  denote the singular points of  $X_{C,0}$ .

**Remark 3.10.** It can be seen that  $X_{C,0}$  is a Gorenstein non  $\mathbb{Q}$ -factorial variety. Hence push-forward arguments we used in the previous section no longer apply.

In order to perform the computation to follow, we need a better understanding of the components  $E_i^*$ 's of  $E^*$ .

**Proposition 3.11. (The components  $E_i^*$ ,  $i = 1, \dots, n$ )**

- i)  $E_i^* = \mathbb{P}_{L_i}(N_{C_0/X|L_i}^\vee)$ .
- ii) The conormal bundle of  $E_i^*$  is given by the extension:

$$0 \longrightarrow \mathcal{O}_{E_i^*} \longrightarrow N_{E_i^*/X_C}^\vee \longrightarrow \mathcal{J}_{Q_i} \otimes \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f) \longrightarrow 0,$$

where, by  $\mathcal{J}_{Q_i}$  we denoted the ideal sheaf of  $Q_i$  on  $E_i^*$ ,  $\mathcal{O}_{E_i^*}(1)$  is the dual of the tautological bundle of the ruled surface  $E_i^*$ , and  $f$  is its fiber.

*Proof.* i) This is a well-known fact. We include a short proof for convenience. The general theory of blowing-ups tells us that, since  $L_i \subset C$ ,  $E_i^* = \mathbb{P}_{L_i}(N_{C/X \times \Delta|L_i}^\vee)$ . To compute

$N_{\mathcal{C}/X \times \Delta|_{L_i}}^\vee$  we use the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & \mathcal{O}_{L_i} & \xlongequal{\quad} & \mathcal{O}_{L_i} & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & N_{\mathcal{C}/X \times \Delta|_{L_i}}^\vee & \longrightarrow & N_{L_i/X}^\vee \oplus \mathcal{O}_{L_i} & \longrightarrow & \mathcal{O}_{L_i}(P_i) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N_{C_0/X|_{L_i}}^\vee & \longrightarrow & N_{L_i/X}^\vee & \longrightarrow & \mathcal{O}_{P_i}(P_i) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array} \tag{3.5}$$

The first row is given by the exact sequence of conormal bundles of the inclusions  $L_i \subset \mathcal{C} \subset X \times \Delta$ . We have the obvious isomorphism  $N_{L_i/X \times \Delta}^\vee \simeq N_{L_i/X}^\vee \oplus \mathcal{O}_{L_i}$ .

On the smooth surface  $\mathcal{C}$ , since the  $L_i$ 's are mutually disjoint *rational curves* and meet  $\mathcal{C}$  transversally at exactly one point, we have:

$$0 = L_i \cdot C_t = L_i \cdot C_0 = L_i \cdot (C + L_1 + \dots + L_n) = 1 + L_i^2.$$

Therefore, the  $L_i$ 's are actually  $(-1)$ -curves and  $N_{L_i|_{\mathcal{C}}}^\vee \simeq \mathcal{O}_{L_i}(P_i)$ .

The second row is the exact sequence of Andreatta-Wisniewski [AnWi98, page 265]. From the snake lemma, we can see now that

$$N_{\mathcal{C}/X \times \Delta|_{L_i}}^\vee \simeq N_{C_0/X|_{L_i}}^\vee.$$

ii) Let  $\bar{F} : \bar{\mathcal{X}} \rightarrow \mathcal{X}_{\mathcal{C}}$  be the blowing-up of  $\mathcal{X}_{\mathcal{C}}$  at the points  $Q_i$ , for  $i = 1, \dots, n$ , and  $\bar{\Pi} : \bar{\mathcal{X}} \rightarrow \Delta$  the projection onto  $\Delta$ . We denote by  $\bar{X}$  the strict transform of  $X_{\mathcal{C},0}$ , and by  $Z_i$ ,  $i = 1, \dots, n$ , the exceptional divisors of  $\bar{F}$ .  $\bar{\Pi}$  has the following fibers:

$$\bar{\Pi}^{-1}(t) = \begin{cases} X_{\mathcal{C}_t}, & \text{if } t = 0 \\ \bar{X} + 2Z_1 + \dots + 2Z_n, & \text{if } t \neq 0. \end{cases}$$

Since  $\mathcal{X}_{\mathcal{C}}$  is smooth, we have  $Z_i \simeq \mathbb{P}_3$ , and  $N_{Z_i/\bar{\mathcal{X}}} \simeq \mathcal{O}_{\mathbb{P}_3}(-1)$ . The multiplicity of the  $Z_i$ 's in the central fiber are caused by the ordinary double point singularities of  $X_{\mathcal{C},0}$ . The reduced component  $\bar{X}$  is a big resolution of  $X_{\mathcal{C},0}$ . The induced map  $\bar{X} \rightarrow X_{\mathcal{C},0}$  has  $n$  exceptional divisors,  $T_i = \bar{X} \cap Z_i$ ,  $i = 1, \dots, n$ , each of them isomorphic to  $\mathbb{P}_1 \times \mathbb{P}_1$ , with  $N_{T_i/\bar{X}} \simeq \mathcal{O}(-1, -1)$ . Moreover,  $N_{T_i/Z_i} \simeq \mathcal{O}(1, 1)$ , for all  $i = 1, \dots, n$ .

To compute the conormal bundle  $N_{E_i^*/\mathcal{X}_{\mathcal{C}}}^\vee$ , we need a good understanding of the main component  $\bar{X}$  of  $\bar{\Pi}^{-1}(0)$ . Let  $\bar{p} : \bar{X} \rightarrow X$  be the natural morphism onto  $X$ . This has the following alternative description :

- Consider  $p_L : X_L \rightarrow X$ , the blowing up of  $X$ , along the disjoint union of curves  $L_1, \dots, L_n$ . Let  $E_1, \dots, E_n$  denote the exceptional divisors and  $\bar{C}$  denote the strict transform of  $C$  and  $\{x_i\} = \bar{C} \cap E_i$ . The  $E_i$ 's are rational ruled surfaces over  $L_i$ ,  $E_i = \mathbb{P}_{L_i}(N_{L_i/X}^\vee)$ , with  $N_{E_i/X_L} \simeq \mathcal{O}_{E_i}(-1)$ . Consider  $f_i \in E_i$ , the fiber of  $E_i$  through  $x_i$ , for all  $i = 1, \dots, n$ .

- Let  $p_{\bar{C}} : X_{L,\bar{C}} \rightarrow X_L$  be the blowing-up of  $X_L$  along  $\bar{C}$ . We denote by  $E_{\bar{C}}$  the exceptional divisor, and by  $\bar{E}_i$  the strict transforms of  $E_i$ , for all  $i = 1, \dots, n$ . Each of the  $\bar{E}_i$ 's is the blowing-up of  $E_i$  at  $x_i$ . Let  $\ell_i$  denote the exceptional divisor of these blowing-up.  $E_{\bar{C}}$  and  $\bar{E}_i$  meet transversally along  $\ell_i$ , and  $\ell_i$  sits in  $E_{\bar{C}}$  as fiber. Moreover, we have  $N_{\bar{E}_i/X_{L,\bar{C}}} = p_{\bar{C}}^* N_{E_i/X_L}$  (see [Ful]). In each  $\bar{E}_i$ , we denote by  $\bar{f}_i$  the strict transform of  $f_i$ .

We can immediately see that  $N_{\bar{E}_i/X_{L,\bar{C}}|_{\bar{f}_i}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ , and the exact sequence:

$$0 \rightarrow N_{\bar{f}_i/\bar{E}_i} \rightarrow N_{\bar{f}_i/X_{L,\bar{C}}} \rightarrow N_{\bar{E}_i/X_{L,\bar{C}}|_{\bar{f}_i}} \rightarrow 0$$

yields

$$N_{\bar{f}_i/X_{L,\bar{C}}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

- We blow-up now  $X_{L,\bar{C}}$  along  $\bar{f}_i$ , for all  $i = 1, \dots, n$ . The resulting 3-fold is isomorphic to  $\bar{X}$ , where the exceptional divisors of the last blowing-up coincide with  $T_i$ ,  $i = 1, \dots, n$ . Let  $p_{\bar{F}} : \bar{X} \rightarrow X_{L,\bar{C}}$  be the blowing-up map. The map  $\bar{p}$  is the composition:

$$\bar{p} = p_L \circ p_{\bar{C}} \circ p_{\bar{F}}.$$

Denote by  $R_i = p_{\bar{F}}^* \bar{E}_i - T_i$  the strict transforms of  $\bar{E}_i$ . Since  $\bar{f}_i \subset \bar{E}_i$ ,  $R_i$  is isomorphic to  $\bar{E}_i$ , the blowing-up of  $E_i$  at  $x_i$ . Let  $\bar{E}$  be the strict transform of  $E_{\bar{C}}$ , and  $\bar{\ell}_i$  be the strict transform of  $\ell_i$ , for all  $i = 1, \dots, n$ .  $\bar{E}$  is isomorphic to  $E_{\bar{C}}$  blown-up at the intersection points of  $E_{\bar{C}}$  with the curves  $\bar{f}_i$ , and intersects  $R_i$  transversally along  $\bar{\ell}_i$  for all  $i = 1, \dots, n$ .

First we need to determine  $N_{R_i/\bar{X}}$ . To do this, we have to analyze more closely the position of the exceptional divisors of the map  $\bar{p}$ .

