On the nef cone of symmetric products of a generic curve

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Abstract. The nef cone of the symmetric product $C^{(k)}$ of a curve C with general moduli is easily determined when k is at least equal to the gonality $gon(C)$ of the curve. In this paper we describe the nef cone for $k = gon(C) - 1$, when the genus of C is even. In this case the boundary of the nef cone has rational slope and is determined using curves on $C^{(k)}$ associated to the g^1_{k+1}'s of C.

1 Introduction

Let C be a smooth irreducible curve of genus $g \geq 1$. For any $k \geq 2$, denote by $C^{(k)}$ its k-th symmetric product that parametrizes effective degree k divisors $Z = x_1 + x_2 + \ldots + x_k$ on C. Let $N_1(C^{(k)})$ (resp. $N^1(C^{(k)})$) be the set of $\mathbb{R}$-linear combinations of 1-cycles (resp. divisors) on $C^{(k)}$ modulo numerical equivalence. $N^1(C^{(k)})$ is called the Néron-Severi group of $C^{(k)}$ and denoted $NS(C^{(k)})$. $N_1(C^{(k)})$ and $N^1(C^{(k)})$ are finite dimensional vector spaces, and the bilinear form given by the intersection pairing is non-degenerate. If C has general moduli, $N_1(C^{(k)})$ and $N^1(C^{(k)})$ have rank two. An interesting problem is to try to determine, inside $N_1(C^{(k)})$, the convex cone of classes of effective 1-cycles, denoted by $NE(C^{(k)})$ and called the *cone of curves* or *Mori cone*. Dually, one can study, inside $N^1(C^{(k)})$, the *nef cone* $Nef(C^{(k)})$, *i.e.* the closure of the convex cone of classes of ample divisors. By Kleiman's criterion of ampleness ([Kl]), $Nef(C^{(k)}) = \overline{NE(C^{(k)})^*}$.

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If k is at least equal to the gonality $gon(C)$ of C (recall that $gon(C) := min\{d : \exists \ \ a\ covering\ C \xrightarrow{d:1} \mathbf{P}^1\}$), then $NE(C^{(k)})$ is easily determined (see §2 for the details). The first two interesting cases to look at are then provided by the second symmetric product $C^{(2)}$, and by $C^{(k)}$, with $k = gon(C) - 1$.

The former has been studied by A. Kouvidakis and C. Ciliberto ([Kou1],[CK]). In [CK], they degenerate $C^{(2)}$ to the symmetric product of a rational g-nodal curve, and reduce the problem of determining $NE(C^{(2)})$ to the Nagata conjecture, concerning the existence of plane curves of fixed degree passing through given points in general position with prescribed multiplicities (see [N] for the precise statement). In such a way, they succeed in describing $NE(C^{(2)})$, when the genus of C is a square $g = m^2 \geq 9$. In this case, the cone of curves turns out to have rational slope, and to be closed on one side, and open on the other one. Let us also mention that the description of the nef cone of $C^{(2)}$ was used by B. Shiffman and M. Zaidenberg in [SZ] to construct new examples of hyperbolic surfaces of low degree in $\mathbb{P}^3$.

In this work, we focus our attention on $C^{(k)}$, for $k = gon(C)-1$. If C is a curve with general moduli and that has even genus $g = 2k$, by Brill-Noether theory (see [ACGH]) its gonality is equal to $k+1$, *i.e.* $k = gon(C) - 1$. Let δ be the small diagonal in $C^{(k)}$ defined by $\{x_1 = \ldots = x_k\}$. Let $L' \to C$ be a g^1_{k+1} on C, and consider the curve $Y \subset C^{(k)}$ defined as follows:

$$Y := \{Z \in C^{(k)} :\ \exists\ x \in C\ s.t.\ x + Z \in |L'|\}.$$

The class $c_Y \in N_1(C^{(k)})$ of the curve Y is independent from the g^1_{k+1} chosen, can be explicitly computed (cf. [ACGH], 3.2, p.342) and is not proportional to the class $c_\delta \in N_1(C^{(k)})$ of the small diagonal. These two classes will enable us to give a description of the nef cone of $C^{(k)}$. More precisely, we will prove the following

Theorem 1.1. *Let C be a curve with general moduli, of genus $g = 2k$. Then the nef cone of $C^{(k)}$*

$$Nef(C^{(k)}) := \{\alpha \in N^1(C^{(k)}) : \alpha\ is\ numerically\ effective\}$$

has rational slope and is determined by the conditions

$$(i)\ \alpha \cdot c_\delta \geq 0, \quad (ii)\ \alpha \cdot c_Y \geq 0.$$

The first step in the proof of the theorem will be to refine a criterion, due to L. Göttsche ([BS]), for certain line bundles on the punctual Hilbert scheme (in our case $C^{(k)}$) to be nef (see Lemma 3.3). The second key ingredient will be to specialize C to a curve on a $K3$ surface S. By [L], if $C \subset S$ generates $Pic(S)$, then C is generic in the sense of Brill-Noether theory. In the spirit of C. Voisin [V], we will then make use of the rich geometrical constructions introduced by R. Lazarsfeld [L1] to check the criterion in this case.

The paper is organized as follows: we start recalling in §2 the basic definitions and proving some preliminary results; in section 3 we recall Göttsche's criterion for nefness, and give a refinement of it. Then, we put the stress on two sufficient conditions (A) and (B) for the class of a set of numerically equivalent divisors on $C^{(k)}$ to be on the boundary of the nef cone. We exhibit in §4 a set of "natural candidates" to satisfy (A) and (B), when $k = gon(C) - 1$, and check they verify condition (B). In section 5, after having recalled Lazarsfeld's construction, we see, following [V], how condition (A) can be expressed, for curves on a K3 surface, in terms of the cohomology of a certain sheaf on a Grassmannian. Finally, we prove the last technical lemma and conclude our proof.

Throughout this paper we work on the field of complex numbers $\mathbb{C}$.

