HYPERBOLIC APPROXIMATION OF THE FOURIER TRANSFORMED VLASOV EQUATION

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Résumé. Nous construisons une approximation hyperbolique de l'équation de Vlasov dans laquelle la dépendance dans la variable de vitesse a été supprimée. Nous commençons par transformer l'équation de Vlasov par rapport à la variable de vitesse comme dans [9]. Puis nous semi-discrétisons l'équation par des éléments finis dans la variable spectrale. Nous présentons des résultats préliminaires sur des cas tests monodimensionnels classiques en physique des plasmas : amortissement Landau, instabilité double faisceau.

Abstract. We construct an hyperbolic approximation of the Vlasov equation in which the dependency on the velocity variable is removed. The model is constructed from the Vlasov equation after a Fourier transformation in the velocity variable [9]. A well-chosen finite element semi-discretization in the spectral variable leads to an hyperbolic system.The resulting model enjoys interesting conservation and stability properties. It can be numerically solved by standard schemes for hyperbolic systems. We present numerical results for one-dimensional classical test cases in plasma physics: Landau damping, two-stream instability.

Introduction

Solving the Vlasov-Poisson equation is challenging. Some popular methods for studying this equation are the Particle-In-Cell (PIC) method [1] or the semi-lagrangian approach [5].

In a previous work [8], we constructed a reduced Vlasov-Poisson model with a velocity basis expansion.

In this paper, we consider a Fourier velocity transformation of the Vlasov equation. We construct a reduced model where the unknown depends on space and time instead of the full phase-space variables. The reduced model is a linear hyperbolic system, with non-linear source terms. We present numerical results for classical plasma physics test cases.

1. Plasma mathematical model

In our work, we consider the one-dimensional Vlasov equation

$$\partial_t f + v \partial_x f + E \partial_v f = 0, \tag{1}$$

where the unknown distribution function f depends on the space variable $x \in \mathbb{R}/L\mathbb{Z}$, the velocity variable $v \in \mathbb{R}$ and the time variable $t \in \mathbb{R}^+$. The electric field E depends on x and t and is the solution of the Poisson equation

$$\partial_x E = -1 + \int_v f, \quad \int_{x=0}^{L} E = 0. \tag{2}$$

The equations (1-2) are supplemented by an initial condition

$$f(x, v, 0) = f_0(x, v).$$

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Generally, this initial condition is close to a stationary solution satisfying

$$\int_v f = 1.$$

For practical reasons, we will allow that f and E take complex values, however, of course, only the real parts are physically relevant.

We consider a Fourier transformation with respect to the velocity variable (we denote by $I = \sqrt{-1}$)

$$\phi(x, \eta, t) = \int_{v=-\infty}^{+\infty} f(x, v, t) \exp(-I\eta v) dv.$$

The Fourier velocity variable is denoted by $\eta \in \mathbb{R}$. The distribution function $\phi(x, \eta, t)$ satisfies the Fourier transformed Vlasov equation [9]

$$\partial_t \phi + I\partial_x \partial_\eta \phi + IE\eta\phi = 0. \tag{3}$$

In addition, the Poisson equation becomes

$$\partial_x E(x, t) = -1 + \phi(x, 0, t). \tag{4}$$

We call the new model the Vlasov-Fourier equation.

2. Discretization of the Vlasov-Fourier equation with respect to the Fourier velocity variable

We will perform a semi-discretization of (3) with respect to the variable η in order to obtain a first order hyperbolic system set only in (x, t). We shall call this new system of equations the reduced Vlasov-Fourier model. We could expand the function ϕ on a basis of arbitrary functions depending on η. See for instance [2] and included references. For practical numerical reasons, we decide to choose a classical Lagrange finite element interpolation basis, because it leads to a sparse matrix representation of the hyperbolic system.

2.1. Continuous interpolation by the finite element method

In practice, ϕ almost vanishes at the boundaries $\eta \to \pm\infty$. We consider thus a truncated domain $\eta \in [-\eta_{\max}, \eta_{\max}]$ and the following dissipative boundary conditions at $\pm\eta_{\max}$

$$\partial_x \phi(x, \pm\eta_{\max}, t) \pm I\gamma\phi(x, \pm\eta_{\max}, t) = 0, \tag{5}$$

in which $\gamma \geq 0$.