- $R_i$  and  $T_i$  meet transversally along  $h_i$ , one of the rulings of  $T_i$  which coincides with  $\bar{f}_i$ , under the identification of  $R_i$  with  $\bar{E}_i$ ;
- $\bar{E}$  and  $T_i$  meet transversally along  $k_i$ , the other ruling of  $T_i$ , for all  $i = 1, \dots, n$ ;
- $\bar{E} \cap R_i \cap T_i = \{\text{point}\}$ , for all  $i = 1, \dots, n$ ;
- $R_i \cap R_j = \emptyset$ , for  $i \neq j$ .

Let  $p_i : R_i \rightarrow E_i$  be the blowing up of  $E_i$  at  $x_i$ , where  $h_i$  is the strict transform of the fiber through  $x_i$ , and  $\bar{\ell}_i$  denotes the exceptional divisor. Using i) we can see that  $E_i^*$  is actually the elementary transform of  $E_i$  centered at  $x_i$ . Consequently, we denote by  $q_i : R_i \rightarrow E_i^*$ , the blowing-down of  $\bar{f}_i$ , for every  $i = 1, \dots, n$ .

**Lemma 3.12.**  $N_{R_i/\bar{X}} \simeq \mathcal{O}_{R_i}(-h_i) \otimes p_i^* \mathcal{O}_{E_i}(-1)$ .

*Proof of Lemma 3.12.* Since  $R_i$  is the blowing-up of  $E_i$ , we can write  $N_{R_i/\bar{X}}$  as

$$\mathcal{O}_{R_i}(ah_i) \otimes p_i^* \mathcal{O}_{E_i}(b) \otimes p_i^* \mathcal{O}_{E_i}(cf),$$

where  $f$  is the generic fiber of the ruled surface  $E_i$ . Let  $d_i = \deg N_{L_i/X}$ . Let also denote by  $\bar{f}$  the strict transform in  $\bar{X}$  of the generic fiber of  $E_i$ . From the fact that  $R_i = p_{\bar{F}}^* \bar{E}_i - T_i$  and the projection formula, we compute:

$$\begin{aligned} R_i \cdot \bar{f} &= (p_{\bar{F}}^* \bar{E}_i - T_i) \cdot \bar{f} = E_i \cdot f = -1; \\ R_i \cdot \bar{\ell}_i &= (p_{\bar{F}}^* \bar{E}_i - T_i) \cdot \bar{\ell}_i = p_{\bar{C}}^* E_i \cdot \ell_i - T_i \cdot \bar{\ell}_i = -1; \\ R_i^3 &= (p_{\bar{F}}^* \bar{E}_i - T_i)^3 = \bar{E}_i^3 - 3p_{\bar{F}}^* \bar{E}_i \cdot p_{\bar{F}}^* \bar{E}_i \cdot T_i + 3p_{\bar{F}}^* \bar{E}_i \cdot T_i^2 - T_i^3 \\ &= E_i^3 + 3(\bar{E}_i \cdot \bar{f}_i) \cdot (T_i \cdot k_i) + 2 = -d_i + 3 - 2 = -d_i + 1. \end{aligned}$$

On the other hand, computing on the surface  $R_i$ , we have:

- $R_i \cdot \bar{f} = (ah_i + p_i^* \mathcal{O}_{E_i}(b) + cp_i^* f) \cdot p_i^* f = b$ , and so  $b = -1$ .
- $R_i \cdot \bar{\ell}_i = (ah_i + p_i^* \mathcal{O}_{E_i}(b) + cp_i^* f) \cdot \ell_i = a(p_i^* f - \bar{\ell}_i) \cdot \bar{\ell}_i = a$ , and so  $a = -1$ ;
- $R_i^3 = (-h_i - p_i^* \mathcal{O}_{E_i}(1) + cp_i^* f)^2 = -1 - d_i + 2 - 2c = -d_i + 1 - 2c$ , and so  $c = 0$ .

□

We compute now  $N_{R_i/\bar{\mathcal{X}}}^\vee$  from the conormal sequence of the inclusions  $R_i \subset \bar{X} \subset \bar{\mathcal{X}}$ :

$$0 \rightarrow N_{\bar{X}/\bar{\mathcal{X}}|_{R_i}}^\vee \rightarrow N_{R_i/\bar{\mathcal{X}}}^\vee \rightarrow N_{R_i/\bar{X}}^\vee \rightarrow 0. \quad (3.6)$$

In  $\bar{\mathcal{X}}$ , we have  $\bar{X} + 2Z_1 + \cdots + 2Z_n \sim 0$ , (linearly equivalence) and so

$$N_{\bar{X}/\bar{\mathcal{X}}}^\vee \simeq \mathcal{O}_{\bar{X}}(2T_1 + \cdots + 2T_n).$$

Tensoring by  $\mathcal{O}_{R_i}$ , we get  $N_{\bar{X}/\bar{\mathcal{X}}|_{R_i}}^\vee \simeq \mathcal{O}_{R_i}(2h_i)$ . Hence we obtained:

$$0 \rightarrow \mathcal{O}_{R_i}(2h_i) \rightarrow N_{R_i/\bar{\mathcal{X}}}^\vee \rightarrow \mathcal{O}_{R_i}(h_i) \otimes p_i^* \mathcal{O}_{E_i}(1) \rightarrow 0. \quad (3.7)$$

On the other hand, since  $\mathcal{X}_{\mathcal{C}}$  and  $E_i^*$  are smooth,  $R_i$  is the strict transform of  $E_i^*$  in  $\bar{\mathcal{X}}$ . Moreover, the restriction of blowing-up map  $\bar{F}$  to  $R_i$  coincides with the blowing-up  $q_i$  with  $h_i$  as exceptional divisor. Therefore, by [Ful, page 437],

$$N_{R_i/\bar{\mathcal{X}}} \simeq q_i^* N_{E_i^*/\mathcal{X}_{\mathcal{C}}} \otimes \mathcal{O}_{R_i}(-h_i).$$

From (3.7) we obtain:

$$0 \rightarrow \mathcal{O}_{R_i}(h_i) \rightarrow q_i^* N_{E_i^*/\mathcal{X}_{\mathcal{C}}} \rightarrow p_i^* \mathcal{O}_{E_i}(1) \rightarrow 0. \quad (3.8)$$

**Lemma 3.13.** *On the surface  $R_i$ , we have:*

$$p_i^* \mathcal{O}_{E_i}(1) = q_i^* \mathcal{O}_{E_i^*}(1) \otimes q_i^* \mathcal{O}_{E_i^*}(f) \otimes \mathcal{O}_{R_i}(-h_i),$$

where here  $f$  denotes the generic fiber of  $E_i^*$ .

*Proof of Lemma.* Computing the canonical line bundle of  $R_i$  in two ways, we get:

$$p_i^* \mathcal{O}_{E_i}(K_{E_i}) \otimes \mathcal{O}_{R_i}(\bar{\ell}_i) = q_i^* \mathcal{O}_{E_i^*}(K_{E_i^*}) \otimes \mathcal{O}_{R_i}(h_i). \quad (3.9)$$

Using the canonical bundle formula for ruled surfaces, and the fact that  $E_i^*$  is the elementary transform of  $E_i$  centered at  $x_i$ , from (3.9) we have:

$$\begin{aligned} p_i^* \mathcal{O}_{E_i}(-2) \otimes p_i^* \mathcal{O}_{E_i}(-d_i f) \otimes \mathcal{O}_{R_i}(\bar{\ell}_i) = \\ q_i^* \mathcal{O}_{E_i^*}(-2) \otimes q_i^* \mathcal{O}_{E_i^*}((-d_i - 1)f) \otimes \mathcal{O}_{R_i}(h_i). \end{aligned} \quad (3.10)$$

But,  $\mathcal{O}_{R_i}(\bar{\ell}_i) = q_i^* \mathcal{O}_{E_i}(f) \otimes \mathcal{O}_{R_i}(-h_i)$ , and  $p_i^* \mathcal{O}_{E_i}(f) = q_i^* \mathcal{O}_{E_i^*}(f)$ . Simplifying (3.10) we get:

$$p_i^* \mathcal{O}_{E_i}(-2) = q_i^* \mathcal{O}_{E_i^*}(-2) \otimes q_i^* \mathcal{O}_{E_i^*}(-2f) \otimes \mathcal{O}_{R_i}(2h_i),$$

and the proof of the lemma follows. □

To finish the proof of the proposition, notice that we obtained the following exact sequence:

$$0 \rightarrow \mathcal{O}_{R_i}(h_i) \rightarrow q_i^* N_{E_i^*/\mathcal{X}_{\mathcal{C}}}^\vee \rightarrow q_i^* \mathcal{O}_{E_i^*}(1) \otimes q_i^* \mathcal{O}_{E_i^*}(f) \otimes \mathcal{O}_{R_i}(-h_i) \rightarrow 0.$$

By pushing forward on  $E_i^*$ , since  $R^1 q_{i*} \mathcal{O}_{R_i}(h_i) = 0$  and  $q_{i*} \mathcal{O}_{R_i}(h_i) = \mathcal{O}_{E_i^*}$ , the projection formula yields:

$$0 \longrightarrow \mathcal{O}_{E_i^*} \longrightarrow N_{E_i^*/\mathcal{X}_{\mathcal{C}}}^\vee \longrightarrow \mathcal{I}_{Q_i} \otimes \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f) \longrightarrow 0,$$

where,  $\mathcal{I}_{Q_i}$  is the ideal sheaf of the point  $Q_i$ , and we are done. □

**Corollary 3.14.** *The Chern classes of  $N_{E_i^*/X_C}^\vee$  are:*

- $\det(N_{E_i^*/X_C}^\vee) = \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f)$ ;
- $c_2(N_{E_i^*/X_C}^\vee) = 1$ .