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2 Preliminaries

Let π be the projection from the cartesian product C^k to $C^{(k)}$, p_i the projection from C^k to the i-th factor, and $J(C)$ the Jacobian of C. Recall that, fixing a point $x \in C$, there are two maps defined as follows:

$$u_k : C^{(k)} \longrightarrow J(C), \ Z \longmapsto alb(Z - kx),$$

and

$$i_{k-1} : C^{(k-1)} \longrightarrow C^{(k)}, \ Z \longmapsto Z + x.$$

On $C^{(k)}$, associated to these maps, there are three natural divisors:

- the diagonal Δ, that can be thought of as the branch locus of the finite $k! : 1$ map π, and can be shown to be divisible by 2;

- the divisor $\Theta_k := u_k^*\Theta$, where $\Theta \subset J(X)$ is the ample theta divisor;
- the image $i_{k-1}(C^{(k-1)})$, whose class will be denoted by x.

If C has general moduli, its Néron-Severi group is two-dimensional, and any two of the above divisors provide a basis for $NS(C^{(k)})$.

Lemma 2.1. *(i) There is a relation among the classes x, Θ_k and $\Delta/2$ in $C^{(k)}$ given by the formula*

$$\frac{\Delta}{2} = (k+g-1)x - \Theta_k. \tag{1}$$

(ii) For $0 \leq r \leq k \leq g$ we have on $C^{(k)}$

$$\Theta_k^r x^{k-r} = \frac{g!}{(g-r)!}. \tag{2}$$

Proof. (i) is a special case of [ACGH], Prop. 5.1, p. 358, while (ii) is a direct consequence of Poincaré's formula (see [Kou1], Lemma 1). □

For any line bundle L on C, consider the line bundle

$$L^{\boxtimes k} := p_1^*L \otimes \ldots \otimes p_k^*L$$

on the k-th cartesian product of C. By abuse of notation, when no confusion is possible, we will still denote by $L^{\boxtimes k}$ the unique line bundle on $C^{(k)}$ such that its pull-back via $\pi : C^k \to C^{(k)}$ is equal to $p_1^*L \otimes \ldots \otimes p_k^*L$. Then $i_{k-1}(C^{(k-1)})$ is equal to $\mathcal{O}_C(p)^{\boxtimes k}$.

With this notation, the canonical bundle $K_{C^{(k)}}$ of $C^{(k)}$ can be shown to be equal to $K_C^{\boxtimes k} - \Delta/2$, where K_C is the canonical bundle of C (see, for instance, [Kou2], Prop. 2.6).

One side of the nef cone can be described as follows. Consider the map from the cartesian product of C to the cartesian product of order $\binom{k}{2}$ of its Jacobian:

$$\begin{aligned} \varphi \quad &: C^k \longrightarrow J(C)^{\binom{k}{2}} \\ &(x_1, \ldots, x_k) \mapsto (x_i - x_j)_{i<j}. \end{aligned}$$

This map contracts exactly the small diagonal $\delta := \{x_1 = \ldots = x_k\}$. Let π be the projection $C^k \to C^{(k)}$, and

$$p_i : J(C)^{\binom{k}{2}} \to J(C)$$

the projection on the i-th factor. Then the *unique* divisor E on $C^{(k)}$ such that

$$\pi^* E = \varphi^*(\otimes_i p_i^* \Theta) \tag{3}$$

is nef, and $E.\delta = 0$, since φ contracts δ. Hence we have proved

Lemma 2.2. *For any $k \geq 2$, the class c_δ belongs to the boundary of the Mori cone of $C^{(k)}$. Dually, the class of E belongs to the boundary of the nef cone of $C^{(k)}$.*

□

A standard way to produce curves in $C^{(k)}$ is to consider, for a given g_h^1 on C, with $h \geq k$, the curve

$$\Gamma_k(g_h^1) := \{Z \in C^{(k)} : \ D - Z \geq 0 \ \textit{for some} \ D \in g_h^1\}, \tag{4}$$

(see [ACGH], pp. 341-342). We have

Lemma 2.3. *If a curve C possesses a g_h^1 then Θ_k is a nef but not ample divisor on $C^{(k)}$, for any $k \geq h$.*

Proof. Fix a point $p \in C$. Consider the rational curve

$$\Gamma_h(g_h^1) + (k-h) \cdot p \subset C^{(k)}.$$

This $\mathbf{P^1}$ is contracted by the morphism $u_k : C^{(k)} \to J(C)$, since there are no holomorphic non constant maps from $\mathbf{P^1}$ to $J(C)$. Hence $\Theta_k \subset C^{(k)}$ is numerically effective but not ample. □

We briefly indicate how to compute all these classes. The class of $\Gamma_h(g_h^1)$ in $C^{(h)}$ does not depend on the choice of the g_h^1 and is given by the general formula

$$\textit{class}(\Gamma_k(g_h^1)) = \sum_{i=0}^{k-1} \binom{k-g-1}{i} \frac{x^i \Theta_k^{k-1-i}}{(k-1-i)!} \ \in N_1(C^{(k)}), \ \forall h \geq k, \tag{5}$$

(cf. [ACGH], Lemma 3.2, p. 342). Then the class $c_{\mathbf{P^1}} \in N_1(C^{(k)})$ of the rational curve $\Gamma_h(g_h^1) + (k-h) \cdot p \subset C^{(k)}$ can be computed using (5) and the push formula (cf. [ACGH], p. 369)

$$\textit{class}\big(A_{k-h}(x^a \Theta_k^b)\big) = \sum_{j=0}^{k-h} \binom{b}{j} \binom{g-b+j}{j} \binom{k-a-2b}{k-h-j} j! x^{a+j} \Theta_k^{b-j}, \tag{6}$$

where A_{k-h} is the map sending any $Z \in C^{(h)}$ to $Z + (k-h)x \in C^{(k)}$.

The class $c_\delta \in N_1(C^{(k)})$ can be computed using [ACGH], Prop. 5.1, p. 358. Dually, using (2) and the fact that $E.c_\delta = 0$, one can compute the ray generated by the class of E in $N^1(C^{(k)})$ in terms of the basis $x, \ \Delta/2$.