We recall now how the finite element basis is constructed. We consider an arbitrary polynomial degree d. The reference element is defined by

$$\hat{Q} = [-1, 1].$$

We define the $d + 1$ reference nodes by

$$\hat{N}_i = -1 + 2\frac{i-1}{d}, \quad i = 1 \cdots d + 1.$$

We mesh the interval $[-\eta_{\max}, \eta_{\max}]$ with N finite elements $(Q_i)_{i=1\ldots N}$ and nodes $(N_j)_{j=1\ldots P}$. The total number of nodes in this interval is $P = d \cdot N + 1$. In practice, we suppose that the nodes are equally spaced in $[-\eta_{\max}, \eta_{\max}]$

$$N_j = -\eta_{\max} + \frac{2\eta_{\max}}{dN}(j-1).$$

We introduce a connectivity array for detecting that node N_j is the kth local node of a given element Q_i

$$j = \text{connec}(k, i) = k + (i-1)d, \quad 1 \leq k \leq d+1, \quad 1 \leq i \leq N.$$

We also use the notation
$$N_j = N_{k,i}$$
and then, element Q_i has its support in the interval $[N_{1,i}, N_{d+1,i}]$.

We construct a transformation τ_i that maps element $\hat{Q}$ onto Q_i. For this purpose we consider the Lagrange polynomials on $\hat{E}$, defined by

$$\hat{L}_k(\hat{\eta}) = \prod_{l \neq k} \frac{\hat{\eta} - \hat{N}_l}{\hat{N}_k - \hat{N}_l}. \tag{6}$$

The transformation is then given by

$$\tau_i(\hat{\eta}) = \sum_{k=1}^{d+1} \hat{L}_k(\hat{\eta}) N_{k,i}. \tag{7}$$

Because the nodes of the mesh are equally spaced in our application, the transformation τ_i is linear. We construct the interpolation basis in such a way that each basis function φ_j is associated to a node N_j of the mesh and satisfies

$$\varphi_j(N_i) = \delta_{ij},$$

where δ_{ij} denotes the Kronecker symbol. We recall how to compute the basis function φ_j. Let $\eta \in [-\eta_{\max}, \eta_{\max}]$. Necessarily, η belongs at least to one finite element Q_i. Two cases are possible

(1) Node N_j belongs to finite element Q_i, i.e. $\exists k, N_j = N_{k,i}$, then

$$\varphi_j(\eta) = \hat{L}_k(\hat{\eta}), \quad \text{where } \eta = \tau_i(\hat{\eta}). \tag{8}$$

(2) Node N_j does not belong to Q_i, then
$$\varphi_j(\eta) = 0.$$

2.2. Application to Vlasov-Fourier discretization

We suppose that the function $\phi(x, \eta, t)$ is well approximated by an expansion on the basis $\{\varphi_j\}_{j=1\dots P}$

$$\phi(x, \eta, t) = \sum_{j=1}^{P} w_j(x, t)\varphi_j(\eta), \tag{9}$$

we shall also use the convention of sum on repeated indices

$$\phi(x, \eta, t) = w_j(x, t)\varphi_j(\eta). \tag{10}$$

Because of the interpolation property of the basis $\{\varphi_j\}_{j=1\dots P}$

$$\varphi_i(N_j) = \delta_{ij},$$

we have

$$\phi(x, N_i, t) = \sum_{j=1}^{P} w_j(x, t)\varphi_j(N_i) = w_i(x, t).$$

Therefore, we can approximate the initial condition in the following way

$$w_j(x, 0) = \phi(x, N_j, 0) = \phi_0(x, N_j).$$

Considering the equation (3) and the boundary condition (5) we can consider the following weak formula of the problem : find $\phi(x, \eta, t)$ such that for all (continuous) test function $\varphi(\eta)$ we have

$$\int_\eta \partial_t \phi \varphi + \int_\eta I \partial_x \partial_\eta \phi \varphi + \int_\eta IE\eta \phi \varphi - \frac{1}{2}\varphi(\eta_{\max})I\partial_x\phi(\cdot, \eta_{\max}, \cdot) + \frac{1}{2}\varphi(-\eta_{\max})I\partial_x\phi(\cdot, -\eta_{\max}, \cdot)$$

$$+ \frac{1}{2}\varphi(\eta_{\max})\gamma\phi(\cdot, \eta_{\max}, \cdot) + \frac{1}{2}\varphi(-\eta_{\max})\gamma\phi(\cdot, -\eta_{\max}, \cdot) = 0 \tag{11}$$