**Remark 3.15.** The description of  $\bar{X}$  in the proof of the above proposition is of local nature and it comes from the well-known diagram below [EiHa00, pages 178-179]. This commutative diagram exhibits the relation between the two small resolutions and the natural big resolutions of a three-dimensional ordinary double points, as those appearing as singularities of  $X_{C,0}$ .

$$(3.11) \quad \begin{array}{ccccc} & & \bar{X} & & \\ & \swarrow \bar{p} & \downarrow \bar{f} & \searrow \bar{q} & \\ X_{C,\bar{L}} & & X_{C \cup L} & & X_{L,\bar{C}} \\ \downarrow p_L & \swarrow r_C & \downarrow f_{C \cup L} & \swarrow r_L & \downarrow q_C \\ X_C & & X & & X_L \\ \downarrow p_C & & & & \downarrow p_L \end{array}$$

To simplify the notation and explain the diagram (3.11), let  $X$  be an arbitrary threefold, and  $C, L \subset X$  be two smooth curves intersecting transversally at exactly one point  $\{x\} = C \cap L$ .

- $f_{C \cup L} : X_{C \cup L} \rightarrow X$  is the blowing-up of  $X$  along the ideal sheaf of  $C \cup L$ ;
- $\bar{f} : \bar{X} \rightarrow X_{C \cup L}$  is the big resolution of  $\bar{X}$  obtained by blowing-up the singular point.
- $p_C : X_C \rightarrow X$  is the blowing-up of  $X$  along  $C$ . Let  $f_C$  be the fiber of the exceptional divisor over  $x$ .
- $q_L : X_L \rightarrow X$  is the blowing-up of  $X$  along  $L$ . Let  $f_L$  be the fiber of the exceptional divisor over  $x$ .
- $p_{\bar{L}} : X_{C,\bar{L}} \rightarrow X$  is the blowing-up of  $X_C$  along  $\bar{L}$ , the proper transform of  $L$  in  $X_C$ . Let  $\bar{f}_C$  denote the proper transform of  $f_C$ .
- $q_{\bar{C}} : X_{L,\bar{C}} \rightarrow X$  is the blowing-up of  $X_L$  along  $\bar{C}$ , the proper transform of  $C$  in  $X_L$ ; Let  $\bar{f}_L$  denote the proper transform of  $f_L$ .
- $\tilde{p} : \tilde{X} \rightarrow X_{C,\bar{L}}$  is the blowing-up of  $X_{C,\bar{L}}$  along  $\bar{f}_C$ .
- $\tilde{q} : \tilde{X} \rightarrow X_{L,\bar{C}}$  is the blowing-up of  $X_{L,\bar{C}}$  along  $\bar{f}_L$ .
- $r_L : X_{L,\bar{C}} \rightarrow \tilde{X}$  and  $r_C : X_{C,\bar{L}} \rightarrow \tilde{X}$  are the two *small resolutions* of the singular point of  $\tilde{X}$ .

We will modify the family  $\Pi : \mathcal{X}_C \rightarrow \Delta$  to produce a flat, proper map

$$\Phi : \mathcal{X} \rightarrow \Delta$$

with normal crossing central fiber, and with  $X_{C_t}$  as the general fiber.

The map  $\Phi$  is obtained as the composition

$$\mathcal{X} \xrightarrow{F} \mathcal{X}_C \xrightarrow{\Pi} \Delta,$$

where  $F : \mathcal{X} \rightarrow \mathcal{X}_C$  is the blowing-up  $X_C$  along  $E_i^*$ , for  $i = 1, \dots, n$ .

It is easy to see that the generic fiber of  $\Phi$  is  $X_{C_t}$ , the blowing-up of  $X$  along the smooth curve  $C_t$ . The central fiber of  $\Phi$  is a normal crossing threefold

$$X_0 = X_p \cup X_1 \cup \dots \cup X_n,$$

with exactly  $n + 1$  irreducible, smooth components.

To describe the main component, denoted by  $X_p$ , as before, let  $X_L \rightarrow X$  denote the blowing-up of  $X$  along the curves  $L_i$ ,  $i = 1, \dots, n$ . Then  $X_p$  is obtained by blowing-up  $X_L$  along  $\bar{C}$ , the strict transforms of the  $C$ . It actually coincides with the 3-fold  $X_{L, \bar{C}}$ , described in the above proposition. We denote by

$$f_p : X_p \rightarrow X,$$

the projection onto  $X$ .

The other components,

$$X_i = \mathbb{P}_{E_i^*}(N_{E_i^*|_{\mathcal{X}_C}}^\vee) \xrightarrow{f_i} E_i^*,$$

are the exceptional divisors of the of the blowing-up of  $\mathcal{X}_C$  along  $E_i^*$ , for  $i = 1, \dots, n$ .

Moreover, the intersections  $S_i := X_p \cap X_i$  are smooth surfaces, the blowing-up of  $E_i^*$  at  $Q_i$ . Again, the  $S_i$  actually coincides with the surfaces  $\bar{E}_i$  described in the above proposition.

It is easy to notice that all of the components of the central fiber and their intersections are in fact rationally connected manifolds.

**3.3. Extensions of line bundles.** In this section we will show that any line bundle on the central fiber of  $\Phi$  extends to a line bundle on an open neighborhood of the central fiber. The results are probably standard and well-known, e.g. [Per77, Theorem Q, page 50]. However, since we couldn't find any reference for the proof, we include it for completeness. But first, we must describe the line bundles on  $X_0$ . To do this we follow [Fr83].

Since the components  $X_p$  and  $X_i$  of  $X_0$  intersect transversally along  $S_i$ , and  $X_i \cap X_j = \emptyset$ , for  $i \neq j$ , a line bundle  $H_{X_0}$  on  $X_0$  consists of a  $(n + 1)$ -uple of line bundles  $(H_p, H_1, \dots, H_n)$  on  $X_p$  and  $X_i$ , for  $i = 1, \dots, n$ , respectively, such that

$$H_p|_{S_i} \simeq H_i|_{S_i},$$

for any  $i = 1, \dots, n$ .

As an example, we consider [Fr83] the case of dualizing sheaf of  $\omega_{X_0}$ , an example which will be needed later. We have:

$$\begin{aligned} \omega_p &= \omega_{X_0|_{X_p}} = \mathcal{O}_{X_p}(K_{X_p} + S_1 + \dots + S_n), \\ \omega_i &= \omega_{X_0|_{X_i}} = \mathcal{O}_{X_i}(K_{X_i} + S_i). \end{aligned}$$

By adjunction, we have the canonical isomorphisms:

$$\omega_p|_{S_i} = \mathcal{O}_{X_p}(K_{X_p} + S_i)|_{S_i} = \mathcal{O}_{R_i}(K_{S_i}) = \mathcal{O}_{X_i}(K_{X_i} + S_i)|_{S_i} = \omega_i|_{S_i},$$

showing that  $\omega_p$  agrees with  $\omega_i$  on  $S_i$ , and so

$$\omega_{X_0} = (\mathcal{O}_{X_p}(K_{X_p} + R_1 + \dots + R_n), \mathcal{O}_{X_1}(K_{X_1} + R_1), \dots, \mathcal{O}_{X_n}(K_{X_n} + R_n)). \quad (3.12)$$

**Lemma 3.16.**  $H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = 0$ .

*Proof.* Recall that the central fiber  $X_0$  has  $(n + 1)$  smooth components

$$X_0 = X_p \cup X_1 \cup \cdots \cup X_n,$$

where  $S_i = X_p \cap X_s$ , is the blowing up of a rational ruled surface, and  $X_i \cap X_j = \emptyset$  for  $i \neq j$ . Their structure sheaves are related by the Mayer-Vietoris sequence:

$$0 \rightarrow \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{X_p} \oplus \mathcal{O}_{X_1} \oplus \cdots \oplus \mathcal{O}_{X_n} \rightarrow \mathcal{O}_S \rightarrow 0, \quad (3.13)$$

where  $S$  is the reduced, reducible surface  $S_1 + \cdots + S_n$ .

The relevant part of the cohomology sequence is:

$$\begin{aligned} \cdots &\rightarrow H^0(X_p, \mathcal{O}_{X_p}) \oplus H^0(X_1, \mathcal{O}_{X_1}) \oplus \cdots \oplus H^0(X_n, \mathcal{O}_{X_n}) \xrightarrow{s} H^0(S, \mathcal{O}_S) \\ &\rightarrow H^1(X_0, \mathcal{O}_{X_0}) \rightarrow H^1(X_p, \mathcal{O}_{X_p}) \oplus H^1(X_1, \mathcal{O}_{X_1}) \oplus \cdots \oplus H^1(X_n, \mathcal{O}_{X_n}) \\ &\rightarrow H^1(S, \mathcal{O}_S) \rightarrow H^2(X_0, \mathcal{O}_{X_0}) \rightarrow H^2(X_p, \mathcal{O}_{X_p}) \oplus H^2(X_1, \mathcal{O}_{X_1}) \oplus \cdots \\ &\oplus H^2(X_n, \mathcal{O}_{X_n}) \rightarrow \cdots \end{aligned} \quad (3.14)$$

Under the identification

$$H^k(S, \mathcal{O}_S) \simeq H^k(S_1, \mathcal{O}_{S_1}) \oplus \cdots \oplus H^k(S_n, \mathcal{O}_{S_n}), \quad \forall k \geq 0,$$

the map:

$$s : H^0(X_p, \mathcal{O}_{X_p}) \oplus H^0(X_1, \mathcal{O}_{X_1}) \oplus \cdots \oplus H^0(X_n, \mathcal{O}_{X_n}) \rightarrow H^0(S, \mathcal{O}_S)$$

is given by  $(s_p, s_1, \dots, s_n) \mapsto (s_p - s_1, \dots, s_p - s_n)$ , and is obviously surjective.

