Rational curves on general projective hypersurfaces

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Abstract. In this article, we study the geometry of k-dimensional subvarieties with geometric genus zero of a general projective hypersurface $X_d \subset \mathbf{P}^n$ of degree $d = 2n - 2 - k$, where k is an integer such that $1 \leq k \leq n - 5$. As a corollary of our main result we obtain that the only rational curves lying on the general hypersuface $X_{2n-3} \subset \mathbf{P}^n$, for $n \geq 6$, are the lines.

1 Introduction

It was shown by H. Clemens [C] that the general (in the countable Zariski topology) hypersurface of degree d in $\mathbf{P}^n$ does not contain any rational curve, if d is sufficiently large. More precisely, he proved the following:

Theorem (Clemens). *There is no rational curve on the general hypersurface $X_d \subset \mathbf{P}^n$ of degree $d \geq 2n - 1$, $n \geq 3$.*

L. Ein studied more generally (see [E1] and [E2]) the geometric genus of subvarieties contained in complete intersections $X_{(d_1,\ldots,d_r)} \subset M$, where M is an arbitrary smooth and projective ambient variety. He proved that if $d_1 + \ldots + d_r \geq 2n - r - k + 1$, any k-dimensional subvariety $Y \subset X$ has desingularization with effective canonical bundle. However, in the case of a hypersurface $X_d \subset \mathbf{P}^n$, he obtained the lower bound $d \geq 2n - k$ on the degree of X, which was the same as Clemens' for $k = 1$, and was not optimal. Indeed, it was classically known that the lines lying on the general hypersurface $X_d \subset \mathbf{P}^n$ of degree $d = 2n - 2 - k$ cover a k-dimensional subvariety, which then has geometric genus zero. Thus nothing was known yet about the canonical bundle of subvarieties of dimension k on the general $X_{2n-1-k} \subset \mathbf{P}^n$. Voisin ([V2], [V3]) showed then that it was possible to sharpen Ein's bound by one, as conjectured by Clemens himself, by proving:

Theorem (Voisin). *Let $X_d \subset \mathbf{P}^n$ be a general hypersurface of degree $d \geq 2n-1-k$, where k is an integer such that $1 \leq k \leq n-3$. Then any k-dimensional subvariety Y of X has desingularization $\tilde{Y}$ with effective canonical bundle.*

For $k=1$ we immediately obtain that the general $X_{2n-2} \subset \mathbf{P}^n,\ n \geq 4$, contains no rational curves. Taking $k=2$ we get another very interesting corollary whose analogue in the case $n=4$ would solve Clemens' conjecture on the finiteness of rational curves of fixed degree on the general quintic threefold in $\mathbf{P}^4$:

Corollary (Voisin). *For each integer $\delta \geq 1$, the general hypersurface $X_{2n-3} \subset \mathbf{P}^n,\ n \geq 5$, contains at most a finite number of rational curves of degree δ.*

The goal of our work is to investigate, for the general $X_{2n-2-k} \subset \mathbf{P}^n$, $1 \leq k \leq n-4$, the geometry of its k-dimensional subvarieties having geometric genus zero. Since the locus covered by the lines of X is the only known example of such a subvariety, it seems natural to start with the following:

Question. *Is the variety covered by the lines the only subvariety of dimension k with geometric genus zero on the general hypersurface $X_{2n-2-k} \subset \mathbf{P}^n,\ 1 \leq k \leq n-4$?*

Remark that the numerical hypothesis $1 \leq k \leq n-4$ implies the positivity of the canonical bundle of $X_{2n-2-k} \subset \mathbf{P}^n$, and gives meaning to the question in contrast to the case of the Calabi-Yau hypersurface $X_{n+1} \subset \mathbf{P}^n$. The main result of this paper gives actually a positive answer to the previous question for $1 \leq k \leq n-5$:

Theorem. *Let $X_d \subset \mathbf{P}^n$ be a general hypersurface of degree $d = 2n-2-k$, where k is an integer such that $1 \leq k \leq n-5$. Then any subvariety $Y \subset X$ of dimension k, whose desingularization $\tilde{Y}$ has $h^0(\tilde{Y}, K_{\tilde{Y}}) = 0$, is a component of the (k-dimensional) subvariety covered by the lines lying on X.*

Taking $k = 1$ we get, for $n \geq 6$, a corollary on rational curves on the general projective hypersurface of degree $d = 2n - 3$. Voisin's corollary already implies that, for each fixed integer $\delta \geq 1$, there are only a finite number of such curves of degree δ. Here we prove that there are only lines - whose number is easily computed as the top Chern class of a certain vector bundle on the Grassmannian of lines in $\mathbf{P}^n$:

Corollary. *There is no rational curve of degree $\delta \geq 2$ on the general hypersurface $X_{2n-3} \subset \mathbf{P}^n$, $n \geq 6$.*

Throughout this paper we work on the field of complex number $\mathbb{C}$.

