On the uniruledness of stable base loci

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§1. Introduction & Main Result.

Conjecture. (Ueno, ..., around '75)

Let X : a smooth projective variety of general type.

Then every irred component V of the stable base locus of K_X has negative Kodaira dim; $\kappa(V) = -\infty$.

Mori's theory, MMP ('80s) rephrase

V should be uniruled, i.e., $\exists$ dominant map : $\mathbb{P}^1 \times W^{d-1} \dashrightarrow V^d$.

Definition. D : a divisor on X.

(1) Stable base locus : $\mathrm{SBs}\,(D) = \bigcap_{m>0} \mathrm{Bs}\,|mD|$.

(2) Non-ample locus : $\mathrm{NAmp}\,(D) = \bigcap_{m>0} \mathrm{SBs}\,(mD - A)$, for any given ample divisor A.

Ample locus : $\mathrm{Amp}\,(D) = X - \mathrm{NAmp}\,(D)$,

$x \in \mathrm{Amp}\,(D) \iff \exists m > 0$ s.t. $\Phi_{|mD|}$ gives an embedding around x.

(3) Non-nef locus :

$\mathrm{NNef}\,(D) = \bigcup_{m>0} \mathrm{SBs}\,(mD + A)$.

- at most countable union of subvars
- $\mathrm{NNef}\,(D) = \emptyset \iff D$: nef
- $D \cdot C < 0 \Longrightarrow C \subset \mathrm{NNef}\,(D)$

◇ $\mathrm{NNef}\,(D) \subset \mathrm{SBs}\,(D) \subset \mathrm{NAmp}\,(D)$.

◇ Since $\mathrm{SBs}\,(mD) = \mathrm{SBs}\,(D)$, ... for $m \in \mathbb{Z}_{>0}$,

one can define $\mathrm{SBs}\,(D)$, $\mathrm{NAmp}\,(D)$, $\mathrm{NNef}\,(D)$ for $\mathbb{Q}$-divisors.

Theorem 1. Assume K_X is big.

Then every irred comp of SBs(K_X), NAmp(K_X) or NNef(K_X) is uniruled.

Theorem 2. Assume $K_X = 0$ (numerically). Let L: a big divisor on X.

Then every irred comp of SBs(L), NAmp(L) or NNef(L) is uniruled.

Theorem 3. Assume $-K_X$ is big.

Then every irred comp of SBs$(-K_X)$ or NAmp$(-K_X)$,

which is not contained in NNef$(-K_X)$, is uniruled.

NNef$(-K_X)$ can be non-uniruled.

Related known results:

Classic : study of linear series on surfaces.

Mori's bend and break.

Kawamata : In case when the relevant divisor

$D : K_X,\ L\ (\& K_X = 0),\ -K_X$ is nef & big

($\Longrightarrow \Phi_{|mD|}$ birat mor, by Kawamata-Shokurov)

$\Longrightarrow$ NAmp$(D) = \mathrm{Exc}(\Phi_{|mD|})$ is covered by rational curves.

K_X : nef & big, and let Z a fiber of $\Phi_{|mK_X|}$ with $\dim Z > 0$.

$\Longrightarrow K_X|_Z = 0,\ N_{Z/X} < 0$ ($\because$ contractible)

$\Longrightarrow K_Z = K_X|_Z + N_{Z/X} < 0 \Longrightarrow Z$: uniruled, by MM.

Huybrechts, Boucksom :

Exceptional divisors in hyper-Kähler manifolds are uniruled.

Preview of §5 : Outline of proof.

Assume L : big, (i) $V \subset \mathrm{SBs}\,(L)$: an irred component, &

(ii) $V \subset \mathrm{SBs}\,(K_X + aL)$ for some or any $a > 0$.

(i) $\Longrightarrow$ (ii), if $L = K_X$, $K_X = 0$, or $L = -K_X$.

(i) $\Longrightarrow$ §4 : $0 < \exists\, a \in \mathbb{Q}$, $\exists\, D \succeq 0$ $\mathbb{Q}$-div s.t. $D \sim_{\mathbb{Q}} a\,L$, &

V is a maximal log-canonical center for the pair (X, D).

(regard $a = 1$)

$\Longrightarrow$ Kawamata's subadjunction : $(K_X + D)|_V \succeq K_V$

$\Longrightarrow$ §3 Extension theorem :

$$H^0(X, m(K_X + D)) \cdots \twoheadrightarrow H^0(V, mK_V)$$

Then, (ii) $\Longrightarrow$ $H^0(V, mK_V) = 0$ for $\forall\, m > 0$.

Refinement + §2 MM, BDPP $\Longrightarrow$ V : uniruled.

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§2. Uniruledness Criterion.

Miyaoka - Mori :

X : uniruled $\Longleftrightarrow$ $\exists$ covering family of curves $\{C_\lambda\}_\lambda$ s.t. $K_X \cdot C_\lambda < 0$.

Boucksom, Demailly, Paun - Peternell :

a divisor D : pseudo-effective

$\Longleftrightarrow$ $D \cdot C_\lambda \geq 0$ for $\forall$ covering family of curves $\{C_\lambda\}_\lambda$.