Putting together Lemma 2.2 and Lemma 2.3 with $h = gon(C)$, we obtain the following

Proposition 2.4. *Let C be a curve with general moduli. For any $k \geq gon(C)$ the nef cone of $C^{(k)}$ is determined by the conditions*

$$(i)\ \alpha \cdot c_\delta \geq 0, \quad (ii)\ \alpha \cdot c_{\mathbf{P}^1} \geq 0,$$

and its boundary rays are generated by the class of Θ_k, and the class of the divisor E defined by (3).

□

From the above, the first two interesting cases to look at are then provided by the second symmetric product $C^{(2)}$, and by $C^{(k)}$, with $k = gon(C) - 1$.

Our purpose is to study the case of $C^{(k)}$, for $k = gon(C) - 1$. If C has general moduli and $g(C) = 2k$ then, by the classical Brill-Noether theory, the gonality of C is equal to $k+1$, and there are a finite number of g^1_{k+1}'s, say $L'_1, \ldots, L'_s$, where s is the Castelnuovo number

$$s = \frac{g!}{(g-k)!(g-k+1)!},$$

(see [ACGH], p. 211). We will consider the curves Y_i in $C^{(k)}$ associated to the L'_i:

$$Y_i := \Gamma_k(L'_i) = \{Z \in C^{(k)} : L'_i - Z \geq 0\}, \tag{7}$$

whose class is given by (5), and the numerically equivalent line bundles on $C^{(k)}$

$$L_i^{\boxtimes k} - \frac{\Delta}{2}, \tag{8}$$

where L_i are the line bundles on C defined as $L_i := K_C - L'_i$. Notice that their numerical equivalence class is equal to

$$deg(L_i)x - \frac{\Delta}{2} = (3k-1)x - \frac{\Delta}{2},$$

which, by (1) and by the particular numerology considered, is the same class of Θ_k. Our goal is to prove that

$$L_i^{\boxtimes k} - \frac{\Delta}{2} \ \textit{is nef} \tag{9}$$

and that, moreover,

$$c_1(L_i^{\boxtimes k} - \frac{\Delta}{2}).Y_j = 0 \qquad (10)$$

for any $i \neq j$. This will automatically provide the description given in Theorem 1.1 of the nef cone of $C^{(k)}$, and dually, of the effective cone of curves in $C^{(k)}$.

3 The nef cone of $C^{(k)}$: a first reduction

In the beginning of this section, following Göttsche (see his appendix to [BS]), we associate to any $L \in Pic(C)$ a line bundle $\mathcal{G}_{k,L}$ on $C^{(k)}$.

Let X be a complete algebraic variety defined over an algebraically closed field. Let $Z \subset X$ be a zero dimensional subscheme of length $k := dim\ H^0(\mathcal{O}_Z)$, defined by an ideal sheaf $\mathcal{I}_Z \hookrightarrow \mathcal{O}_X$.

Let L be an invertible sheaf over X and, for any zero dimensional subscheme $Z \subset X$, consider the restriction map

$$r_Z\ : H^0(X, L) \to H^0(L \otimes \mathcal{O}_Z).$$

Definition 3.1. *L is said to be k-very ample if the restriction map r_Z is surjective, for any zero dimensional subscheme $Z \subset X$ of length less than or equal to $k+1$.*

The classical notions of global generation and very ampleness correspond in this way to, respectively, 0-very ampleness and 1-very ampleness.

For any $(k-1)$-very ample line bundle L one can then define a morphism from $X^{[k]}$, the Hilbert scheme of 0-dimensional subschemes of length k of X, to the Grassmannian $G(k, H^0(X, L))$ of codimension k subspaces of $H^0(X, L)$:

$$\varphi_{k,L}\ :\ X^{[k]} \to G(k, H^0(X, L)). \qquad (11)$$

This morphism associates to $Z \in X^{[k]}$ the kernel $H^0(X, L \otimes \mathcal{I}_Z)$ of the surjective map

$$H^0(X, L) \xrightarrow{r_Z} H^0(X, L \otimes \mathcal{O}_Z) \to 0.$$

In order to simplify the notation, we will now focus our attention on the case when $X = C$ is a curve: this is the one we need in the rest of the paper. Recall that the Hilbert scheme of points of length k on a curve C is given by the symmetric product $C^{(k)}$.

Consider the incidence variety in $C \times C^{(k)}$

$$\begin{array}{ccc} \Sigma_k & \xrightarrow{q} & C \\ \big\downarrow{\scriptstyle p} & & \\ C^{(k)} & & \end{array} \tag{12}$$

and p and q its natural projections. Let L be a line bundle on C. Consider the rank k vector bundle $\mathcal{E}_{k,L}$ over $C^{(k)}$ defined as

$$\mathcal{E}_{k,L} := p_* q^* L. \tag{13}$$

As the referee pointed out, the vector bundle $\mathcal{E}_{k,L}$ was introduced by A. Mattuck in [Ma]. We give the following

Definition 3.2. *The line bundle*

$$\mathcal{G}_{k,L} := det\ \mathcal{E}_{k,L} \to C^{(k)}, \tag{14}$$

will be called the Göttsche line bundle on $C^{(k)}$ associated to L.