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2 Preliminaries and sketch of the proof

To motivate our approach and render our proof more transparent, we will briefly sketch the key ideas contained in [E1], [V2] and [V3]. We start with some

Notation.
$S^d := H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$;
$S^d_x := H^0(\mathbf{P}^n, \mathcal{I}_x \otimes \mathcal{O}_{\mathbf{P}^n}(d))$;
$N := h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) = dim\ S^d$;
$\mathcal{X} \subset \mathbf{P}^n \times S^d$ will denote the universal hypersurface of degree d;
$X_F \subset \mathbf{P}^n$ the fiber of the family $\mathcal{X}$ over $F \in S^d$, i.e. the hypersurface defined by F.

Let $U \to S^d$ be an étale map and $\mathcal{Y} \subset \mathcal{X}_U$ a universal, reduced and irreducible subscheme of relative dimension k (in the following, by abuse of notation, we will often omit the étale base change). We may obviously assume $\mathcal{Y}$ invariant under some lift of the natural action of $GL(n+1)$ on

$\mathbf{P}^n \times S^d$: $g(x,F) = (g(x),(g^{-1})^* F))$, $g \in GL(n+1)$. Let $\tilde{\mathcal{Y}} \to \mathcal{Y}$ be a desingularization and $\tilde{\mathcal{Y}} \xrightarrow{j} \mathcal{X}_U$ the natural induced map. Let $\pi : \mathcal{X} \to \mathbf{P}^n$ be the projection on the first component and $T_{\mathcal{X}}^{vert}$ (resp. $T_{\mathcal{Y}}^{vert}$) the vertical part of $T_{\mathcal{X}}$ (resp. $T_{\mathcal{Y}}$) w.r.t. π, i.e. $T_{\mathcal{X}}^{vert}$ (resp. $T_{\mathcal{Y}}^{vert}$) is the sheaf defined by

$$0 \to T_{\mathcal{X}}^{vert} \to T\mathcal{X} \xrightarrow{\pi_*} T\mathbf{P}^n \to 0$$
$$(resp. \quad 0 \to T_{\mathcal{Y}}^{vert} \to T\mathcal{Y} \xrightarrow{\pi_*} T\mathbf{P}^n).$$

The hypothesis of $GL(n+1)$-invariance for $\mathcal{Y}$ has two easy but very important consequences that will be frequently used in the rest of the paper:

Lemma 2.1. *Let $\mathcal{Y}$ be a subvariety of $\mathcal{X} \times \mathbf{P}^n$ of relative dimension k and invariant under the action of $GL(n+1)$. Then:*

(i) codim $_{T^{vert}_{\mathcal{X},(y,F)}} T^{vert}_{\mathcal{Y},(y,F)} = codim_{\mathcal{X}}\mathcal{Y} = n-k-1$; in particular, since we are assuming $1 \le k \le n-5$, we have that

$$codim\ _{\mathcal{X}}\ \mathcal{Y}\ \geq 4.$$

(ii) $T^{vert}_{\mathcal{Y},(y,F)} \supset < S^1_y \cdot J^{d-1}_F, F >$, where J^{d-1}_F is the Jacobian ideal of F.

Proof. (i) Use the surjectivity of the map $T\mathcal{Y} \xrightarrow{\pi_*} T\mathbf{P}^n$.
(ii) It follows from the fact that, by $GL(n+1)$-invariance, $T^{vert}_{\mathcal{Y},(y,F)}$ contains the vertical part of the tangent space to the orbit of (y,F) under the action of $GL(n+1)$. □

Assume $d \geq 2n-k$ and let $Y_F \subset X_F$ be a general fiber of the subfamily $\mathcal{Y} \subset \mathcal{X}_U$. Then to prove Clemens' result (which corresponds to the case $k=1$) we have to show that $h^0(\tilde{Y}_F, K_{\tilde{Y}_F}) \neq 0$, where $\tilde{Y}_F \to Y_F$ is the desingularization.

Ein's idea is to produce, by adjunction, a non-zero section in $H^0(\tilde{Y}_F, K_{\tilde{Y}_F})$ by restricting to $\tilde{\mathcal{Y}}$ holomorphic forms on $\mathcal{X}_U$ defined along X_F. The main technical difficulty consists in controlling the positivity of the twisted "vertical" tangent bundle to the universal hypersurfaces. We start then recording, in the first subsection, the needed positivity result, and an equivalent one for a bundle on the Grassmannian of lines in $\mathbf{P}^n$ that will be used later.

2.1 Positivity results

Let d be a positive integer. Consider the bundle $M^d_{\mathbf{P}^n}$ defined by the exact sequence

$$0 \to M^d_{\mathbf{P}^n} \to S^d \otimes \mathcal{O}_{\mathbf{P}^n} \xrightarrow{ev} \mathcal{O}_{\mathbf{P}^n}(d) \to 0, \tag{1}$$

whose fiber at a point x identifies by definition with S^d_x. From the inclusion $\mathcal{X} \hookrightarrow \mathbf{P}^n \times S^d$ we get the exact sequence

$$0 \to T\mathcal{X}_{|X_F} \to T\mathbf{P}^n{}_{|X_F} \oplus (S^d \otimes \mathcal{O}_{X_F}) \to \mathcal{O}_{X_F}(d) \to 0,$$

which combined with (1) gives us

$$0 \to M^d_{\mathbf{P}^n|X_F} \to T\mathcal{X}_{|X_F} \to T\mathbf{P}^n{}_{|X_F} \to 0.$$

In other words $M^d_{\mathbf{P}^n|X_F}$ identifies to the vertical part of $T_\mathcal{X} \otimes \mathcal{O}_{X_F}$ with respect to the projection to $\mathbf{P}^n$.