This "semi-weak" formula is equivalent with the initial problem (3) supplemented with the boundary conditions (5).

Indeed, if ϕ is a solution of (3) with the conditions (5), it is evident that ϕ is also a solution of (11).

Reciprocally, if we suppose that ϕ is a solution of (11), because (11) is true for arbitrary test function. Thus for every function φ such that $\varphi(-\eta_{\max}) = \varphi(\eta_{\max}) = 0$, we obtain

$$\int_\eta \partial_t \phi \varphi + \int_\eta I \partial_x \partial_\eta \phi \varphi + \int_\eta I E \eta \phi \varphi = 0$$

and thus (3).

Then taking test functions φ that do not vanish at $\eta = \pm\eta_{\max}$ we obtain the boundary conditions (5).

Morever we show now that the choice of boundary conditions (5) ensures the hyperbolicity of the system and ensures that the energy is not increasing.

We can introduce the following matrices of dimension $P \times P$

$$M = (\int_\eta \varphi_i \varphi_j), \quad A = I\tilde{A}, \quad B = B_E + D, \tag{12}$$

where

$$\tilde{A}_{ij} = \int_\eta \varphi_i \varphi'_j - \frac{1}{2}\varphi_i(\eta_{\max})\varphi_j(\eta_{\max}) + \frac{1}{2}\varphi_i(-\eta_{\max})\varphi_j(-\eta_{\max}),$$

$$(B_E)_{ij} = IE \int_\eta \eta \varphi_i \varphi_j, \quad D_{ij} = \frac{1}{2}\gamma(\varphi_i\varphi_j(\eta_{\max}) + \varphi_i\varphi_j(-\eta_{\max})).$$

We obtain the following equation

$$M\partial_t W + A\partial_x W + BW = 0, \tag{13}$$

in which $W(x,t)$ is the complex vector of P components

$$W = (w_1, w_2, ..., w_P)^T.$$

Obviously, the mass matrix M is positive hermitian. An integration by parts in η shows that A is hermitian. Finally, B_E is skew-hermitian and D is diagonal non-negative. It is then classical to prove that system (13) is hyperbolic (i.e. that $M^{-1}A$ is diagonalizable with real eigenvalues [6]) and energy dissipative.

In practice, to compute the matrices M, A, B we use the Gauss-Legendre integration and sparse matrix representations.

3. Finite volume schemes

We describe now the numerical approximation. We assume that the spatial domain $]0, L[$ is split into N_x cells. The cell C_i is the interval $]x_{i-1/2}, x_{i+1/2}[$, $i = 1..N_x$. For practical reasons, we also consider two virtual cells C_0 and C_{N_x+1} for applying the periodic boundary condition. At the beginning of a time step, we copy the values of the cell C_{N_x} to the cell C_0, and the values of the cell C_1 to the cell C_{N_x+1}. The center of the cell C_i is $x_i = i\triangle x - \frac{\triangle x}{2}$. The space step is $\triangle x = L/N_x$. We also consider a sequence of times t_n, $n \in \mathbb{N}$, such that $t_0 = 0$ and $t_n = n\triangle t$, where $\triangle t$ satisfies the following CFL condition

$$\triangle t = \alpha \frac{\triangle x \triangle \eta}{2d}, \ 0 < \alpha \leq 1.$$

We consider a finite volume approximation of (13). We denote by $W_i(t)$ a piecewise constant approximation of W in each cell

$$W_i(t) \simeq W(x,t), \quad x \in C_i.$$

We obtain the following semi-discrete (in space) approximation

$$M\partial_t W_i = -\frac{F(W_i, W_{i+1}) - F(W_{i-1}, W_i)}{\triangle x} - BW_i.$$

where $(W_L, W_R) \mapsto F(W_L, W_R)$ denotes the numerical flux.