Let $\mathcal{O}_G(1)$ be the line bundle on the Grassmannian $G := G(k, H^0(C, L))$ giving its Plücker polarization. Let $\mathcal{S}$ be the universal subbundle over G, so that $det\ \mathcal{S}^* \ = \mathcal{O}_G(1)$. By definition, whenever L is $(k-1)$-very ample, we have

$$\mathcal{E}_{k,L} = \varphi_{k,L}^* \mathcal{S}^*, \tag{15}$$

and then the Göttsche line bundle associated to L is given by

$$\mathcal{G}_{k,L} = \varphi_{k,L}^* \mathcal{O}_G(1). \tag{16}$$

Göttsche, in his appendix to [BS], proved the following

Theorem (Göttsche). *For any $L \in Pic(C)$ we have*

$$L^{\boxtimes k} \otimes \mathcal{O}_{C^{(k)}}(-\frac{\Delta}{2}) = \mathcal{G}_{k,L}.$$

This theorem together with (16) gives a sufficient condition for $L^{\boxtimes k} \otimes \mathcal{O}_{C^{(k)}}(-\frac{\Delta}{2})$ to be nef:

(Göttsche's criterion for nefness): *For any $L \in Pic(C)$, the line bundle $L^{\boxtimes k} \otimes \mathcal{O}_{C^{(k)}}(-\frac{\Delta}{2})$ over $C^{(k)}$ is nef whenever L is $(k-1)$-very ample.*

Let L be a line bundle over C, and suppose $h^0(C,L) = k$. The hypothesis $h^0(C,L) = k$ implies that $\mathcal{E}_{k,L}$ is generically globally generated. Hence the Göttsche line bundle $\mathcal{G}_{k,L} \to C^{(k)}$ is effective, since k global sections of L generating $H^0(C,L)$ will be independent at the generic set of k points $x_1, \dots, x_k$ of C and then their wedge product gives a nonzero global section of $\mathcal{G}_{k,L}$.

The k-th exterior power of the evaluation map $H^0(C,L) \otimes \mathcal{O}_C \to L$ induces a morphism (which is actually an isomorphism)

$$\bigwedge^k H^0(C,L) \to H^0(C^{(k)}, \mathcal{G}_{k,L}), \tag{17}$$

and we will denote by σ_L a nonzero global section of $\mathcal{G}_{k,L}$ generating the image of this morphism. Notice that, by construction,

$$V(\sigma_L) = \{Z \in C^{(k)} : H^0(L) \xrightarrow{r_Z} H^0(L_{|Z}) \textit{ is not surjective}\}, \tag{18}$$

i.e.

> *the zeroes of σ_L are given by the k-tuples of points of C which fail to be separated by the sections of L.*

Now, we have the following

Lemma 3.3. *Let $L_1, \dots, L_s$ be a finite number of line bundles of fixed degree d on C, with $h^0(C, L_i) = k$. If they verify the property*

(A) $\bigcap_i V(\sigma_{L_i}) = \emptyset$,

then the class in $NS(C^{(k)})$ of the numerically equivalent line bundles $\mathcal{G}_{k,L_i}$ is nef.

Proof. For any curve $X \subset C^{(k)}$, by **(A)** we can find an index i such that $\sigma_{L_i|X} \neq 0$. Then $(det\ \mathcal{E}_{L_i})_{|X}$ is effective, i.e.

$$c_1(\mathcal{G}_{k,L_i}).X \geq 0,$$

and the lemma is proved. □

Remark 3.4. *The reader may observe that the conclusion of Lemma 3.3 holds even under the weaker condition that the intersection in* **(A)** *is just finite.*

Remark 3.5. *From (18) it follows that condition* **(A)** *is satisfied if and only if for any (possibly coinciding) k points $x_1, \dots, x_k \in C$, there is one of the L_i's separating them. In this sense Lemma 3.3 can be seen as a refinement of Göttsche's criterion.*

Notice that if, moreover, there existed an irreducible curve $Y \subset C^{(k)}$ such that

(B) $c_1(\mathcal{G}_{k,L_i}).Y = 0$,

the class of $\mathcal{G}_{k,L_i}$ in $NS(C^{(k)})$ would belong to the boundary of the nef cone.

We summarize the above in the following

Proposition 3.6. *Let $L_1, \dots, L_s$ be a finite number of line bundles of fixed degree d on C, with $h^0(C, L_i) = k$, and $Y \subset C^{(k)}$ an irreducible curve whose class $c_Y \in N_1(C^{(k)})$ is not proportional to the class c_δ of the small diagonal. If the following properties are verified*

(A) $\bigcap_i V(\sigma_{L_i}) = \emptyset$,

(B) $c_1(\mathcal{G}_{k,L_i}).c_Y = 0$,

then

$$Nef(C^{(k)}) = \{\alpha \in NS(C^{(k)}) : \alpha \cdot c_\delta \geq 0 \text{ and } \alpha \cdot c_Y \geq 0\},$$

and the boundary rays of the nef cone are generated by the class of $\mathcal{G}_{k,L_i}$ and the class of the divisor E defined by (3).

4 The case of $C^{(k)}$, for $k = gon(C) - 1$

From now on C will be a generic curve of genus $g(C) = 2k$. According to the notation introduced in §2, we denote by L'_i the g^1_{k+1}'s on C, and by L_i the line bundles

$$L_i := K_C - L'_i. \tag{19}$$

By Riemann-Roch, we have $h^0(C, L_i) = k$. Thus, as we saw in the previous section, we have an effective line bundle on $C^{(k)}$, given by $\mathcal{G}_{k,L_i} = L_i^{\boxtimes k} - \frac{\Delta}{2}$. Notice that the line bundles $\mathcal{G}_{k,L_i}$ are numerically equivalent since the L_i's all have the same degree.

For any L'_i, consider the irreducible curve

$$Y_i := \Gamma_k(L'_i) \subset C^{(k)}$$

defined by (7), which is actually a copy of C in its symmetric product $C^{(k)}$.

Our purpose is to prove the following

Theorem 4.1. *Let C be a curve with general moduli, of genus $g = 2k$. Then the line bundles $L_1, \ldots, L_s$ defined in (19), and the class $c_Y \in N_1(C^{(k)})$ of the curves $Y_i \subset C^{(k)}$ verify the properties:*

(A) $\bigcap_i V(\sigma_{L_i}) = \emptyset$;

(B) $c_1(\mathcal{G}_{k,L_i}).c_Y = 0$.

By Proposition 3.6, the previous theorem implies Theorem 1.1. The proof of **(A)** will occupy the rest of the paper, while in the remainder of this section we will verify **(B)**.

Proposition B. *The section $\sigma_{L_i} \in H^0(C^{(k)}, \mathcal{G}_{k,L_i})$ does not vanish along the curve Y_i. In particular this implies*

(B) $c_1(\mathcal{G}_{k,L_i}).c_Y = 0$.