Let $G := Grass(1, n)$ be the Grassmannian of lines in $\mathbf{P}^n$, $\mathcal{O}_G(1)$ the line bundle on G giving its Plücker polarization, and $\mathcal{E}_d$ be the d^{th}-symmetric power of the dual of the tautological subbundle on G. Recall that the fibre of $\mathcal{E}_d$ at a point $[\ell]$ is, by definition, given by $H^0(\ell, \mathcal{O}_\ell(d))$.

Let M^d_G be the vector bundle on G defined as the kernel of the evaluation map:

$$0 \to M^d_G \to S^d \otimes \mathcal{O}_G \to \mathcal{E}_d \to 0.$$

Notice that the fiber of M^d_G at a point $[\ell]$ is equal to $I_\ell(d) := H^0(\mathcal{I}_\ell(d))$.

Then we have the following

Proposition 2.2. [1] *(i) $M^d_{\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}(1)$ is generated be its global sections; (ii) $M^d_G \otimes \mathcal{O}_G(1)$ is generated be its global sections.*

Proof. (i) We start observing that the sheaf $\Omega^s_{\mathbf{P}^n}(s+1)$ is globally generated. Indeed, by Bott's vanishing theorem [B] we have

$$H^i(\Omega^p_{\mathbf{P}^n}(q)) = 0, \ \forall i > 0 \ q \geq p + 1 - i. \tag{2}$$

Therefore, by Mumford's m-regularity theorem ([M1], page 99) the maps

$$H^0(\Omega^p_{\mathbf{P}^n}(q)) \otimes H^0(\mathcal{O}_{\mathbf{P}^n}(1)) \to H^0(\Omega^p_{\mathbf{P}^n}(q+1)),$$

are surjective for $q \geq p + 1$. The result now follows immediately recalling the isomorphism

$$M^1_{\mathbf{P}^n} \cong \Omega^1_{\mathbf{P}^n}(1)$$

and the surjection

$$S^{d-1} \otimes M^1_{\mathbf{P}^n} \twoheadrightarrow M^d_{\mathbf{P}^n}.$$

[1] The quick proof of (i) we reproduce here is due to the referee, who pointed out to us the possibility of using Bott's and Mumford's results. The referee also suggested us the proof of (ii), which is much simpler than the one we originally proposed.

(ii) Again it suffices to check the case $d = 1$. Using the irreducibility of the action of $GL\,(n+1)$ on the Grassmannian, it suffices to construct a single non-trivial meromorphic section of M^1_G with simple pole along the zero set of a Plücker coordinate. To do this, for all lines ℓ not meeting

$$X_1 = X_2 = 0$$

the Plücker coordinate $p_{12}(\ell) \neq 0$ so there is, by Cramer's rule, a unique $L = X_0 + aX_1 + bX_2 \in M^1_G$ containing ℓ.

□

2.2 Proof of Ein's and Voisin's results

Following Ein [E1], [E2], one can use the positivity result (2.2), (i), to produce holomorphic forms on the (vertical) tangent space to the family $\mathcal{X}_U$. Then, by pulling back them to $\tilde{Y}_F$ and using the adjunction formula, it will be possible to provide a non zero section of $H^0(\tilde{Y}_F, K_{\tilde{Y}_F})$. To make this more precise, first recall the following elementary facts:

(i) $\Omega^{N+k}_{\tilde{\mathcal{Y}}\ |\tilde{Y}_F} \cong K_{\tilde{Y}_F}$;

(ii) $(\wedge^{n-1-k} T\mathcal{X}_{U|X_F}) \otimes K_{X_F} \cong \Omega^{N+k}_{\mathcal{X}_U\ |X_F}$.

Therefore, from the natural morphism $\Omega^1_{\mathcal{X}_U} \to \Omega^1_{\tilde{\mathcal{Y}}}$, we get a map

$$(\wedge^{n-1-k} T\mathcal{X}_{U|X_F}) \otimes K_{X_F} \cong \Omega^{N+k}_{\mathcal{X}_U\ |X_F} \to \Omega^{N+k}_{\tilde{\mathcal{Y}}\ |\tilde{Y}_F} \cong K_{\tilde{Y}_F}. \tag{3}$$

Since $K_{X_F} = \mathcal{O}_{X_F}(d-n-1) = \mathcal{O}_{X_F}((n-k-1)+(d-2n+k))$ and

$$\wedge^{n-1-k} T\mathcal{X}_{U|X_F}(n-k-1) = \wedge^{n-1-k}(T\mathcal{X}_{U|X_F}(1)),$$

we have

$$(\wedge^{n-1-k} T\mathcal{X}_{U|X_F}) \otimes K_{X_F} = \wedge^{n-1-k}(T\mathcal{X}_{U|X_F}(1)) \otimes \mathcal{O}_{X_F}(d-2n+k). \tag{4}$$