Since, as discussed in Theorem 3.3, the structural sheaf of a rationally connected manifold has no higher cohomology, it follows that:

$$H^k(X_p, \mathcal{O}_{X_p}) = H^k(X_i, \mathcal{O}_{X_i}) = H^k(S_i, \mathcal{O}_{S_i}) = 0,$$

for  $k = 1, 2$  and  $i = 1, \dots, n$  and the lemma follows now from a simple inspection of (3.14).  $\square$

**Corollary 3.17.**  $R^1\Phi_*\mathcal{O}_{\mathcal{X}} = R^2\Phi_*\mathcal{O}_{\mathcal{X}} = 0$ .

*Proof.* Since  $\Phi$  is proper and flat, and we have just proved that  $H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = 0$ , from the semi-continuity theorem it follows that

$$R^1\Phi_*\mathcal{O}_{\mathcal{X}} = R^2\Phi_*\mathcal{O}_{\mathcal{X}} = 0.$$

Of course, we shrink  $\Delta$  whenever necessary.  $\square$

**Theorem 3.18.** Any line bundle  $\mathcal{L}_{X_0} \in \text{Pic}(X_0)$  extends to a line bundle over  $\mathcal{X}$ .

*Proof.* Applying  $\Phi_*$  over the exponential sequence

$$0 \rightarrow \mathbb{Z}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}}^* \rightarrow 0$$

we obtain

$$0 = R^1\Phi_*\mathcal{O}_{\mathcal{X}} \rightarrow R^1\Phi_*\mathcal{O}_{\mathcal{X}}^* \rightarrow R^2\Phi_*\mathbb{Z}_{\mathcal{X}} \rightarrow R^2\Phi_*\mathcal{O}_{\mathcal{X}} = 0.$$

Hence  $R^1\Phi_*\mathcal{O}_{\mathcal{X}}^* \simeq R^2\Phi_*\mathbb{Z}_{\mathcal{X}}$ , and passing to global sections we get

$$\text{Pic}(\mathcal{X}) \simeq H^2(\mathcal{X}, \mathbb{Z}).$$

However,  $X_0$  is a deformation retract of  $\mathcal{X}$ , and therefore

$$H^2(X_0, \mathbb{Z}) \simeq H^2(\mathcal{X}, \mathbb{Z}).$$

But, from Lemma 3.16 and the exponential sequence we can see that

$$\text{Pic}(X_0) \simeq H^2(X_0, \mathbb{Z}),$$

and the proposition follows.  $\square$

The following proposition is well-known. It is the key that allows us to solve the ampleness issue discussed in Remark 2.10. For its proof we refer the interested reader to [KoMo98, Proposition 1.41].

**Proposition 3.19.** *Let  $f : X \rightarrow Y$  be a proper morphism, and  $D$  a Cartier divisor on  $X$ . Let  $y \in Y$  be a point and  $X_y$  the fiber of  $f$  over  $y$ . If  $\mathcal{O}_{X_y}(D)$  is ample, then  $D$  is ample over some open set  $U \ni y$  of  $Y$ .*

**3.4. Intersection Theory I.** In this section we set up the intersection theory of the central fiber  $X_0$  of  $\Phi : \mathcal{X} \rightarrow \Delta$ . To do this, we will adopt a cohomological approach, as discussed in Appendix A. In our situation we need the following:

**Proposition 3.20.** *If  $L_0^i = (L_p^i, L_1^i, \dots, L_n^i)$ ,  $i = 1, 2, 3$  are three line bundles on  $X_0$ , then*

$$L_0^1 \cdot L_0^2 \cdot L_0^3 = L_p^1 \cdot L_p^2 \cdot L_p^3 + \sum_{k=1}^n L_k^1 \cdot L_k^2 \cdot L_k^3.$$

*Proof.* Let  $L_0 = (L_p, L_1, \dots, L_n)$  be an arbitrary line bundle on  $X_0$ . Of course,  $L_p \in \text{Pic}(X_p)$ ,  $L_i \in \text{Pic}(X_i)$  for  $i = 1, \dots, n$ , with the property that:

$$L_{S_i} \stackrel{\text{def}}{=} L_p|_{S_i} \simeq L_i|_{S_i}.$$

Tensoring the Mayer-Vietoris sequence (3.14) by  $L_0^\vee$  we get:

$$0 \rightarrow L_0^\vee \rightarrow L_p^\vee \oplus L_1^\vee \oplus \dots \oplus L_n^\vee \rightarrow L_{S_1}^\vee \oplus \dots \oplus L_{S_n}^\vee \rightarrow 0 \quad (3.15)$$

From (3.15) we immediately obtain:

$$\chi(-L_0) = \chi(-L_p) + \sum_{k=1}^n \chi(-L_k) - \sum_{k=1}^n \chi(-L_{S_k}). \quad (3.16)$$

Now using (3.16) in (A.1) we see that

$$L_0^1 \cdot L_0^2 \cdot L_0^3 = L_p^1 \cdot L_p^2 \cdot L_p^3 + \sum_{k=1}^n L_k^1 \cdot L_k^2 \cdot L_k^3 - \sum_{k=1}^n L_{S_k}^1 \cdot L_{S_k}^2 \cdot L_{S_k}^3.$$

But, since  $S_k$  is a smooth surface  $L_{S_k}^1 \cdot L_{S_k}^2 \cdot L_{S_k}^3 = 0$ , and the conclusion follows.  $\square$

**Corollary 3.21.** *Let  $\mathcal{L}^i$ ,  $i = 1, 2, 3$  be three line bundles on  $\mathcal{X}$ , and denote by  $L_t^i$  their restriction to the fiber  $\Phi^{-1}(t)$ . If  $L_0^i = (L_p^i, L_1^i, \dots, L_n^i)$ ,  $i = 1, 2, 3$ , then:*

$$L_t^1 \cdot L_t^2 \cdot L_t^3 = L_p^1 \cdot L_p^2 \cdot L_p^3 + \sum_{k=1}^n L_k^1 \cdot L_k^2 \cdot L_k^3, \quad \forall t \in \Delta.$$

*Proof.* Let  $\mathcal{L}$  be an arbitrary line bundle on  $\mathcal{X}$ . We will denote by  $L_t$  its restriction to  $\Phi^{-1}(t)$  an arbitrary fiber of  $\Phi$ . Since the Euler characteristic is constant in flat families, we have:

$$\chi(L_0) = \chi(L_t), \quad \forall t \in \Delta. \quad (3.17)$$

From (A.1), (3.17) and the above proposition, the result follows.  $\square$

**Remark 3.22.** The proposition and its corollary are nothing but the three dimensional version of [Per77, Proposition 2.4.1]. There, U. Persson sets up the intersection theory on the central fiber of degenerations of surfaces for extendable line bundles. However, our cohomological approach is different than his topological approach.

**3.5. Construction of the line bundle.** Recall now the initial package. We have a smooth threefold  $X$  containing a smooth curve  $C$ , with  $K_X \cdot C \geq 0$ , and let  $p : X_C \rightarrow X$  be the blowing-up of  $X$  along  $C$ . On  $X_C$  we have an ample line bundle  $H_{X_C}$  such that  $K_{X_C} \cdot H_{X_C}^2 < 0$ .

We want to find an ample line bundle  $H_0$  on  $X_0$  such that  $K_{X_0} \cdot H_{X_0}^2 < 0$ . What we are going to do is to start with the line bundle on the main component  $X_p$ , and by studying its restriction to  $S_i$  we will extend it across  $X_0$ .

Remember that  $f_p : X_p \rightarrow X$  is gotten by first blowing-up  $X$  along the curves  $L_1, \dots, L_n$ , followed by blowing-up  $\bar{C}$ , the strict transform of  $C$ . We denoted the exceptional divisors by  $S_1, \dots, S_n$ , and  $E_{\bar{C}}$ , respectively. The  $S_i$ 's can be either viewed as blowing-ups of  $E_i$ , the exceptional divisors of  $X$  along the  $L_i$ 's, at one point or as blowing-ups of  $E_i^*$  at one point. The components  $X_i$  of  $X_0$  are pairwise disjoint and meet  $X_p$  transversally along  $S_i$ . For the moment, we pick an arbitrary line bundle  $H_p$  on  $X_p$ .