Theorem. (MM, BDPP)

X : uniruled $\Longleftrightarrow$ K_X is not pseudo-effetive.

$\Longrightarrow \kappa(X) = -\infty$ ($\Longleftarrow$ Open).

§3. Application of Extension Theorem.

Theorem. Let $V \subset X$: a subvar, $\dim V > 0$, L : a big divisor $/X$.

Assume $\exists$ a decomposition

$L \sim_{\mathbb{Q}} A + D$ s.t. (i) A : ample $\mathbb{Q}$-div,

(ii) D : effective $\mathbb{Q}$-div &

V is a maximal log-can center for (X, D).

In case $\operatorname{codim} V = 1$, one can rephrase (ii) as

(ii) $D = V + E$ with E : effective $\mathbb{Q}$-div, $\operatorname{Supp} E \not\supset V$.

Moreover assume $V \subset \operatorname{SBs}(K_X + L)$.

Then $K_{\widetilde{V}}$ is <u>not</u> pseudo-effective, and hence V is uniruled by MM, BDPP.

Here $\widetilde{V}$ is any smooth model of V.

% Hereafter "$=$" means "$\sim_{\mathbb{Q}}$", "$E \not\supset V$" means "$\operatorname{Supp} E \not\supset V$".

<u>Proof</u> (in case $\operatorname{codim} V = 1$): By taking an embedded resol of $D = V + E$

..., we may assume V is smooth, and D is SNC.

$K_X + L - \llcorner E \lrcorner = K_X + A + V + (E - \llcorner E \lrcorner)$, let $F = E - \llcorner E \lrcorner$.

the pair $(X, V + F)$ is exactly log-can. along V.

$(K_X + L - \llcorner E \lrcorner)|_V = K_V + A|_V + F|_V$, the pair $(V, F|_V)$ is klt.

Then the restriction map

$$(*) \quad H^0(X,\ m(K_X + L - \llcorner E \lrcorner)) \longrightarrow H^0(V,\ m(K_X + L - \llcorner E \lrcorner)|_V)$$

is surjective for all $m > 0$. $\leftrightarrow m(K_V + A|_V + F|_V)$.

If K_V is pseudo-effective $\Longrightarrow K_V + A|_V + F|_V$: big

(by $(*)$) $\Longrightarrow V \not\subset \operatorname{SBs}(K_X + L - \llcorner E \lrcorner)$

(since $V \not\subset E$) $\Longrightarrow V \not\subset \operatorname{SBs}(K_X + L)$

§4. Decompositions of big divisors.

Definition. (Asymptotic vanishing order, only for big divisors)

Let L : a pseudo-effective divisor $/X$, & $V \subset X$: a subvar.

$\mathrm{mult}_V|mL| :=$ the multiplicity of a general $D \in |mL|$ along V.

(1) L : big. $\sigma_V(L) := \exists \lim_{m\to\infty} \frac{1}{m}\mathrm{mult}_V|mL| \quad \in \mathbb{R}_{\geq 0}$.

$\sigma_V(qL) := q\,\sigma_V(L)$ for $0 < q \in \mathbb{Q}$.

(2) L : pseudo-effective. $\sigma_V(L) := \lim_{\varepsilon\to 0} \sigma_V(L+\varepsilon A)$ for an ample A.

Goodman, Wilson, Nakayama, Boucksom, ELMNP,...

The following are equivalent (by the effective global generation of MPI)

(i) $\sigma_V(L) = 0$,

(ii) $V \not\subset \mathrm{NNef}(L)$, i.e., $V \not\subset \mathrm{SBs}(L+\varepsilon A)$ for $\forall\varepsilon > 0$ and ample A,

(iii) $\exists\, C > 0$ s.t. $\mathrm{mult}_V|mL| < C$ for $\forall m > 0$.

Lemma. (a refinement of Kodaira's lemma)

Let L : a big divisor $/X$, & V an irred comp of $\mathrm{SBs}(L)$.

Then $\exists$ a decomposition $\alpha L = A + D$ s.t.

(o) $0 < \alpha \in \mathbb{Q}$, $\alpha \gg 1$ if $\sigma_V(L) = 0$,

$1/\sigma_V(L) - \varepsilon < \alpha \leq \mathrm{codim}\, V/\sigma_V(L)$ if $\sigma_V(L) \neq 0$,

(i) A : ample $\mathbb{Q}$-div,

(ii) D : effective $\mathbb{Q}$-div, V is a maximal log-can center for (X, D).

In case $\mathrm{codim}\, V = 1$, one can rephrase as

(o) $0 < \alpha \fallingdotseq \sigma_V(L)^{-1}$,

(ii) $D = V + E$ with E : effective $\mathbb{Q}$-div, $E \not\supset V$.

§5. Outline of Proof.

X : of general type, $V \subset \mathrm{SBs}\,(K_X)$: an irred component, $\operatorname{codim} V = 1$.

Case: $\sigma_V(K_X) = 0$

(1) By §4, $\alpha K_X = A + V + E$ for $1 \ll \exists\, \alpha \in \mathbb{Q}$, A : ample $\mathbb{Q}$-div,

E : effective $\mathbb{Q}$-div, $E \not\supset V$.