Now, since we are supposing $d \geq 2n-k$, Proposition (2.2) (i) implies that the vertical part of

$$\wedge^{n-1-k}(T\mathcal{X}_{U|X_F}(1)) \otimes \mathcal{O}_{X_F}(d-2n+k) \cong \Omega^{N+k}_{\mathcal{X}_U\ |X_F},$$

namely, the subsheaf

$$\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F}) \otimes K_{X_F} = \wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F}(1)) \otimes \mathcal{O}_{X_F}(d-2n+k),$$

is globally generated. Composing the inclusion

$$\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F})\otimes K_{X_F}\hookrightarrow \wedge^{n-1-k}(T\mathcal{X}_{U|X_F})\otimes K_{X_F}$$

with the restriction map defined in (3), we have a natural morphism

$$\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F})\otimes K_{X_F}\to K_{\tilde{Y}_F}. \tag{5}$$

Ein's result is then given by the following

Lemma 2.3. *Let F be a general polynomial of degree $2n-k$. The map*

$$H^0(\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F})\otimes K_{X_F})\to H^0(K_{\tilde{Y}_F}),$$

induced in cohomology by (5), is non zero.

Proof. By Lemma (2.1), (i), we have

$$codim\ _{T^{vert}_{\mathcal{X},(y,F)}}\ T^{vert}_{\mathcal{Y},(y,F)}=codim_{\mathcal{X}}\mathcal{Y}.$$

Let (y,F) be a smooth point of $\mathcal{Y}$. Since the bundle $\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F})\otimes K_{X_F}$ is generated by its global sections, there exists a section

$$s\in H^0(\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F})\otimes K_{X_F})$$

such that

$$< s(y), T^{vert}_{\mathcal{Y},(y,F)} >\neq 0. \tag{6}$$

Since $j:\tilde{\mathcal{Y}}\to\mathcal{X}_U$ is generically an immersion, we obtain from the above a non zero element in $H^0(K_{\tilde{Y}_F})$ coming from $H^0(\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F})\otimes K_{X_F})$. □

(For other proofs of Clemens' theorem see, of course, [C] and also [CLR]).

In order to try and improve the bound on the degree by one, we observe that, if $d=2n-1-k$, then $K_{X_F}=\mathcal{O}_{X_F}(n-2-k)$, so we have, as in (3), a map

$$\wedge^{n-1-k}T\mathcal{X}_{U|X_F}(n-k-2)\cong\Omega^{N+k}_{\mathcal{X}_U\ |X_F}\to\Omega^{N+k}_{\tilde{\mathcal{Y}}\ |\tilde{Y}_F}\cong K_{\tilde{Y}_F}. \tag{7}$$

As we saw in Lemma 2.1, (i), by the hypothesis of $GL(n+1)$-invariance on $\mathcal{Y}$, the relevant part of the tangent space to look at is the vertical one, hence we focus our attention on the map

$$\wedge^{n-1-k}M^d_{\mathbf{P}^n|X_F}(n-k-2)\to\Omega^{N+k}_{\tilde{\mathcal{Y}}\ |\tilde{Y}_F}\cong K_{\tilde{Y}_F}. \tag{8}$$

Now, because of the shift between the exterior power and the degree of the canonical bundle we are tensoring with on the lefthand side of (8), the global generation of the sheaf $\wedge^{n-1-k} M^d_{\mathbf{P}^n|X_F}(n-k-2)$ will not follow from the global generation of $M^d_{\mathbf{P}^n}(1)$. Voisin's idea is then to study the positivity of $H^0(\wedge^2 M^d_{\mathbf{P}^n}(1))$, to produce holomorphic forms on the (vertical) tangent space to the universal hypersurface, and use the commutative diagram below to produce sections in $H^0(K_{\tilde{Y}_F})$:

$$\begin{array}{ccc} & H^0(\wedge^{n-1-k} M^d_{\mathbf{P}^n|X_F}(n-2-k)) \longrightarrow & H^0(K_{\tilde{Y}_F}) \\ & \uparrow & \nearrow \\ H^0(\wedge^{n-3-k} M^d_{\mathbf{P}^n|X_F}(n-3-k)) \otimes H^0(\wedge^2 M^d_{\mathbf{P}^n|X_F}(1)) & & \end{array} \tag{9}$$

(the vertical map in (9) is simply obtained by wedging the sections of the sheaves $\wedge^{n-3-k} M^d_{\mathbf{P}^n|X_F}(n-3-k)$ and $\wedge^2 M^d_{\mathbf{P}^n|X_F}(1)$). Unfortunately the following fact holds:

Fact (Amerik-Voisin). $\wedge^2 M^d_{\mathbf{P}^n}(1)$ *is not generated by its global sections.*

Indeed, in [V3] the following counterexample to the global generation of $\wedge^2 M^d_{\mathbf{P}^n}(1)$ is given. Consider the subvariety

$$\Delta_{d,F} := \{x \in X_F : \textit{there exists a line } \ell \textit{ s.t. } \ell \cap X_F = d \cdot x\}. \tag{10}$$