Proof. Recall first that, by the base-point-free pencil trick (see [ACGH], p. 126), the space of sections $H^0(K_C - 2L'_i)$ identifies to the kernel of the multiplication map

$$\mu_0 : H^0(L'_i) \otimes H^0(K_C - L'_i) \to H^0(K_C). \tag{20}$$

By Gieseker [G] (or Lazarsfeld [L1]) we know that μ_0 is injective for generic C and hence we have

$$H^0(K_C - 2L'_i) = 0. \tag{21}$$

Now, we check

Lemma 4.2. *For any $x \in C$ such that $L'_i - x \geq 0$,*

$$H^0(L_i - (L'_i - x)) = H^0(K_C - 2L'_i + x) = H^0(K_C - 2L'_i).$$

Proof. By Riemann-Roch, the equality

$$H^0(K_C - 2L'_i + x) = H^0(K_C - 2L'_i).$$

is equivalent to

$$h^0(2L'_i - x) = h^0(2L'_i) - 1. \tag{22}$$

But $|L'_i|$ is base-point-free, hence $|2L'_i|$ is base-point-free and (22) immediately follows. □

We conclude the proof of Proposition B. By (21), Lemma 4.2, and recalling that

$$Z \in Y_i \ \Leftrightarrow \ \exists x : \ Z \sim L_i' - x,$$

we see that, for any $Z \in Y_i$,

$$ker(r_Z) = H^0(C, L_i - Z) = 0.$$

By (18), this implies that $\sigma_{L_i} \in H^0(C^{(k)}, \mathcal{G}_{k,L_i})$ does not vanish along Y_i. $\square$

5 Proof of condition (A)

In this section we introduce a map ϕ_k associated to the line bundles $L_i = K_C - L_i'$, and prove that its surjectivity implies condition (A) (see Proposition 5.1 below). Then we specialize to the case of a curve C on a $K3$ surface, with C generating the Picard group of the surface. After recalling some constructions due to Lazarsfeld [L1] and Voisin [V], and some results from [V], we will prove Proposition 5.1 for such a curve. By [L1], such a curve is Brill-Noether-Petri generic. Hence, since all our constructions and results only depend on the pencils of minimal degree on the curve, this will conclude the proof of Theorem 1.1.

5.1 A second reduction

Let L_i' be a g^1_{k+1} on C. The choice of a section $\tau \in H^0(C, L_i')$ furnishes an inclusion

$$H^0(K_C - L_i') = H^0(L_i) \stackrel{\times\tau}{\hookrightarrow} H^0(K_C), \tag{23}$$

which induces a $\mathbb{C}$-linear inclusion

$$H^0(C^{(k)}, \mathcal{G}_{k,L_i}) = \bigwedge^k H^0(C, L_i) \hookrightarrow H^0(C^{(k)}, \mathcal{G}_{k,K_C}) = \bigwedge^k H^0(C, K_C). \tag{24}$$

Fix a base $s_1, ..., s_k$ of $H^0(C, L_i)$, so that $s_1 \wedge ... \wedge s_k = \sigma_{L_i} \in H^0(C^{(k)}, \mathcal{G}_{k,L_i})$. For any i, consider more generally the map

$$\phi_{k,i} : \ S^k H^0(C, L_i') \rightarrow H^0(C^{(k)}, \mathcal{G}_{k,K_C}), \tag{25}$$

which sends τ^k to the image of $s_1 \wedge ... \wedge s_k$ via the inclusion (24). We will denote this image by $\tau^k \cdot \sigma_{L_i}$. Dually, the map

$$\phi_{k,i}^* : \ \bigwedge^k H^0(\mathcal{O}_C(K_C))^* \cong \bigwedge^k H^0(\mathcal{O}_C(K_C)) \rightarrow S^k H^0(C, L_i')^*$$

sends $w \in \wedge^k H^0(\mathcal{O}_C(K_C))^*$ to the polynomial $Q_w \in S^k H^0(C, L'_i)^*$ defined as follows

$$Q_w(t) := (t^k \cdot \sigma_{L_i}) \bullet w,$$

where $\bullet$ denotes the natural pairing between a vector space and its dual.

We want to prove

Proposition 5.1. *The elements $\tau^k \cdot \sigma_{L_i}$, $\forall i$, $\tau \in H^0(C, L'_i)$, generate the whole space of sections $H^0(C^{(k)}, \mathcal{G}_{k,K_C}) = \wedge^k H^0(C, K_C)$, i.e. the map*

$$\phi_k \ : \ \bigoplus_i S^k H^0(C, L'_i) \longrightarrow \bigwedge^k H^0(C, K_C), \tag{26}$$

whose components $\phi_{k,i}$ have been described above, is surjective.

Assuming Proposition 5.1 for the moment, we see how it implies

Proposition A. *Let C be a curve with general moduli, of genus $g = 2k$. Consider the line bundles $L_i = K_C - L'_i$. Then the sections $\sigma_{L_i} \in H^0(C^{(k)}, \mathcal{G}_{k,L_i})$ verify the property:*

(A) $\bigcap_i V(\sigma_{L_i}) = \emptyset$.

Proof. The space of global sections $H^0(C^{(k)}, \mathcal{G}_{k,K_C})$ has no base points. Indeed, for any $Z \in C^{(k)}$, we have a surjection

$$H^0(K_C) \twoheadrightarrow H^0(K_{C|Z}).$$

Otherwise, by Riemann-Roch, $h^0(\mathcal{O}_C(Z)) = h^1(\mathcal{O}_C(K_C(-Z))) \geq 2$, contradicting the fact that $gon(C) = k+1$. This, in turn, implies the surjectivity of the evaluation map

$$H^0(C^{(k)}, \mathcal{G}_{k,K_C}) = \bigwedge^k H^0(C, K_C) \twoheadrightarrow \bigwedge^k H^0(K_{C|Z}) = (\mathcal{G}_{k,K_C})_{|Z}. \tag{27}$$