(2) $\sigma_V(K_X) = 0 \Longrightarrow V \not\subset \mathrm{SBs}\,|(\ulcorner\alpha\urcorner - \alpha)K_X + A/2|$,

$\exists\, F$: effective $\mathbb{Q}$-div s.t. $F \sim_{\mathbb{Q}} (\ulcorner\alpha\urcorner - \alpha)K_X + A/2$, and $F \not\supset V$.

(3) $L := \ulcorner\alpha\urcorner K_X = \alpha K_X + (\ulcorner\alpha\urcorner - \alpha)K_X$

$= A/2 + V + E + (\ulcorner\alpha\urcorner - \alpha)K_X + A/2 \;=\; A/2 + V + (E + F)$.

(4) $\mathrm{SBs}\,(K_X + L) = \mathrm{SBs}\,((1 + \ulcorner\alpha\urcorner)K_X) = \mathrm{SBs}\,(K_X) \supset V$.

(3), (4) $\Longrightarrow K_{\tilde{V}}$ is not pseudo-effective $\Longleftrightarrow$ V is uniruled.

(by §3) (by §2 MM, BDPP)

Case: $\sigma_V(K_X) > 0$

(1) By §4, $\alpha K_X = A + V + E$ with $0 < \alpha \fallingdotseq \sigma_V(K_X)^{-1}$.

(2) Take H : an ample div s.t. $\ulcorner\alpha\urcorner K_X + H$ is ample. Consider

$$t_0 := \sup\{\, 0 \le t \in \mathbb{Q}\,;\ \sigma_V(tK_X + H) = 0 \,\} \in \mathbb{R}_{\ge 0}.$$

($\ulcorner\alpha\urcorner K_X + H$ is ample $\Longrightarrow$) $\ulcorner\alpha\urcorner < t_0 < +\infty$ ($\Longleftarrow \sigma_V(K_X) > 0$)

$V \not\subset \mathrm{SBs}\,(tK_X + H)$ for $\forall t < t_0$, $V \subset \mathrm{SBs}\,(tK_X + H)$ for $\forall t > t_0$.

Let $m_0 = \llcorner t_0 \lrcorner$, $\Longrightarrow \exists\, F$: effective $\mathbb{Q}$-div s.t.

$0 \le m_0 - \alpha < t_0$ $F \sim_{\mathbb{Q}} (m_0 - \alpha)K_X + H$, and $F \not\supset V$.

(3) $L := m_0 K_X + H = \alpha K_X + (m_0 - \alpha)K_X + H$

$= A + V + (E + F)$.

(4) $K_X + L = (m_0 + 1)K_X + H$ & $m_0 + 1 > t_0 \Longrightarrow V \subset \mathrm{SBs}\,(K_X + L)$.

(3), (4) $\Longrightarrow K_{\tilde{V}}$ is not pseudo-effective $\Longleftrightarrow$ V is uniruled.

§6. Supplement.

Theorem 1'. If K_X is pseudo-effective,

every irred comp of $\mathrm{NNef}\,(K_X)$ is uniruled.

<u>Comment by M^{c}Kernan:</u> Assume K_X is big (or pseudo-effective).

Run MMP, then $\exists$ a sequence of flips

$$\begin{array}{ccccccccccc} X = X_1 & & \dashrightarrow & & X_2 & & \dashrightarrow & & X_3 & \dashrightarrow \cdots \dashrightarrow & X_m : \text{minimal} \\ & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow \quad \swarrow & \\ & & S_1 & & & & S_2 & & \cdots & S_{m-1} & \end{array}$$

$\mathrm{NNef}\,(K_X) \fallingdotseq \mathrm{Exc}(X \dashrightarrow X_m)$

$= \bigcup \{$ extremal rational curves (in each step) $\}$.

K_{X_m}: nef & big, $\mathrm{NAmp}\,(K_X) \fallingdotseq \mathrm{NAmp}\,(K_{X_m}) \leftarrow$ uniruled by Kawamata.

It should be $\mathrm{SBs}\,(K_X) = \mathrm{NNef}\,(K_X)$ (by abundance ?).

Theorem 2'. Assume $K_X = 0$ (numerically).

If L is pseudo-effective, every irred comp of $\mathrm{NNef}\,(L)$ is uniruled.

<u>Comment by M^{c}Kernan:</u>

Assume $K_X = 0$ (numerically), L : big (or pseudo-effective).

Run log-MMP for $K_X + \Delta$ with $\Delta \sim_{\mathbb{Q}} \varepsilon L$, ...

Theorem 3'. If $-K_X$ is pseudo-effective,

every irred comp V of NNef$(-K_X)$ with $0 < \sigma_V(-K_X) < 1$ is uniruled.

<u>Example.</u>

$-K_X = \mu^*\mathcal{O}(1) + E$ is big, not nef.

$\mathrm{NNef}(-K_X) = \mathrm{SBs}(-K_X) = \mathrm{NAmp}(-K_X) = E,$

$\sigma_E(-K_X) = 1, \ E \cong C.$