An elementary dimension count shows that, for generic F,

$$dim\ \Delta_{d,F} = 2n - 2 - (d-1) = 2n - 2 - (2n - 1 - k - 1) = k$$

(these subvarieties are generically empty for $d \geq 2n-1$, which is the reason they don't come into play in Clemens' and Ein's case). Let Δ_d be the family of the $\Delta_{d,F}$'s, let $\tilde{\Delta}_d \to \Delta_d$ be a desingularization, and $j : \tilde{\Delta}_d \to \mathcal{X}$ the natural morphism. Notice that $\Delta_{d,F}$ parametrizes 0-cycles of X_F which are all rationally equivalent since, by definition, $d \cdot x \equiv H^{n-1}.X_F$, $\forall x \in \Delta_{d,F}$, where H is the hyperplane divisor in $\mathbf{P}^n$. Thus, the variational (and higher dimensional) version of Mumford's fundamental result on 0-cycles on surfaces applies (see [M2], and [V1] for the variational generalization in dimension 2), so we have

$$j^* s = 0 \ \ in \ \ H^0(\Omega^{N+k}_{\tilde{\mathcal{Y}}}{}_{|\tilde{Y}_F}), \ \forall s \in H^0(\Omega^{N+k}_{\mathcal{X}}{}_{|X_F}),$$

i.e. the map

$$H^0(\wedge^{n-1-k} T\mathcal{X}_{|X_F}(n-2-k)) \cong H^0(\Omega^{N+k}_{\mathcal{X}}{}_{|X_F}) \to H^0(\Omega^{N+k}_{\tilde{\mathcal{Y}}}{}_{|\tilde{Y}_F}) \cong H^0(K_{\tilde{Y}_F})$$

is identically zero and then so is

$$H^0(\wedge^{n-1-k} M^d_{\mathbf{P}^n|X_F}(n-2-k)) \to H^0(K_{\tilde{Y}_F}).$$

In particular, by (9) and Proposition 2.2, (i), we have that, at a smooth point $(y, F) \in \mathcal{Y}$, all the global sections of the bundle $\wedge^2 M^d_{\mathbf{P}^n}(1)_{|X_F}$, seen as a line bundle on the Grassmannian of codimension two subspaces of $T\mathcal{X}^{vert}_{|X_F}$, vanish on the codimension two subspaces of $T^{vert}_{\mathcal{X},(y,F)}$ containing $T^{vert}_{\mathcal{Y},(y,F)}$.

Voisin's alternative approach to the problem, as developped in [V3], consists then in studying the base locus of $H^0(\wedge^2 M^d_{\mathbf{P}^n}(1)_{|X_F})$, to investigate the geometry of the subvarieties for which the composite map in (9) fails to provide non-zero sections of their canonical bundle. She shows in [V3] that, in the case $d = 2n-1-k$, the subvariety $\Delta_{d,F}$ defined in (10) is the only one for which this phenomenon occurs. Then, she completes her proof by verifying that each component of $\Delta_{d,F}$ has positive geometric genus.

2.3 The strategy of our proof

Our purpose is to study, for $d = 2n-2-k$, the geometry of k-dimensional subvarieties of $X_F \subset \mathbf{P}^n$, having geometric genus equal to zero. Recall that, since $d = 2n-2-k$, we have $K_{X_F} = \mathcal{O}_{X_F}(n-3-k)$, and note that the composite map

$$\begin{array}{ccc} H^0(\wedge^{n-1-k} M^d_{\mathbf{P}^n|X_F}(n-3-k)) & \longrightarrow & H^0(K_{\tilde{Y}_F}) \\ \uparrow & \nearrow & \\ H^0(\wedge^{n-5-k} M^d_{\mathbf{P}^n|X_F}(n-5-k)) \otimes H^0(\wedge^4 M^d_{\mathbf{P}^n|X_F}(2)) & & \end{array} \tag{11}$$

is obviously zero, since we are supposing $h^0(K_{\tilde{Y}_F}) = 0$. Then the proof of our theorem will naturally be divided into two steps. In section 3, analysing the base locus of $H^0(\wedge^4 M^d_{\mathbf{P}^n|X_F}(2))$, considered as the space of sections of a line bundle on the Grassmannian of codimension four subspaces of $T\mathcal{X}^{vert}_{|X_F}$, we will prove

Proposition A. *Let $X_F \subset \mathbf{P}^n$ be a general hypersurface of degree $d = 2n-2-k$, and $Y_F \subset X_F$ a subvariety of dimension k such that $H^0(\tilde{Y}_F, K_{\tilde{Y}_F}) = 0$, where $\tilde{Y}_F$ is a desingularization of Y_F. Then Y_F has to be contained in*

$$\Delta_{d,F} = \{x \in X_F : \text{there exists a line } \ell \text{ s.t. } \ell \cap X_F = d \cdot x\},$$

a subvariety of X_F of dimension $2n-2-(d-1) = k+1$.