We then introduce a time discretization to compute W_i^n

$$W_i^n \simeq W(x, t^n), \quad x \in C_i.$$

We use a time second order scheme given by the following algorithm

$$M \frac{W_i^{n+1/2} - W_i^n}{\triangle t/2} = -\frac{F(W_i^n, W_{i+1}^n) - F(W_{i-1}^n, W_i^n)}{\triangle x} - B W_i^n,$$

$$M \frac{W_i^{n+1} - W_i^n}{\triangle t} = -\frac{F(W_i^{n+1/2}, W_{i+1}^{n+1/2}) - F(W_{i-1}^{n+1/2}, W_i^{n+1/2})}{\triangle x} - B W_i^{n+1/2}. \quad (14)$$

We consider several choices for the numerical flux $F(W_L, W_R)$. We consider the centered flux or a numerical flux with small numerical viscosity ("slightly upwinded flux"). The centered flux is given by

$$F(W_L, W_R) = A \frac{W_L + W_R}{2},$$

and the slightly upwinded flux with $\delta > 0$

$$F(W_L, W_R) = A \frac{W_L + W_R}{2} - \frac{\delta}{2}(W_R - W_L). \quad (15)$$

In practice, for saving CPU time and memory, we use two subroutines for computations with the sparse matrices in skyline format. The first one computes the product of a sparse matrix and a vector. The other one is used to solve a linear system (by the LU method).

4. Test cases

In this section, we will apply our numerical scheme, (14-15) and (12), on two test cases: the Landau damping and the two stream instability. The electric field, solution of the Poisson equation (4), is computed with the FFT (Fast Fourier Transform) algorithm.

We will compute the electric energy defined by

$$\mathcal{E}(t) = \sqrt{\int_0^L E(x, t)^2 dx}.$$

We are also interested in the distribution function in physical variable (x, v). The formula of the inverse Fourier transform reads

$$f(x, v, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(x, \eta, t) e^{I\eta v} d\eta \simeq \frac{1}{2\pi} \int_{-\eta_{\max}}^{\eta_{\max}} \phi(x, \eta, t) e^{I\eta v} d\eta. \quad (16)$$

We apply the rectangle method with oversampling for computing (16), in order to avoid Shannon aliasing. For this computation we can use a naive DFT computation instead of the FFT algorithm, because this step is applied only at the beginning and the end of the simulation.

In our numerical experiments, the discretization parameters are $N = 40$, $d = 5$, $N_x = 256$, $\gamma = 0$. Numerical investigations for $\gamma > 0$ will be carried out in future work.

4.1. The Landau damping

In this test case, the initial distribution function and the initial electric field are given by

$$f_0(x, v) = (1 + \varepsilon \cos(kx)) \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}},$$

$$E_0(x) = \frac{\varepsilon}{k} \sin(kx),$$

where $\varepsilon > 0$ and $k \in \mathbb{N}^*$, and the domain size is $L = 2\pi/k$.