If $Z \in C^{(k)}$ belongs to $\bigcap_i V(\sigma_{L_i})$, with $\sigma_{L_i} \in H^0(C^{(k)}, \mathcal{G}_{k,L_i})$, then, *a fortiori*, $Z \in \bigcap_{\tau \in H^0(L'_i),\, i \in I} V(\tau^k \cdot \sigma_{L_i})$. But this is absurd, since by Proposition 5.1 the elements $\tau^k \cdot \sigma_{L_i}$ generate the base-point-free space of sections $H^0(C^{(k)}, \mathcal{G}_{k,K_C})$. □

5.2 Curves on a $K3$ surface

Let C be a curve of genus $g(C) = 2k$ contained in a K3 surface S, such that C generates the Picard group of S. By Lazarsfeld [L1], the curve $C \subset S$ is Brill-Noether-Petri generic, and hence in particular its gonality is equal to $k+1$, and the number of g^1_{k+1}'s on C is finite. Let L' be a g^1_{k+1} on C. In what follows we will recall a classical construction due to Lazarsfeld, which associates to the data (C, L') a vector bundle E on S. This, along the lines of Voisin [V], will allow us to put a structure of complete intersection on the set of g^1_{k+1}'s over C, seen as points of a certain Grassmannian.

The line bundle L' is obviously base-point-free. Then we can consider the sheaf F defined by the exact sequence

$$0 \to F \to H^0(L') \otimes \mathcal{O}_S \xrightarrow{ev} L' \to 0, \tag{28}$$

where, by abuse of notation, we still denote by L' the sheaf on S obtained by extending L' to zero away from C. The sheaf F is locally free of rank 2 and we define $E := F^*$. The rank 2 vector bundle E on S sits inside the short exact sequence

$$0 \to H^0(L')^* \otimes \mathcal{O}_S \to E \to K_C - L' \to 0, \tag{29}$$

obtained by dualizing (28). We record now a series of facts about the bundle E:

(i) $c_1(E) = C$, $c_2(E) = deg\ L' = k+1$, $h^0(E) = k+2$;

(ii) E is stable;

(iii) E is independent from the choice of C in its linear system, and from the g^1_{k+1} considered on C.

The numerical conditions in (i) follow easily from (29). The stability of E, as remarked in [Mu], follows from the fact that $detE = L$ generates the Picard group of S and from the vanishing $H^0(S, E(-L)) = H^0(S, F) = 0$ (this second fact follows from (28)). Finally, (iii) is a consequence of (i) and (ii). Indeed, if there were another stable rank two bundle E' on S satisfying (i) and (ii), then, as computed in [L1], $\chi(E, E') = 2$, and hence, by Riemann-Roch, either $Hom(E, E') \neq 0$ or $Hom(E', E) \neq 0$. But such a non trivial homomorphism would furnish a destabilizing subbundle of either E or E', or would be an isomorphism.

The exact sequence (29) realizes $H^0(C,L')^* \cong H^0(C,L')$ as a 2-dimensional subspace of $H^0(S,E)$. Let s_1 and s_2 be two generators of $H^0(C,L')$. Then, by (29) restricted to C, we have that C is the zero locus of $s_1 \wedge s_2 \in H^0(S, detE)$, and the image of the morphism

$$H^0(C,L') \otimes \mathcal{O}_C \to E_{|C}$$

identifies to L'.

Consider now the Grassmannian $G_2 := Grass(2, H^0(S,E))$ of dimension two subspaces of $H^0(S,E)$, which, by (i), has dimension equal to 2k, and let $\mathcal{O}_{G_2}(1)$ be the line bundle on G_2 giving its Plücker embedding. Notice that the determinant

$$det\ : H^0(\mathcal{O}_{G_2}(1))^* = \bigwedge^2 H^0(S,E) \to H^0(S, det\ E)$$

is not zero on any rank two element W of $H^0(\mathcal{O}_{G_2}(1))^*$. Indeed, if this were the case, then the subspace $W \subset H^0(S,E)$ would generate a rank one subsheaf of E with at least two sections, in contradiction with (ii).

By the above we can then define a map

$$d\ :\ G_2 \to \mathbf{P}H^0(S, \mathcal{O}_S(C)) = \mathbf{P}H^0(S, det(E)),$$

sending a point $w \in G_2$, that corresponds to a subspace $W \subset H^0(S,E)$, to the element in $|C|$ defined by the zeroes of $\bigwedge^2 W$. The target space has dimension 2k (since $g(C) = 2k$) and the map d is finite. Its fibre over C, which we will denote Z_C, is in 1-1 correspondence with the set of the g^1_{k+1}'s over C. Finally, by construction, we have the commutativity of the following diagram

$$\begin{array}{ccc} G_2 & \hookrightarrow & \mathbf{P}^N = \mathbf{P}H^0(\mathcal{O}_{G_2}(1)) \\ & \searrow^{d} & \downarrow \pi \\ & & \mathbf{P}H^0(\mathcal{O}_S(C)) \end{array} \qquad (30)$$

Hence the set of the g^1_{k+1}'s over C, seen as a finite subset of G_2, is endowed, by this projection, with a scheme-theoretic structure of complete intersection of elements of $|\mathcal{O}_{G_2}(1)|$.

5.3 Proof of Proposition 5.1

Let C be a curve of genus $2k$ on a K3 surface S, generating the Picard group of S, and $L'_1, \dots L'_s$ the g^1_{k+1}'s on C, parametrized by the finite complete intersection $Z_C \subset G_2$. Consider the bundle $E \to S$ defined in the previous section. By the exact sequence (29), there is a natural map from $H^0(C, L_i')^* \cong H^0(C, L_i')$ to $H^0(S, E)$. Denote by ψ_l the induced map

$$\psi_l \ : \ \bigoplus_{L_i' \in Z_C} S^l H^0(C, L_i') \to S^l H^0(S, E). \tag{31}$$

The maps ψ_l have been considered by Voisin in [V], where she proves

Proposition [V]. *The maps ψ_l are surjective for any $l \leq k-1$.*

We will prove in the next section the following

Proposition 5.2. *The image of*

$$\psi_k \ : \ \bigoplus_{L_i' \in Z_C} S^k H^0(C, L_i') \to S^k H^0(S, E),$$

has dimension $\geq dim \ \wedge^k H^0(C, K_C)$.