In § 4, we will study an explicit desingularization $\tilde{\Delta}_{d,F}$ of $\Delta_{d,F}$, given by the zeroes of a section of a bundle on the incidence variety in $\mathbf{P}^n \times Grass(1,n)$. Denote by $\tilde{\Delta}_d$ the family $(\Delta_{d,F})_{F\in S^d}$, and recall the isomorphism

$$T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}} \cong \Omega^{N+k}_{\tilde{\Delta}_d \ |\tilde{S}_F}.$$

The positivity result (2.2), (ii), for the bundle $M^d_G \otimes \mathcal{O}_G(1)$ on $Grass(1,n)$, will allow us to construct a subbundle of $T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}$ generated by its global sections. Using this fact, together with the vanishing of the natural restriction map

$$H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}) = H^0(\Omega^{N+k}_{\tilde{\Delta}_d \ |\tilde{S}_F}) \to H^0(K_{\tilde{Y}_F}), \tag{12}$$

we will prove

Proposition B. *Let F be a general polynomial of degree $d = 2n-2-k$. Let $Y_F \subset \Delta_{d,F}$ be a subvariety of codimension 1 such that $H^0(\tilde{Y}_F, K_{\tilde{Y}_F}) = 0$, where $\tilde{Y}_F$ is a desingularization of Y_F. Then Y_F has to be a component of the k-dimensional subvariety of $\Delta_{d,F}$ covered by the lines lying on X_F.*

These propositions will combine to prove our main theorem.

3 Base locus of $\wedge^4 M^d_{\mathbf{P}^n}(2)$ and osculating lines

Let $\mathcal{X} \subset \mathbf{P}^n \times S^d$ be the universal hypersurface of degree $d = 2n-2-k$, $U \to S^d$ an étale map and $\mathcal{Y} \subset \mathcal{X}_U$ a universal, reduced and irreducible subscheme of relative dimension k (to simplify the notation, in what follows we will occasionally omit the étale base change). Assume $\mathcal{Y}$ invariant under some lift of the action of $GL(n+1)$, denote by $\tilde{\mathcal{Y}} \to \mathcal{Y}$ a desingularization, and suppose that the fibres of $\tilde{\mathcal{Y}}$ verify $h^0(\tilde{Y}_F, K_{\tilde{Y}_F}) = 0$.

Consider the bundle $M^d_{\mathbf{P}^n}$ defined by the exact sequence

$$0 \to M^d_{\mathbf{P}^n} \to S^d \otimes \mathcal{O}_{\mathbf{P}^n} \xrightarrow{ev} \mathcal{O}_{\mathbf{P}^n}(d) \to 0,$$

whose fiber at a point x identifies by definition with S^d_x. Recall from §2.1 that

$$M^d_{\mathbf{P}^n|X_F} = T^{vert}_{\mathcal{X}} \otimes \mathcal{O}_{X_F}, \tag{13}$$

where $T^{vert}_{\mathcal{X}}$ is the sheaf defined by

$$0 \to T^{vert}_{\mathcal{X}} \to T\mathcal{X} \xrightarrow{\pi_*} T\mathbf{P}^n \to 0.$$

From the vanishing of the composite map

$$\begin{array}{ccc} H^0(\wedge^{n-1-k}T\mathcal{X}_{U|X_F}(n-3-k)) & \longrightarrow & H^0(K_{\tilde{Y}_F}) \\ \uparrow & \nearrow & \\ H^0(\wedge^{n-5-k}T\mathcal{X}_{U|X_F}(n-5-k))\otimes H^0(\wedge^4 T\mathcal{X}_{U|X_F}(2)) & & \end{array} \quad (14)$$

and (13) we deduce that the composite map

$$\begin{array}{ccc} H^0(\wedge^{n-1-k}M^d_{\mathbf{P}^n|X_F}(n-3-k)) & \longrightarrow & H^0(K_{\tilde{Y}_F}) \\ \uparrow & \nearrow & \\ H^0(\wedge^{n-5-k}M^d_{\mathbf{P}^n|X_F}(n-5-k))\otimes H^0(\wedge^4 M^d_{\mathbf{P}^n|X_F}(2)) & & \end{array} \quad (15)$$

is also zero. Since, by Lemma 2.2, (i), $\wedge^{n-5-k}M^d_{\mathbf{P}^n|X_F}(n-5-k)$ is generated by its global sections, the vanishing of the composite map in (15) and the $GL(n+1)$ invariance of $\mathcal{Y}$ implies that, at a smooth point $(y,F)\in\mathcal{Y}$, any codimension four subspace of $T^{vert}_{\mathcal{X}_U,(y,F)} = S^d_y$ containing $T^{vert}_{\mathcal{Y},(y,F)}$ is in the base locus of $H^0(\wedge^4 M^d_{\mathbf{P}^n|X_F}(2))$, considered as a space of sections of a line bundle over the Grassmannian of codimension four subspaces of $T^{vert}\mathcal{X}_{|X_F}$. Studying this base locus we will see how, at each point y of a subvariety $Y_F\subset X_F$ with zero geometric genus, the ideal of a line through y naturally comes into play. More precisely, we will prove

Proposition 3.1. *Let $\mathcal{Y}\subset\mathcal{X}_U$ be such that the composite map in (15) is zero. Then, at a smooth point (y,F), the vertical tangent space $T^{vert}_{\mathcal{Y},(y,F)}$, which is a subspace of $T^{vert}_{\mathcal{X}_U,(y,F)} = S^d_y$, has to contain (at least) a hyperplane $H_{\ell_{(y,F)}}\subset I_{\ell_{(y,F)}}(d)$, where $\ell_{(y,F)}$ is a line passing through y.*

We will then study the distribution $\mathcal{H}\subset T^{vert}_{\mathcal{Y}}$, pointwise defined by $H_{\ell_{(y,F)}}$, and prove its integrability. The description of the corresponding foliation and the $GL(n+1)$-invariance of $\mathcal{Y}$ will allow us to conclude that the line $\ell_{(y,F)}$ is such that $\ell_{(y,F)}\cap X_F = d\cdot y$, thus proving Proposition A.