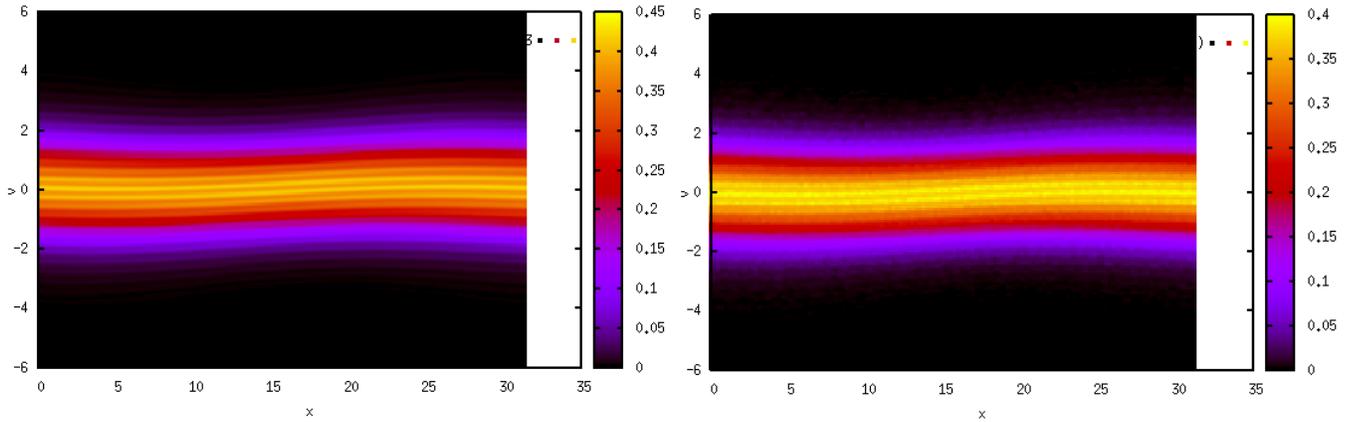So, the initial distribution function in velocity Fourier variable writes

FIGURE 1. The distribution function of the Landau damping test case at time $t = 100$. Left: Vlasov-Fourier method. Right: the PIC method.
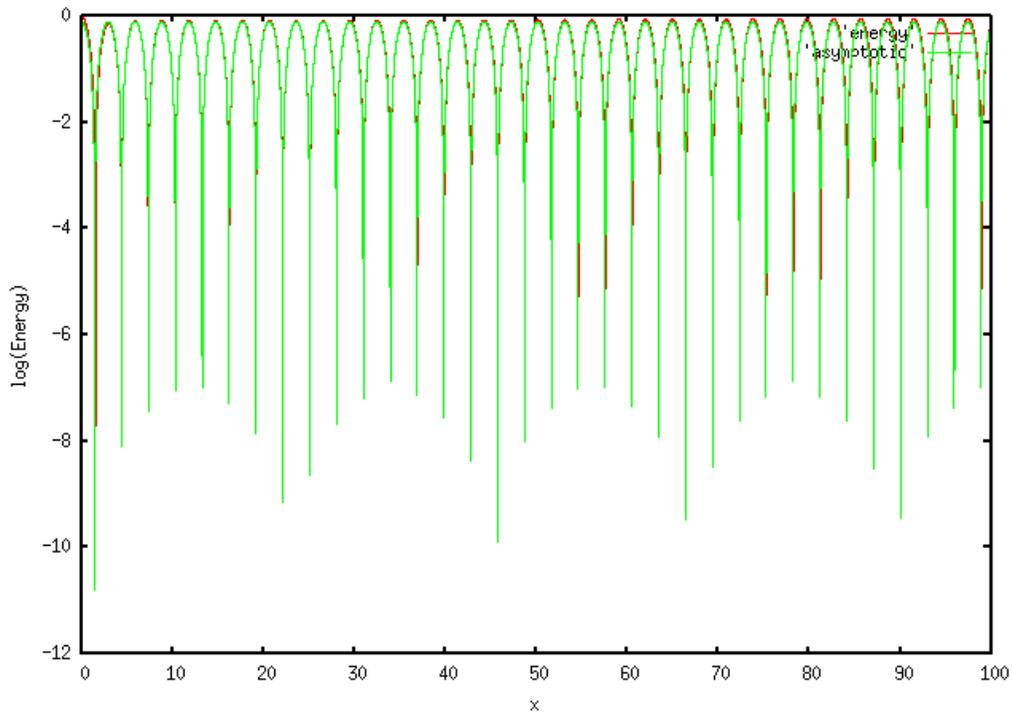

FIGURE 2. The electric energy of the Landau damping test case up to time $t = 100$, the green curve "asymptotic" is the analytical solution and the red curve "energy" is computed with the Vlasov-Fourier method.

$$\phi_0(x, \eta) = \int_{-\infty}^{+\infty} f_0(x, v) e^{-I\eta v} dv = (1 + \varepsilon \cos(kx)) e^{-\frac{\eta^2}{2}}.$$

For small ε, thanks to a linear approximation of the non-linear Vlasov-Poisson system, it is possible to compute an approximate analytical solution of the electric field. The details of the computation are given in [7].