To simplify the exposition, we introduce the following notations:

- $d_i = \deg_{L_i}(N_{L_i/X}) > 2$ , since  $L_i$  is a very free rational curve;
- $p_i : S_i \rightarrow E_i$  is the blowing-up of  $E_i$  at  $P_i = \bar{C} \cap E_i$  with  $e_i$  its exceptional divisor.
- $k_i$  be the strict transform in  $S_i$  of the fiber of the ruled surface  $E_i$  that passes through  $P_i$ ;
- $q_i : S_i \rightarrow E_i^*$  is the contraction of  $k_i$ ;
- $h_i = \mathcal{O}_{E_i}(1)$ ,  $h_i^* = \mathcal{O}_{E_i^*}(1)$ ;
- $f$  denotes the generic fiber of either  $E_i$  or  $E_i^*$ . We will still abusively denote by  $f$  its strict transform in  $S_i$ .
- $c_1 = \det(N_{E_i^*/X_C}^\vee) = h_i^* + f$ ;
- $\xi_i = \mathcal{O}_{X_i}(1)$ .

What we do next is to find the restriction of  $H_p$  to  $S_i$ , for each  $i = 1, \dots, n$ . For convenience we will denote the restriction of  $H_p$  to  $S_i$  by  $H_{S_i}$ .

Since  $\text{Pic}(S_i) \simeq q_i^* \text{Pic}(E_i^*) \oplus \mathbb{Z}[k_i]$ , we can write:

$$H_{S_i} = q_i^* M_i - x_i k_i, \quad (3.18)$$

for some line bundle  $M_i \in \text{Pic}(E_i^*)$ , and some integer  $x_i$ .

To construct the line bundle on  $X_i$ , recall that

$$X_i = \mathbb{P}(N_{E_i^*/X_C}^\vee) \xrightarrow{f_i} E_i^*$$

is the exceptional divisor of the blowing-up of  $X$  along  $E_i^*$ , and  $X_i$  intersects transversally  $X_p$ . This transversality can be used, as in [Per77, page 41], to see that

$$N_{X_i/X_p} = N_{S_i/X_p}.$$

In our notations and using Lemma 3.13 this translates into:

$$\xi_{i|S_i} = p_i^* h_i = q_i^* h_i^* + q_i^* f - k_i. \quad (3.19)$$

We can write now:

$$H_{S_i} = q_i^*(M_i - x_i h_i^* - x_i f) + x_i \xi_{i|S_i}. \quad (3.20)$$

Thus, if we let  $N_i = M_i - x_i h_i^* - x_i f = M_i - x_i c_1$  we can see that we can write  $H_{S_i}$  as

$$H_{S_i} = q_i^* N_i + x_i \xi_{i|S_i}. \quad (3.21)$$

With this is now clear that, we have a unique choice for  $H_i$ , namely:

$$H_i = f_i^* N_i + x_i \xi_i. \quad (3.22)$$

As can be easily seen from the above consideration, the choice of the line bundle  $H_p \in \text{Pic}(X_p)$  is not necessary to extend it across  $X_0$ . Thus, we have just proved:

**Proposition 3.23.** *Let  $H_p$  be an arbitrary line bundle on  $X_p$ . Then  $H_p$  extends uniquely to a line bundle  $H_0$  on  $X_0$ .*

For further considerations, we need to show the ampleness of the line bundles  $H_i$ , provided the ampleness of  $H_p$ . Before we proceed we need the following:

**Lemma 3.24.**  $\mathcal{O}_{X_i}(S_i) = \xi_i$ .

*Proof.* In additive notations, we can write  $\mathcal{O}_{X_i}(S_i) = a\xi_i + bf_i^*h_i^* + cf_i^*\mathcal{O}_{E_i^*}(f)$ , with  $a, b, c \in \mathbb{Z}$ . Restricting to  $S_i$ , and using (3.19) we obtain:

$$\begin{aligned} \mathcal{O}_{S_i}(S_i) &= a\xi_{i|S_i} + bq_i^*h_i^* + cq_i^*\mathcal{O}_{E_i^*}(f) \\ &= (a+b)q_i^*h_i^* + (a+c)q_i^*\mathcal{O}_{E_i^*}(f) - ak_i. \end{aligned} \quad (3.23)$$

On the other hand, the restriction of  $\mathcal{O}_{X_i}(S_i)$  to  $S_i$  is the normal bundle  $N_{S_i/X_i}$ . But, from the transversality of the intersection of  $X_i$  with  $X_p$  we have [Per77, page 41],  $N_{S_i/X_i} \simeq N_{S_i/X_p}^\vee = p_i^*h_i$ , and so:

$$\mathcal{O}_{S_i}(S_i) = p_i^*h_i = q_i^*h_i^* + q_i^*\mathcal{O}_{E_i^*}(f) - k_i$$

Comparing with (3.23) in  $\text{Pic}(S_i)$ , we can immediately see that  $a = 1$  and  $b = c = 0$ .  $\square$

For the computations involved in proving the ampleness of the  $H_i$ 's, we will work with the more traditional basis of  $\text{Pic}(E_i^*)$  described in Appendix C.  $\text{Pic}(E_i^*)$  is generated by  $\{\mathcal{O}_{E_i^*}(C_0), \mathcal{O}_{E_i^*}(f)\}$ , where  $\mathcal{O}_{E_i^*}(C_0)$  is the line bundle associated to the section of  $E_i^*$  of minimal self-intersection, and  $f$  is the class of a fiber. Since  $E_i^*$  is rational,  $C_0^2 = -e \leq 0$ . It is easy to see that  $h_i^* = C_0 - \frac{d_i+1-e}{2}f$ . Using this basis, any line bundle  $L \in \text{Pic}(X_i)$  can be written as  $L = a\xi_i + bf_i^*\mathcal{O}_{E_i^*}(C_0) + cf_i^*\mathcal{O}_{E_i^*}(f)$ , where  $a, b, c \in \mathbb{Z}$ . We have:

**Proposition 3.25.** *Any line bundle  $L \in \text{Pic}(X_i)$  which admits sections, can be written as*

$$L = \lambda(\xi_i - f_i^*c_1) + \beta f_i^*\mathcal{O}_{E_i^*}(C_0) + (\gamma + \beta e)f_i^*\mathcal{O}_{E_i^*}(f),$$

with  $\lambda, \beta, \gamma \geq 0$ .

*Proof.* Let  $L = a\xi_i + bf_i^*\mathcal{O}_{E_i^*}(C_0) + cf_i^*\mathcal{O}_{E_i^*}(f)$  be an arbitrary line bundle on  $X_i$ . For simplicity, we will abusively denote  $\mathcal{O}_{E_i^*}(C_0)$  by  $C_0$  and  $\mathcal{O}_{E_i^*}(f)$  by  $f$ . If  $L$  has sections, its restriction to an arbitrary fiber  $f$  of  $f_i$  has sections. But  $L|_f \simeq \mathcal{O}_{\mathbb{P}^1}(a)$ , and so  $a \geq 0$ .

Next we restrict our line bundle to  $S_i$ . Using again (3.19) have:

$$\begin{aligned} L|_{S_i} &= a\xi_{i|S_i} + bf_i^*C_0|_{S_i} + cf_i^*f|_{S_i} \\ &= a(q_i^*h_i^* + q_i^*f - k_i) + bq_i^*C_0 + cq_i^*f \\ &= (a+b)q_i^*C_0 + (c+a - a\frac{d_i+1-e}{2})q_i^*f - ak_i. \end{aligned} \quad (3.24)$$

Now, if  $L$  has sections,  $L|_{S_i}$  restricted to  $f$ , the strict transform in  $S_i$  of a generic fiber of  $E_i^*$ , still has sections. But this can happen only if  $a+b \geq 0$ . We restrict now to the strict transform  $\bar{C}_0$  of  $C_0$ . Since  $E_i^*$  is the elementary transform of  $E_i$ ,  $\bar{C}_0$  will intersect  $k_i$  transversally at one point. If we assume that  $L$  has sections, it will follow as before that  $c - a\frac{d_i+1-e}{2} - (a+b)e \geq 0$ . By a change of notation, if we let  $\lambda = a \geq 0$ ,  $\beta = a+b \geq 0$  and  $\gamma = c - a\frac{d_i+1-e}{2} - (a+b)e \geq 0$ , we see that we can write any line bundle  $L \in \text{Pic}(X_i)$  with sections as:

$$L = \lambda(\xi - f_i^*h_i^*) + \beta f_i^*C_0 + (\gamma + \beta e)f_i^*f,$$

where  $\lambda, \beta, \gamma \geq 0$ .  $\square$

**Remark 3.26.** One could study if the necessary conditions we found are also sufficient for line bundles on  $X_i$  to admit sections. However, the result we proved is enough for our purpose.

**Lemma 3.27.** *If  $H_p$  is ample, then the line bundles  $H_i$  constructed above are ample for any  $i = 1, \dots, n$ .*

*Proof. Step 1:* We show that for any  $i = 1, \dots, n$ , the line bundle  $N_i = M_i - x_i(h_i^* + f) \in \text{Pic}(E_i^*)$  is ample.

Writing the line bundle  $M_i = ah_i^* + bf \in \text{Pic}(E_i^*)$   $a, b \in \mathbb{Z}$  in our preferred basis  $\{C_0, f\}$ , we see that

$$M_i - x_i(h_i^* + f) = (a - x_i)C_0 + [(b - x_i) - (a - x_i)\frac{d_i + 1 - e}{2}]f.$$

To check the ampleness of this this line bundle, we use the conditions of Proposition C.3. First, if in (3.18) we take the intersection with the fiber of  $E_i^*$  passing through the center of the blowing-up, we immediately see that  $a - x_i > 0$ . The other condition to check is:

$$(b - x_i) - (a - x_i)\frac{d_i + 1 - e}{2} > (a - x_i)e \iff (b - x_i) > (a - x_i)\frac{d_i + 1 + e}{2} \quad (3.25)$$

Now, using the Nakai-Moishezon criterion is easy to check that since  $H_{S_i} = q_i^*M_i - x_i k_i$  is ample, then  $M_i \in \text{Pic}(E_i^*)$  is ample, too. Proposition C.3 yields that:

$$b > a\frac{d_i + 1 + e}{2}, \quad (3.26)$$

which at its turn implies (3.25).