Now, recall that $h^0(S, E) = k+2$, and $det E = \mathcal{O}_S(C)$. Let γ be a global section of E and $e_1, \ldots, e_{k+1} \in H^0(E)$ such that $\gamma, e_1, \ldots, e_{k+1}$ form a basis of $H^0(E)$. For any $i = 1, \ldots, k+1$, the element $\gamma \wedge e_i$ gives a global section of $\mathcal{O}_S(C)$, and then $(\gamma \wedge e_1) \wedge \ldots \wedge (\gamma \wedge e_{k+1})$ divided by $\gamma \wedge e_1 \wedge e_2 \wedge \ldots \wedge e_{k+1} \in \wedge^{k+2} H^0(S, E) \cong \mathbb{C}$ gives a well defined element in $\wedge^{k+1} H^0(\mathcal{O}_S(C))$. By abuse of notation we will call this element $det(\gamma \wedge H^0(E))$. Notice that, for any $0 \neq \lambda \in \mathbb{C}$, we have

$$det(\lambda \cdot \gamma \wedge H^0(E)) = \frac{\lambda^{k+1}}{\lambda} \cdot \frac{(\gamma \wedge e_1) \wedge \ldots \wedge (\gamma \wedge e_{k+1})}{\gamma \wedge e_1 \wedge e_2 \wedge \ldots \wedge e_{k+1}} = \lambda^k \cdot det(\gamma \wedge H^0(E)).$$

Recall that $h^0(S, \mathcal{O}_S(C)) = 2k+1$, and hence we have an isomorphism:

$$\bigwedge^{k+1} H^0(\mathcal{O}_S(C))^* \cong \bigwedge^{k} H^0(\mathcal{O}_S(C)).$$

We define a morphism

$$\nu_k \ : \ S^k H^0(E) \to \bigwedge^{k+1} H^0(\mathcal{O}_S(C)), \tag{32}$$

by describing its dual

$$\nu_k^* : \ \bigwedge^{k+1} H^0(\mathcal{O}_S(C))^* \to S^k H^0(E)^*.$$

The map ν_k^* sends $u \in \bigwedge^{k+1} H^0(\mathcal{O}_S(C))^*$ to the polynomial $P_u \in S^k H^0(E)^*$ defined as follows:

$$P_u(\gamma) := det(\gamma \wedge H^0(E)) \bullet u$$

(recall that $\bullet$ is the dual pairing). We have

Theorem ([V] 3.18). *The map ν_k is an isomorphism.*

This result, together with Proposition 5.2 yields the

Proof of Proposition 5.1. Consider the short exact sequence defining $C \subset S$

$$0 \to \mathcal{O}_S(-C) \xrightarrow{\cdot\sigma_C} \mathcal{O}_S \to \mathcal{O}_C \to 0,$$

where σ_C is the section defining C. By tensoring it with $\mathcal{O}_S(C)$, recalling that $K_S = \mathcal{O}_S$ and $h^1(\mathcal{O}_S) = 0$, and using the adjunction formula, we obtain

$$0 \to H^0(\mathcal{O}_S) \xrightarrow{\cdot\sigma_C} H^0(\mathcal{O}_S(C)) \to H^0(K_C) \to 0.$$

We then have an injective morphism

$$\iota \ : \ \bigwedge^k H^0(K_C) \overset{\wedge\sigma_C}{\hookrightarrow} \bigwedge^{k+1} H^0(\mathcal{O}_S(C)). \tag{33}$$

The maps defined in (26), (31), (32) and (33) sit inside the following diagram:

$$\begin{array}{ccc} S^k H^0(S,E) & \xrightarrow[\sim]{\nu_k} & \bigwedge^{k+1} H^0(S, \mathcal{O}_S(C)) \\ \uparrow{\scriptstyle \psi_k} & & \uparrow{\scriptstyle \iota} \\ \bigoplus_{L_i' \in Z_C} S^k H^0(C, L_i') & \xrightarrow{\phi_k} & \bigwedge^k H^0(C, K_C). \end{array} \tag{34}$$

We check the following

Lemma 5.3. *The composed maps $\nu_k \cdot \psi_k$ and $\iota \cdot \phi_k$ coincide up to a scalar factor.*

Proof of Lemma 5.3. It is of course sufficient to verify that, for any i, the maps $\nu_k \cdot \psi_{k,i}$ and $\iota \cdot \phi_{k,i}$ coincide up to the same scalar factor, where $\psi_{k,i}$ and $\phi_{k,i}$ are the restrictions of ψ_k and ϕ_k to $S^k H^0(C, L_i')$. By duality, we have to prove that, for any $w \in \wedge^k H^0(\mathcal{O}_C(C))^*$ and for any $t \in H^0(C, L_i')$, up to a scalar factor, we have

$$\phi_{k,i}^*(w)(t) := Q_w(t) = P_u(t) =: \nu_k^*(u)(t),$$

with $u = w \wedge \sigma_C^* \in \wedge^{k+1} H^0(\mathcal{O}_S(C))^*$.