In addition, the distribution function can be computed by a well-validated method, such as the PIC method. Let us take the values of parameter $k = 0.2$ and $\varepsilon = 5 \times 10^{-2}$. We compare our numerical results with the PIC results and also with the analytical solution.

We compare the distribution function and the electric energy of the Vlasov-Fourier method and of the PIC method (taken from [3]), for example at time $t = 100$ on Figure 1. We also compare the time evolution of the electric energy in the domain obtained by our method with the analytical solution (see Figure 2).
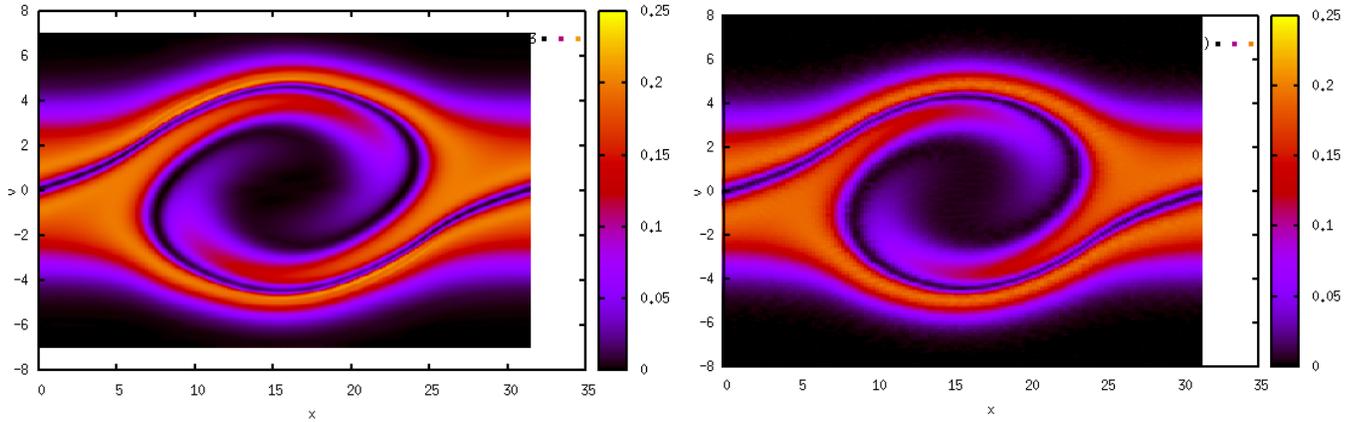
FIGURE 3. The distribution function of the two-stream test case at time $t = 25$. Left: Vlasov-Fourier method (with the centered flux). Right: the PIC method.
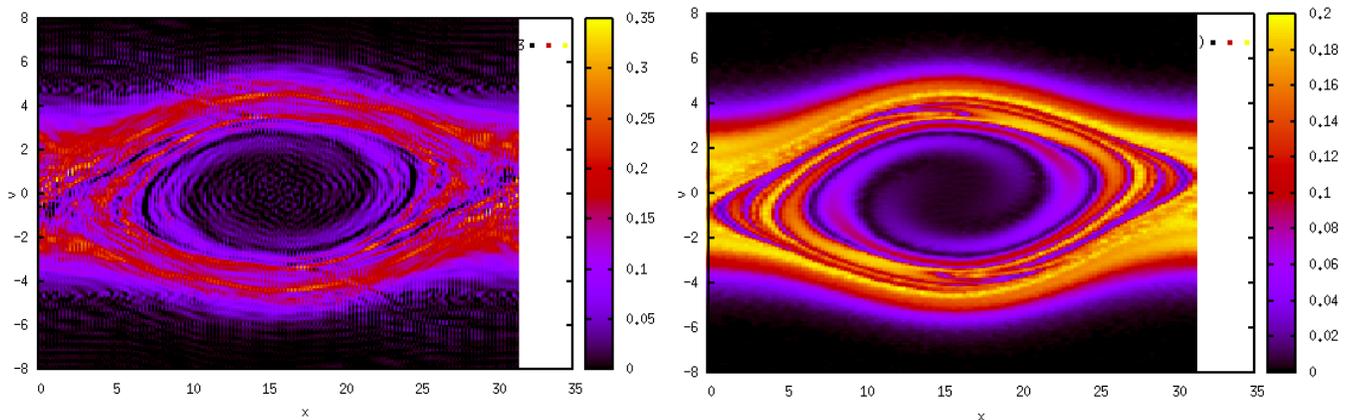