**Step 2:** We show that  $H_i$  is strictly nef. Let  $C \subset X_i$  be an arbitrary irreducible curves. Depending on the position of the curve  $C$  with respect to the surface  $S_i \subset X_i$  we distinguish three cases.

- (1)  $C \subseteq S_i$ . In this case the intersection pairing can be computed on  $S_i$ , where

$$H_i \cdot C = H_{S_i} \cdot C > 0,$$

since  $H_{S_i}$  is ample.

- (2)  $C = f$  is a fiber of  $X_i \rightarrow E_i^*$ . Then

$$H_i \cdot C = (f_i^*N_i + x_i\xi_i) \cdot f = x_i(\xi_i \cdot f) = x_i > 0.$$

- (3) When  $C \not\subseteq S_i$ , using Lemma 3.23, we have:

$$H_i \cdot C = (f_i^*N_i + x_i\xi_i) \cdot C = f_i^*N_i \cdot C + x_iS_i \cdot C \geq f_i^*N_i \cdot C > 0,$$

as  $N_i$  is ample and  $C$  is not contained in a fiber of  $f_i$ .

**Step 3:** We show that  $H_i$  is ample. Using the Nakai-Moishezon, all we need to prove are the following:

- (1)  $H_i^2 \cdot S > 0$  for any irreducible surface  $S \subseteq X_i$ . Using Lemma 3.25, we can write the line bundle associated to  $S$  as  $\lambda(\xi - f_i^*h_i^*) + \beta f_i^*C_0 + (\gamma + \beta e)f_i^*f$ , where  $\lambda, \beta, \gamma \geq 0$ , are not simultaneously zero. We have:

$$\begin{aligned} H_i^2 \cdot S &= (f_i^*N_i + x_i\xi_i)^2 \cdot (\lambda(\xi - f_i^*h_i^*) + \beta f_i^*C_0 + (\gamma + \beta e)f_i^*f) \\ &= \lambda(f_i^*N_i + x_i\xi_i)^2 \cdot (\xi - f_i^*h_i^*) + \beta(f_i^*N_i + x_i\xi_i)^2 \cdot f_i^*C_0 \\ &\quad + (\gamma + \beta e)(f_i^*N_i + x_i\xi_i)^2 \cdot f_i^*f. \end{aligned} \quad (3.27)$$

We compute now separately the terms involved.

$$\begin{aligned}
(f_i^* N_i + x_i \xi_i)^2 \cdot f_i^* f &= (f_i^* N_i \cdot f_i^* N_i + 2x_i \xi_i \cdot f_i^* N_i + x_i^2 \xi_i^2) \cdot f_i^* f \\
&= 2x_i \xi_i \cdot f_i^* N_i \cdot f_i^* f + x_i^2 \xi_i^2 \cdot f_i^* f \\
&= 2x_i (N_i \cdot f) + x_i^2 (f \cdot c_1) \\
&= 2x_i [(M_i - x_i c_1) \cdot f] + x_i^2 [f \cdot (h_i^* + f)] \\
&= 2ax_i - x_i^2 > 0,
\end{aligned} \tag{3.28}$$

since  $a > x_i$ .

$$\begin{aligned}
(f_i^* N_i + x_i \xi_i)^2 \cdot f_i^* C_0 &= (f_i^* N_i \cdot f_i^* N_i + 2x_i \xi_i \cdot f_i^* N_i + x_i^2 \xi_i^2) \cdot f_i^* C_0 \\
&= 2x_i \xi_i \cdot f_i^* N_i \cdot f_i^* C_0 + x_i^2 \xi_i^2 \cdot f_i^* C_0 \\
&= 2x_i (N_i \cdot C_0) + x_i^2 (C_0 \cdot c_1) \\
&= x_i (N_i \cdot C_0) + x_i (M_i \cdot C_0),
\end{aligned} \tag{3.29}$$

as  $N_i$  and  $M_i$  are both ample line bundles on  $E_i^*$ . Finally,

$$\begin{aligned}
(f_i^* N_i + x_i \xi_i)^2 \cdot (\xi - f_i^* h_i^*) &= (f_i^* N_i + x_i \xi_i)^2 \cdot \xi - (f_i^* N_i + x_i \xi_i)^2 f_i^* h_i^* \\
&= H_{S_i}^2 - 2x_i (N_i \cdot h_i^*) - x_i^2 (h_i^*)^2 \\
&= M_i^2 - 2x_i (M_i \cdot h_i^*) + x_i^2 (h_i^*)^2 + x_i^2.
\end{aligned} \tag{3.30}$$

To show that this last term is positive we use (3.26) to get:

$$\begin{aligned}
&M_i^2 - 2x_i (M_i \cdot h_i^*) + x_i^2 (h_i^*)^2 + x_i^2 \\
&= (ah_i^* + bf)^2 - 2x_i [(ah_i^* + bf) \cdot h_i^*] + x_i^2 (h_i^*)^2 + x_i^2 \\
&= 2b(a - x_i) - a^2(d_i + 1) + 2ax_i(d_i + 1) - d_i x_i^2 \\
&> a(a - x_i)(d_i + 1) - a^2(d_i + 1) + 2ax_i(d_i + 1) - d_i x_i^2 \\
&= ax_i(d_i + 1) - d_i x_i^2 > 0,
\end{aligned} \tag{3.31}$$

since  $a > x_i$ .

From (3.27), (3.28), (3.29), (3.30), (3.31), and since  $\lambda$ ,  $\beta$  and  $\gamma$  cannot simultaneously vanish, this step is concluded.

(2)  $H_i^3 > 0$ . This follows again by a direct computation and Step 1.

$$\begin{aligned}
H_i^3 &= (f_i^* N_i + x_i \xi_i)^3 \\
&= [(f_i^* N_i + x_i \xi_i)^2 \cdot f_i^* N_i] + x_i [\xi_i \cdot (f_i^* N_i + x_i \xi_i)^2] \\
&= 2x_i [\xi_i \cdot (f_i^* N_i)^2] + x_i^2 (\xi_i^2 \cdot f_i^* N_i) + x_i H_{S_i}^2 \\
&= 2x_i (N_i \cdot N_i) + x_i^2 (N_i \cdot c_1) + x_i H_{S_i}^2 \\
&= x_i (N_i \cdot N_i) + x_i (N_i \cdot M_i) + x_i H_{S_i}^2 > 0,
\end{aligned} \tag{3.32}$$

since  $x_i > 0$ , and the line bundles  $M_i$ ,  $N_i \in \text{Pic}(E_i^*)$  are ample.  $\square$

**3.6. Intersection Theory II.** In this subsection we will make a choice for the line bundle  $H_p$  used throughout the previous section. Assuming the validity of Conjecture 0.3, noticing again that  $X_p$  is the flop of  $X_{C,L}$ , for which obviously  $\mathcal{P}_{X_{C,L}}$  holds true, it will follow that  $\mathcal{P}_{X_p}$  also holds true. That is  $X_p$  is endowed with ample line bundle  $H_p$  with  $K_{X_p} \cdot H_{X_p}^2 < 0$ . We conclude our arguments of this entire section, by proving the the following:

**Proposition 3.28.** *If  $H_{X_0} = (H_p, H_1, \dots, H_n)$  is the line bundle constructed above,*

$$K_{X_0} \cdot H_{X_0}^2 < 0.$$

*Proof.* Using Proposition A.3 and Corollary 3.20 we compute now  $K_{X_0} \cdot H_{X_0}^2$  :

$$\begin{aligned} K_{X_0} \cdot H_{X_0}^2 &= (K_{X_p} + S_1 + \dots + S_n) \cdot H_p^2 + \sum_{i=1}^n (K_{X_i} + S_i) \cdot H_i^2 \\ &= K_{X_p} \cdot H_p^2 + \sum_{i=1}^n (K_{X_i} \cdot H_i^2 + 2H_{S_i}^2). \end{aligned} \quad (3.33)$$

We compute separately  $K_{X_i} \cdot H_i^2$ . From Proposition C.1 we have that  $K_{X_i} = f_i^* L_i - 2\xi_i$ , where

$$\begin{aligned} L_i &= K_{E_i^*} + c_1 = -2h_i^* - (d_i + 3)f + h_i^* + f \\ &= -h_i^* - (d_i + 2)f = -c_1 - (d_i + 1)f. \end{aligned} \quad (3.34)$$

A straightforward computation using Proposition C.1 again yields:

$$\begin{aligned} K_{X_i} \cdot H_i^2 &= (f_i^* L_i - 2\xi_i) \cdot (f_i^* N_i + x_i \xi_i)^2 \\ &= (f_i^* L_i - 2\xi_i) \cdot (f_i^* N_i \cdot f_i^* N_i + 2x_i f_i^* N_i \cdot \xi_i + x_i^2 \xi_i^2) \\ &= 2x_i f_i^* L_i \cdot f_i^* N_i \cdot \xi_i + x_i^2 f_i^* L_i \cdot \xi_i^2 \\ &\quad - 2f_i^* N_i \cdot f_i^* N_i \cdot \xi_i - 4x_i f_i^* N_i \cdot \xi_i^2 - 2x_i^2 \xi_i^3 \\ &= 2x_i (L_i \cdot N_i) + x_i^2 (L_i \cdot c_1) - 2(N_i \cdot N_i) \\ &\quad - 4x_i (N_i \cdot c_1) - 2x_i^2 (c_1^2 - c_2). \end{aligned} \quad (3.35)$$