3.1 Proof of Proposition 3.1

We start with the following

Lemma 3.2. *Let T be a codimension four subspace of $S^d_x = (T_{\mathcal{X},(x,F)})^{vert}$ which is in the base locus of $H^0(\bigwedge^4 M^d_{\mathbf{P}^n}(2))$. Then T has to contain (at least) a hyperplane of $I_\ell(d)$, where ℓ is a line passing through x.*

Proof. Recall that $H^0(\bigwedge^2 M^d_{\mathbf{P}^n}(1))$ can be naturally interpreted as the kernel of the Koszul map $\bigwedge^2 S^d \otimes S^1 \to S^d \otimes S^{d+1}$. Hence one easily verifies that $Im\ (H^0(\bigwedge^2 M^d_{\mathbf{P}^n}(1)) \to \bigwedge^2 M^d_{\mathbf{P}^n,x})$ contains $PA_1 \wedge PA_2$, for all $P \in S^{d-1}$ and $A_i \in S^1_x$. Then

$$Im\ (H^0(\bigwedge^4 M^d_{\mathbf{P}^n}(2))) \to \bigwedge^4 M^d_{\mathbf{P}^n,x})$$

contains elements of the form

$$PA_1 \wedge PA_2 \wedge QB_1 \wedge QB_2,$$

for all $P, Q \in S^{d-1}$ and $A_i, B_i \in S^1_x$, coming from the wedge product of elements in $Im\ (H^0(\bigwedge^2 M^d_{\mathbf{P}^n}(1)) \to \bigwedge^2 M^d_{\mathbf{P}^n,x})$. Since we are supposing that T is in the base locus of $H^0(\bigwedge^4 M^d_{\mathbf{P}^n}(2))$, the previous fact implies in particular that the dimension of the subspace $\{AP : A \in S^1_x\}$ *modulo* T is at most 3, i.e. the multiplication map

$$\begin{aligned} m_P :\ S^1_x &\to S^d_x/T \\ A &\mapsto A \cdot P\ mod\ T \end{aligned}$$

cannot be surjective, for any $P \in S^{d-1}$.
Recall that if V and W are vector spaces, and $Z_k := \{\phi \in Hom(V, W) : rank\ \phi \leq k\}$, then

$$T_{Z_k,\phi} = \{\psi \in Hom(V, W) : \psi(ker\phi) \subset Im\phi\}. \tag{16}$$

If, for generic P, the map m_P has rank one, from (16) we obtain that $Q \cdot Ker\ m_P\ mod\ T \subset Im\ m_P$, for any $Q \in S^{d-1}$, i.e. $I_{\ell_P}(d) \subset P \cdot S^1_x + T$, where ℓ_P is the line determined by $Ker\ m_P$. Then T contains a hyperplane of $I_{\ell_P}(d)$ and the Lemma is proved.

Thus, we can assume that, for generic P. the map m_P has rank at least two. Let $A_1, A_2 \in S^1_x$ such that $T' :=< A_1P, A_2P, T >$ is of codimension 2 in S^d_x. For generic $Q \in S^{d-1}_x$, consider the map

$$m_Q : S^1_x \to S^d_x/T',$$

whose rank is then equal to 0 or 1. In the former case T' would then contain $S^{d-1} \cdot Ker\ m_Q = S^{d-1} \cdot S^1_x = S^d_x$, which is absurd since T' has codimension 2. Hence we can suppose $rk\ m_Q = 1$. Then by [V2], Lemma 2.3, T' contains the degree d part of the ideal of a line ℓ_Q passing through x, and hence T

contains at least a codimension two subspace of $I_{\ell_Q}(d)$. Assume first that the line does not vary with Q, and denote it by ℓ. If

$$codim_{I_\ell(d)}\ T\cap I_\ell(d)=2,$$

then the image $\overline{T}$ of T in $H^0(\mathcal{O}_\ell(d))$ has codimension 2 in $H^0(\mathcal{O}_\ell(d)(-x))$. On the other hand, since T' contains $I_\ell(d)$, its reduction $\overline{T'}$ modulo $I_\ell(d)$ has also codimension 2 in $H^0(\mathcal{O}_\ell(d)(-x))$. Hence $\overline{T}=\overline{T'}$. By the genericity of the choice P in S^d_x, this fact would imply that

$$\overline{T}=H^0(\mathcal{O}_\ell(d)(-x)),$$

thus leading to a contradiction.