Let $\tau \in H^0(C, L_i')$ and $s_1, \ldots, s_k \in H^0(C, L_i)$ be such that $< t, \tau >= H^0(C, L_i')$ and $< s_1, \ldots, s_k >= H^0(C, L_i)$. Then $t \wedge \tau \in H^0(S, detE)$ is such that $V(t \wedge \tau) = C \subset S$ and $t, \tau, s_1, \ldots, s_k$, regarded as elements of $H^0(S, E) = H^0(C, L_i') \oplus H^0(C, L_i)$, give a basis for this space of sections. Moreover, by [G] and [L1], we have

$$< t \cdot s_1, \ldots, t \cdot s_k, \tau \cdot s_1, \ldots, \tau \cdot s_k >= H^0(C, \mathcal{O}_C(C)),$$

and

$$< t \wedge s_1, \ldots, t \wedge s_k, \tau \wedge s_1, \ldots, \tau \wedge s_k > \oplus < t \wedge \tau >= H^0(S, \mathcal{O}_S(C)).$$

Then

$$\nu_k^*(u)(t) = P_{w \wedge (t \wedge \tau)^*}(t) = \alpha\big((t \wedge s_1) \wedge \ldots \wedge (t \wedge s_k) \wedge (t \wedge \tau)\big) \bullet \big(w \wedge (t \wedge \tau)^*\big),$$

with $\alpha^{-1} = s_1 \wedge \ldots \wedge s_k \in \wedge^k H^0(C, L_i) \cong \mathbb{C}$. But

$$\big((t \wedge s_1) \wedge \ldots \wedge (t \wedge s_k) \wedge (t \wedge \tau)\big) \bullet \big(w \wedge (t \wedge \tau)^*\big) = \big((t \cdot s_1) \wedge \ldots \wedge (t \cdot s_k)\big) \bullet w,$$

and the term on the right is exactly $\phi_{k,i}^*(w)(t) = Q_w(t)$. □

By the previous lemma and (34), Proposition 5.2 implies Proposition 5.1. □

In order to complete the proof of Theorem 4.1 it remains to prove Proposition 5.2, which we will do in the next subsection.

5.4 Proof of Proposition 5.2

Let $Z_C \subset G_2$ be the finite complete intersection of hyperplane sections parametrizing the g_{k+1}^1's over C. Let $\mathcal{E} \to G_2 = Grass(2, H^0(E))$ be the dual of the universal subbundle. In order to prove Proposition 5.2 we study the cohomology group $H^0(G_2, \mathcal{I}_{Z_C} \otimes Sym^k \mathcal{E})$ via the Koszul resolution of the ideal sheaf of the complete intersection $Z_C \subset G_2$.

Let $K := H^0(C, \mathcal{O}_C(C))^* = H^0(K_C)^*$. Recall the injection

$$H^0(C, \mathcal{O}_C(C))^* \subset H^0(S, \mathcal{O}_S(C))^* \overset{d^*}{\hookrightarrow} \bigwedge^2 H^0(S, E)^* = H^0(G_2, \mathcal{O}_{G_2}(1)).$$

The Koszul resolution of $\mathcal{I}_{Z_C}$ is then given by

$$0 \to \bigwedge^{2k} K \otimes \mathcal{O}_{G_2}(-2k) \to \ldots \to K \otimes \mathcal{O}_{G_2}(-1) \to \mathcal{I}_{Z_C} \to 0. \qquad (35)$$

By tensoring it with $S^k\mathcal{E}$ we get the exact complex

$$\mathcal{K}^\bullet : \; 0 \to \bigwedge^{2k} K \otimes \mathcal{O}_{G_2}(-2k) \otimes S^k\mathcal{E} \to \ldots \to \mathcal{I}_{Z_C} \otimes S^k\mathcal{E} \to 0, \qquad (36)$$

where the term $S^k\mathcal{E}$ is put in degree zero. We have

Lemma 5.4. *Let C be a curve of genus $2k$ on a $K3$ surface S, with $< \mathcal{O}_S(C) >= Pic(S)$. Let $\{L'_i\}_i$ be the finite set of g^1_{k+1}'s on C, which are parametrized by the finite complete intersection $Z_C \subset G_2$. Then the following inequality holds:*

$$h^0(G_2, \mathcal{I}_{Z_C} \otimes S^k\mathcal{E}) \leq \binom{2k}{k+1}.$$

Proof. The proof goes as in [V], Lemma 2. The hypercohomology $\mathbb{H}^0(G_2, \mathcal{K}^\bullet)$ vanishes. Now we have the spectral sequence

$$E_1^{p,q} = H^q(G_2, \mathcal{K}^p) \Rightarrow \mathbb{H}^{p+q}(G_2, \mathcal{K}^\bullet).$$

Using the following facts
- all the differentials d_r starting from $E_r^{0,0}$ vanish;
- $H^q(G_2, \mathcal{O}_{G_2}(-q-1) \otimes S^k\mathcal{E}) = 0$ for all $q \neq k$ (see [V], Prop. 9);
- $H^k(G_2, \mathcal{O}_{G_2}(-k-1) \otimes S^k\mathcal{E}) \cong \mathbb{C}$ (see again [V], proof of Prop. 9);

one constructs a surjective map from a subquotient of $E_1^{-k-1,k} = \wedge^{k+1}K$ to $H^0(G_2, \mathcal{I}_{Z_C} \otimes S^k\mathcal{E})$, and this proves the lemma. □

We can finally give the

Proof of Proposition 5.2. Recall that, by (29), $H^0(C, L'_i)$ identifies to a 2-dimensional subspace of $H^0(S, E)$. Consider $\mathcal{E} \to G_2 = Grass(2, H^0(E))$. The fiber of $\mathcal{E}$ at a point $W \in G_2$ is equal to W^*, and the space of its global sections $H^0(G_2, \mathcal{E})$ is given by $H^0(S, E)^*$. Thus, for reduced Z_C (which is always the case for C generic), the dual of the map

$$\psi_k \; : \; \bigoplus_{L'_i \in Z_C} S^k H^0(C, L'_i) \to S^k H^0(S, E),$$

identifies to the evaluation map

$$S^k H^0(G_2, \mathcal{E}) = H^0(G_2, S^k\mathcal{E}) \stackrel{ev_{Z_C}}{\longrightarrow} S^k\mathcal{E}_{|Z_C},$$

whose kernel is exactly $H^0(\mathcal{I}_{Z_C} \otimes S^k\mathcal{E})$. Notice that

$$dim\ S^kH^0(S,E) - dim\ \bigwedge^k H^0(C,K_C) = \binom{2k+1}{k} - \binom{2k}{k}. \qquad (37)$$

Then Lemma 5.4 and (37) imply that the dimension of the image of ψ_k is greater than or equal to the dimension of $\bigwedge^k H^0(C,K_C)$. □

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