Making use of (3.34), we complete this computation as follows:

$$\begin{aligned} (L_i \cdot c_1) &= [(-c_1 - (d_i + 1)f) \cdot c_1] = -c_1^2 - (d_i + 1)(c_1 \cdot f) \\ &= d_i - 1 - d_i - 1 = -2; \\ (L_i \cdot N_i) &= [L_i \cdot (M_i - x_i c_1)] = (L_i \cdot M_i) - x_i (L_i \cdot c_1) = (L_i \cdot M_i) + 2x_i \\ (N_i \cdot N_i) &= [(M_i - x_i c_1) \cdot (M_i - x_i c_1)] = (M_i \cdot M_i) - 2x_i (M_i \cdot c_1) + x_i^2 c_1^2; \\ &= H_{S_i}^2 + x_i^2 - 2x_i (M_i \cdot c_1) + x_i^2 c_1^2; \\ (N_i \cdot c_1) &= [(M_i - x_i c_1) \cdot c_1] = (M_i \cdot c_1) - x_i c_1^2. \end{aligned} \quad (3.36)$$

From (3.35), we get:

$$\begin{aligned} K_{X_i} \cdot H_i^2 + 2H_{S_i}^2 &= 2H_{S_i}^2 + 2x_i (L_i \cdot M_i) + 2x_i^2 - 2x_i^2 - 2H_{S_i}^2 \\ &\quad - 2x_i^2 + 4x_i (M_i \cdot c_1) - 2x_i^2 c_1^2 - 4x_i (M_i \cdot c_1) \\ &\quad + 4x_i^2 c_1^2 - 2x_i^2 c_1^2 + 2x_i^2 \\ &= 2x_i (L_i \cdot M_i). \end{aligned} \quad (3.37)$$

It is not hard to see, from [Hart77] that on the ruled surface  $E_i^*$ , the line bundle  $-L_i = h_i^* + (d_i + 2)f$  has a section which is also a section of  $E_i^*$ , (actually we can say much more, namely that  $-L_i$  is ample) and since  $M_i$  is ample it follows immediately that

$$K_{X_i} \cdot H_i^2 + 2H_{S_i}^2 < 0.$$

Finally, from (3.33) we obtain:

$$K_{X_0} \cdot H_{X_0}^2 = K_{X_p} \cdot H_p^2 + \sum_{i=1}^n (K_{X_i} \cdot H_i^2 + 2H_{S_i}^2) < K_{X_p} \cdot H_p^2 < 0. \quad (3.38)$$

□

**Corollary 3.29.**  $\mathcal{P}_{X_0}$  holds true.

*Proof.* The line bundle  $H_{X_0} = (H_p, H_1, \dots, H_n)$  has  $H_p$  ample from our assumption and, by Lemma 3.27, the  $H_i$ 's are also ample. Thus  $H_{X_0}$  is ample. With this, Proposition 3.28 actually says that  $\mathcal{P}_{X_0}$  holds true. □

**Remark 3.30.** It would seem that the choice of the line bundle  $H_p$  is not important, as by attaching more curves to  $C$  we could arbitrarily decrease  $K_{X_0} \cdot H_{X_0}^2$ . However, this is not correct, as  $H_p$  depends on the number of curves we attach, which makes the intersection number  $K_{X_0} \cdot H_{X_0}^2$  hard to control.

**Remark 3.31.** From the above considerations, it also follows that our Conjecture 0.3 is much stronger than what we need. The only thing we assumed here is the following special case:

**Conjecture 3.32.**  $\mathcal{P}_{X_C} \implies \mathcal{P}_{X_p}$

#### 4. PROOF OF THE MAIN THEOREM

**Proposition 4.1.** *Let  $X$  be a rationally connected, projective threefold, and let  $X_C$  be the blowing-up of  $X$  along a smooth curve  $C \subset X$ , and assume the validity of Conjecture 0.3. Then  $\mathcal{P}_{X_C}$  holds true if and only if  $\mathcal{P}_X$  holds true.*

*Proof.* The implication  $\mathcal{P}_X \implies \mathcal{P}_{X_C}$  is the conclusion of Proposition 2.5.

Conversely, we have two cases.

- If  $K_X \cdot C < 0$  the result follows from Proposition 2.6.
- If  $K_X \cdot C \geq 0$  applying the results of the previous section we obtain a flat, proper family  $\Phi : \mathcal{X} \rightarrow \Delta$  over the unit disk, such that:

- 1)  $\mathcal{X}$  is smooth;
- 2) For any  $t \in \Delta$ ,  $t \neq 0$ ,  $\Phi^{-1}(t) = X_{C_t}$ , the blowing-up of  $X$  along a smooth curve with  $K_X \cdot C_t < 0$ ;
- 3) The central fiber  $X_0 = \Phi^{-1}(0)$  is a normal crossing divisor in  $\mathcal{X}$  with smooth rationally connected components. Moreover,  $\mathcal{P}_{X_0}$  holds true.

By eventually shrinking  $\Delta$ , we know from Theorem 3.18 and Proposition 3.19 that  $H_{X_0}$  can be extended to a line bundle on  $\mathcal{X}$ , whose restriction  $H_{X_{C_t}}$  to the fiber  $X_{C_t}$  is ample. Moreover, since  $K_{X_0} \cdot H_{X_0}^2 < 0$ , from Corollary 3.21 it follows that  $K_{X_t} \cdot H_{X_{C_t}}^2 < 0$ . Thus,  $\mathcal{P}_{X_{C_t}}$  holds true.

Since for all  $i = 1, \dots, n$ ,  $K_X \cdot L_i < 0$ , by attaching sufficiently many such curves,  $K_X \cdot C_0 < 0$  and by the conservation law we will have  $K_X \cdot C_t < 0$ , for any  $t \in \Delta$ . Thus the blowing-up  $X_{C_t} \rightarrow X$  is along the curve  $C_t$  for which  $K_X \cdot C_t < 0$ . Applying again Proposition 2.6 it follows that  $\mathcal{P}_{X_{C_t}} \implies \mathcal{P}_X$  and we are done. □

From the weak factorization theorem and Propositions 2.4 and 4.1, if we assume the validity of Conjecture 0.3, or of the weaker Conjecture 3.32, we easily get:

**Theorem 4.2.**  $\mathcal{P}$  is a birational property of rationally connected, projective threefolds.

We can turn now to the main result of this paper:

*Proof of Theorem A.* If  $X$  is a rationally connected threefold, its Kodaira dimension is negative, and  $X$  is birational to a Mori fiber space, which we denote by  $X_{min}$ . Then, by Proposition 1.8,  $\mathcal{P}_{X_{min}}$  holds true. Moreover, if  $X'$  is an appropriate desingularization of  $X_{min}$ , by Proposition 2.1  $\mathcal{P}_{X'}$  also holds true. Since  $X$  and  $X'$  are smooth and birational, the conclusion of the Theorem A follows now from Theorem 4.2.  $\square$

## APPENDIX A. INTERSECTION THEORY

Throughout this thesis we used the intersection theory algebraic varieties, from a cohomological viewpoint. This is introduced via the the following theorem due to Snapper [Kle66].

**Theorem A.1.** *Let  $X$  be a proper scheme over a field  $k$ ,  $\mathcal{F}$  a coherent sheaf on  $X$ , and let  $\mathcal{L}_1, \dots, \mathcal{L}_r$  be  $r$  Cartier divisors on  $X$ . Then the Euler-Poincaré function*

$$(m_1, \dots, m_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes m_r})$$

*takes the same values on  $\mathbb{Z}^r$  as a polynomial with rational coefficients of degree at most the dimension of the support of  $\mathcal{F}$ .*

**Definition A.2.** *Let  $L_1, \dots, L_r$  be Cartier divisors on a proper scheme  $X$ ,  $\mathcal{F}$  a coherent sheaf on  $X$ . Suppose  $r \geq \dim \text{Supp } \mathcal{F}$ . The intersection number of  $L_1, \dots, L_r$  with  $\mathcal{F}$ , denoted by*

$$(L_1 \cdots L_r \cdot \mathcal{F})$$

*is the coefficient of  $m_1, \dots, m_r$  in the polynomial*

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes m_r})$$

**Notations:** Let  $X$  be a proper scheme over a field  $k$ . If  $Z \subset X$  a closed subscheme of dimension  $s$  and  $L_1, \dots, L_r$  Cartier divisors on  $X$ , for any  $r \geq s$  we will denote by  $(L_1 \cdots L_r \cdot Z)$  the intersection number  $(L_1 \cdots L_r \cdot \mathcal{O}_Z)$ . If  $L_1 = \dots = L_d$  then we also use the notation  $(L^d \cdot Z)$ . When  $Z = X$  we use  $(L_1 \cdots L_d)$  if no confusion is likely.