FIGURE 4. The distribution function of the two-stream test case at time $t = 50$. Left: Vlasov-Fourier method (with the centered flux). Right: the PIC method.

4.2. Two-stream instability

In this test case, the initial distribution function reads

$$f_0(x, v) = (1 + \varepsilon \cos(kx)) \frac{1}{2\sqrt{2\pi}} \left(e^{-\frac{(v-v_0)^2}{2}} + e^{-\frac{(v+v_0)^2}{2}} \right),$$

in which the velocity $v_0 > 0$ is given. It leads to the following initial function ϕ_0 in the Vlasov-Fourier case

$$\phi_0(x, \eta) = (1 + \varepsilon \cos(kx)) e^{-\frac{1}{2}\eta^2} \cos(\eta v_0).$$

The value of parameters for this test case are $k = 0.2$, $\varepsilon = 5 \times 10^{-3}$ $\delta = 0.05$ and $v_0 = 3$.

The distribution function is plotted at times $t = 25$ and $t = 50$ in Figures 3 and 4. We compare the PIC method and the Vlasov-Fourier method with the centered flux. At time $t = 50$, we remark small oscillations, which are maybe due to the fact that we have almost no upwind mechanism in the resolution of the transport equation. In order to remove these oscillations, we use the slightly upwinded flux (15) with parameter $\delta = 0.05$ instead of the centered flux and we obtain the results of Figure 5 in which the oscillations have disappeared.

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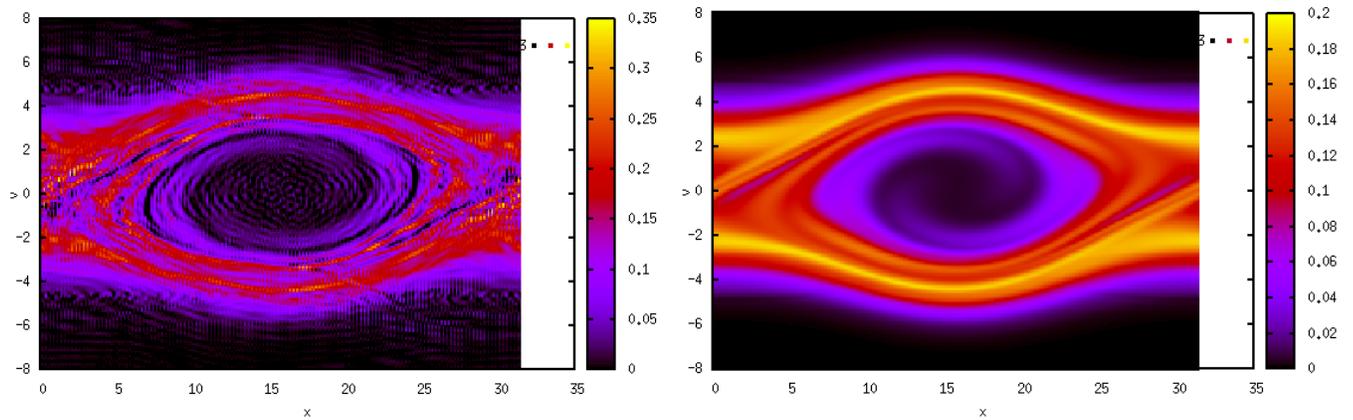
FIGURE 5. The distribution function of the two-stream test case at time $t = 50$ computed with the reduced Vlasov-Fourier method. Left: with the centered flux. Right: with the slightly upwinded flux ($\delta = 0.05$).

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