The basic properties of intersection numbers [Kle66] are summarized in the following:

**Proposition A.3.** *The intersection numbers  $(L_1 \cdots L_r \cdot \mathcal{F})$  are uniquely determined by the following properties:*

- 1) *The intersection number is an integer.*
- 2)  *$(L_1 \cdots L_r \cdot Z)$  is symmetric and multilinear in the  $L_i$ .*
- 3)  *$(L_1 \cdots L_r \cdot Z) = 0$  if  $\dim Z < r$ .*
- 4) *For any exact sequence of coherent sheaves*

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0,$$

*we have*

$$(L_1 \cdots L_r \cdot \mathcal{F}) = (L_1 \cdots L_r \cdot \mathcal{F}') + (L_1 \cdots L_r \cdot \mathcal{F}'').$$

- 5) *If  $j$  denotes the inclusion  $j : Z \rightarrow X$ , we have*

$$(j^* L_1 \cdots j^* L_r \cdot Z)_Z = (L_1 \cdots L_r \cdot Z)_X.$$

- 6) *If  $D$  is an effective Cartier divisor and  $L_r = \mathcal{O}_X(D)$ , we have*

$$(L_1 \cdots L_r) = (L_1 \cdots L_{r-1} \cdot \mathcal{O}_D) = (L_1 \cdots L_{r-1} \cdot D).$$

7) If  $f : X' \rightarrow X$  is a morphism of finite degree, and  $r \geq \dim X = \dim X'$ , then

$$(L_1 \cdots L_r) = \deg(f)(f^*L_1 \cdots f^*L_r).$$

**Corollary A.4.** *We have:*

- 1) If  $L_1 \equiv 0$  then  $(L_1 \cdots L_d \cdot Z) = 0$ .
- 2) If  $X \subset \mathbb{P}_N$  and  $\mathcal{O}_X(L) = \mathcal{O}_X(1)$  then  $(L^d \cdot Z) = \deg Z$ .
- 3) If  $f : X' \rightarrow X$  is a birational morphism, and  $r \geq \dim X = \dim X'$ , then

$$(L_1 \cdots L_r \cdot X) = (f^*L_1 \cdots f^*L_r \cdot X').$$

For our computations we use the following identity (see [Deb01, page 8]), which follows easily from Proposition A.3.4:

$$(L_1 \cdots L_r) = \sum_{I \subset \{1, \dots, r\}} \varepsilon_I \chi \left( X, -\sum_{i \in I} L_i \right) \quad (\text{A.1})$$

where  $\varepsilon_I = (-1)^{\text{Card}(I)}$ .

## APPENDIX B. THE BLOWING UP

Of particular importance for the computations we performed was the multiplication table in the Chow ring of the blowing-up of threefolds. The results [GH78] we used are:

**Proposition B.1.** *Let  $X$  be the a smooth projective threefold, and let  $p : X' \rightarrow X$  be the blow-up at a point. Let  $E \simeq \mathbb{P}_2$  be the exceptional divisor, and  $f$  the class of a line of  $E$  in the Chow ring  $A^*(X')$ . Then*

$$A^*(X') = p^*A^*(X) \oplus \mathbb{Z} \cdot E \oplus \mathbb{Z} \cdot f$$

as an additive group. Moreover  $p_*E = p_*f = 0$ , and  $p_*p^*A^*(X) = A^*(X)$ . The multiplicative structure of  $A^*(X')$  is determined by:

$$E^2 = -f, \quad E^3 = -E \cdot f = 1, \quad E \cdot p^*Z = f \cdot p^*Z = 0$$

for any cycle  $Z \in A^*(X)$ .

**Proposition B.2.** *Let  $X$  be the a smooth projective threefold, and let  $p : X' \rightarrow X$  be the blow-up along a smooth curve  $C \subset X$ . Let  $E \simeq \mathbb{P}(N_{C/X}^\vee)$  be the exceptional divisor, and  $f$  the class of a fiber of  $E$  in the Chow ring  $A^*(X')$ . Then*

$$A^*(X') = p^*A^*(X) \oplus \mathbb{Z} \cdot E \oplus \mathbb{Z} \cdot f$$

as an additive group. Moreover  $p_*E = p_*f = 0$ , and  $p_*p^*A^*(X) = A^*(X)$ . The multiplicative structure of  $A^*(X')$  is determined by:

$$\begin{aligned} E^2 &= -p^*C + \deg_C(N_{C/X}) \cdot f, \quad E^3 = -\deg_C(N_{C/X}), \quad E \cdot f = -1, \\ E \cdot p^*D &= (C \cdot D)f, \quad f \cdot p^*D = 0, \quad \forall D \in A^1(X), \\ E \cdot p^*Z &= f \cdot p^*Z = 0, \quad \forall Z \in A^2(X). \end{aligned}$$

In addition, the usual relation holds:

$$\deg_C(N_{C/X}) = 2g(C) - 2 - K_X \cdot C,$$

where  $g(C)$  is the genus of the curve  $C$ , and  $N_{C/X}$  is the normal bundle of  $C$  in  $X$ .

## APPENDIX C. RULED MANIFOLDS

The computations in Proposition 3.28 are based on the following:

**Proposition C.1.** *Let  $X = \mathbb{P}(\mathcal{E}) \xrightarrow{f} S$  be the projectivization of a rank 2 vector bundle  $\mathcal{E}$  over a smooth surface  $S$ . Let  $\xi = \mathcal{O}_X(1)$ ,  $c_1 = \det \mathcal{E}$ , and  $c_2 = c_2(\mathcal{E})$ . We have:*

- i)  $f^*L \cdot f^*K \cdot f^*M = 0$ ,  $\forall L, K, M \in \text{Pic}(S)$ ;
- ii)  $f^*L \cdot f^*K \cdot \xi = (L \cdot K)$ ,  $\forall L, K \in \text{Pic}(S)$ ;
- iii)  $f^*L \cdot \xi^2 = (L \cdot c_1)$ ,  $\forall L \in \text{Pic}(S)$ ;
- iv)  $\xi^3 = c_1^2 - c_2$ ;
- v)  $K_X = f^*L - 2\xi$ , where  $L = K_S + c_1$ .

*Proof.* *i)* and *ii)* are obvious, while for *iii)* and *iv)* we use the identity [GH85]:

$$\xi^2 - f^*c_1 \cdot \xi + f^*c_2 = 0.$$

Thus:

$$\begin{aligned} f^*L \cdot \xi^2 &= f^*L \cdot f^*c_1 \cdot \xi - f^*L \cdot f^*c_2 = (L \cdot c_1); \\ \xi^3 &= f^*c_1 \cdot \xi^2 - f^*c_2 \cdot \xi = c_1^2 - c_2. \end{aligned}$$

*v)* is also well-known. □

Similarly, for projectivizations of rank three vector bundles we needed the following:

**Proposition C.2.** *Let  $X = \mathbb{P}(\mathcal{E}) \xrightarrow{f} C$  be the projectivization of a rank 3 vector bundle  $\mathcal{E}$  over a smooth curve  $C$ . Let  $\xi = \mathcal{O}_X(1)$ ,  $c_1 = \det \mathcal{E}$ , and  $c_2 = c_2(\mathcal{E})$ . We have:*

- i)  $f^*L \cdot f^*K \cdot M = 0$ ,  $\forall L, K \in \text{Pic}(C), M \in \text{Pic}(X)$ ;
- ii)  $f^*L \cdot \xi^2 = \deg L$ ,  $\forall L \in \text{Pic}(C)$ ;
- iii)  $\xi^3 = \deg \mathcal{E}$ ;
- iv)  $K_X = f^*L - 3\xi$ , where  $L = K_S + c_1$ .

*Proof.* As in the proof of the previous proposition, *i)* and *ii)* are obvious and *iv)* is well known. For two we use again the identity [GH85]:

$$\xi^2 - f^*c_1 \cdot \xi = 0.$$

Multiplying by  $\xi$ , from *ii)* we immediately get  $\xi^3 = \deg \mathcal{E}$ . □

Also, of particular importance are the classical ampleness conditions on ruled surfaces. For the following results, the interested reader is referred to [Hart77].

Recall that if  $\pi : X \rightarrow C$  is a ruled surface, one can write  $X \simeq \mathbb{P}(\mathcal{E})$ , where  $\mathcal{E}$  is a normalized, locally free, rank two sheaf on  $C$ . The integer  $e = -\deg(\mathcal{E})$  is an invariant of  $X$ , and there is a section  $\pi_0 : C \rightarrow X$  with image  $C_0$ , such that  $\mathcal{L}(C_0) \simeq \mathcal{O}_X(1)$ . Any element of  $\text{Pic}(X)$  can be written as  $aC_0 + \pi^*\ell$  with  $a \in \mathbb{Z}$  and  $\ell \in \text{Pic}(C)$ , and any element of the  $\text{Num}(X)$  can be written  $aC_0 + bf$  with  $a, b \in \mathbb{Z}$  and where  $f$  is the class of a fiber of  $X$ . We denote by  $g$  the genus of the curve  $C$ .

**Proposition C.3.** *Let  $\pi : X \rightarrow C$  be a ruled surface, and a divisor  $D \equiv aC_0 + bf$ .*

- (a) *If  $g = 0$ , then  $e \geq 0$ , and  $D$  is ample if and only if  $a > 0$  and  $b > ae$ .*
- (b) *If  $g > 0$ , according to the the sign of  $e$  we have:*
  - i) *If  $e \geq 0$ ,  $D$  is ample if and only if  $a > 0$ ,  $b > ae$ .*
  - ii) *If  $e < 0$ ,  $D$  is ample if and only if  $a > 0$ ,  $b > \frac{1}{2}ae$ .*

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