Assume now that $\ell_Q\neq\ell_{Q'}$, for generic $Q,Q'\in S^{d-1}$. Since T contains a codimension two subspace of $I_{\ell_Q}(d)$, from the exact sequence

$$0\to I_{\ell_Q}(d)\cap I_{\ell_{Q'}}(d)\to I_{\ell_Q}(d)\oplus I_{\ell_{Q'}}(d)\to S^d_x\to 0,$$

and the fact that $T\subset S^d_x$ has codimension 4, it follows that $T\supset I_{\ell_Q}(d)\cap I_{\ell_{Q'}}(d)$. Let $\mathbf{P}^2_{Q,Q'}$ be the span of ℓ_Q and $\ell_{Q'}$: we study the variation of this plane with Q'. If for generic $Q'_1\neq Q'$ the intersection $\mathbf{P}^2_{Q,Q'}\cap\mathbf{P}^2_{Q,Q'_1}$ is equal to the line ℓ_Q, then $T\supset I_{\ell_Q}(d)$ and we are done. If otherwise $\mathbf{P}^2_{Q,Q'}=\mathbf{P}^2_{Q,Q'_1}=\mathbf{P}^2$, then it is immediate to see that T contains

$$\{F\in S^d_x : F_{|\mathbf{P}^2}\ \textit{is singular at the point}\ x\},$$

because ℓ_Q and $\ell_{Q'}$ will vary in this plane. But this is absurd since $T\subset S^d_x$ is of codimension 4. □

From the previous lemma and the vanishing of the composite map

$$H^0(\wedge^{n-5-k}T\mathcal{X}_{U|X_F}(n-5-k))\otimes H^0(\wedge^4 T\mathcal{X}_{U|X_F}(2))\to H^0(K_{\tilde{Y}_F})$$

we have that any codimension four subspace $T\subset T^{vert}_{\mathcal{X}_U,(y,F)}$ containing $T^{vert}_{\mathcal{Y},(y,F)}$ contains a hyperplane $H_{\ell_{(y,F)}}$ of $I_{\ell_{(y,F)}}(d)$, where $\ell_{(y,F)}$ is a line through y. Note that *a priori* the hyperplane $H_{\ell_{(y,F)}}$ could vary with T. We have then to verify that $T_{\mathcal{Y},(y,F)}$ is forced to contain one of those $H_{\ell_{(y,F)}}$.

Proof of Proposition 3.1. Remark that a codimension four subspace $T\subset T^{vert}_{\mathcal{X}_U,(y,F)}$ containing $T^{vert}_{\mathcal{Y},(y,F)}$ cannot contain two hyperplanes H and H' in the ideals $I_\ell(d)$ and $I_{\ell'}(d)$ of different lines $\ell \neq \ell'$. Indeed, if this was the case, by the surjectivity of

$$I_\ell(d) \oplus I_{\ell'}(d) \twoheadrightarrow S^d_y,$$

then T would contain at least a codimension 2 subspace of S^d_y, thus violating the hypothesis on its codimension. Set $V := S^d_y$, and $V_0 := T^{vert}_{\mathcal{Y},(y,F)}$. Denote by ℓ_T the unique line such that T contains an hyperplane in its ideal. Then, by the above, we have a morphism from $G' := Grass(V/V_0, 4)$, the Grassmannian of codimension 4 subspaces of V/V_0, to $G := Grass(1, \mathbf{P}^n)$, the Grassmannian of lines in $\mathbf{P}^n$:

$$\varphi \ : \ G' \to G; \ T \longmapsto \ell_T.$$

Suppose that φ is not constant map. Let ℓ be in its image, and $T \in \varphi^{-1}(\ell)$. Then it is easy to construct from such a T a positive dimensional family of codimension 4 subspaces containing V_0 and a hyperplane in $I_\ell(d)$. Thus φ has positive dimensional fibers and we are done, since in this case the ample line bundle $\varphi^*\mathcal{O}_G(1) = \mathcal{O}'_G(s),\ s > 0$, would have zero intersection with the curves in the fiber, which is absurd. Now, let ℓ be the unique line in the image of φ. A dimension count shows that if $V_0 \cap I_\ell(d)$ had codimension ≥ 2, then it would exist a $T \in Grass(V/V_0, 4)$ with $codim_{I_\ell(d)} T \cap I_\ell(d) \geq 2$, thus contradicting Lemma 3.2.

□

3.2 The vertical contact distribution

We now want to use Proposition 3.1 to construct a (well defined) distribution in $T^{vert}_{\mathcal{Y}}$, and show its integrability.

From (3.1) we know that $T_{\mathcal{Y},(y,F)}$ contains at least a hyperplane $H := H_{\ell_{(y,F)}}$ in $I := I_{\ell_{(y,F)}}(d)$. Remark that $T_{\mathcal{Y},(y,F)}$ cannot contain two different hyperplanes $H \neq H'$ in different ideals $I \neq I'$, otherwise it would contain a codimension two subspace of S^d_y. But this is absurd, since by Lemma 2.1

$$codim_{\ T^{vert}_{\mathcal{X},(y,F)}} \ T^{vert}_{\mathcal{Y},(y,F)} \ = \ codim_{\mathcal{X}} \mathcal{Y} \ \geq 4.$$

Hence the line $\ell_{(y,F)}$ is unique and we have a well defined map

$$\begin{array}{rccc} \phi : & \mathcal{Y} & \longrightarrow & G(1,n) \\ & (y,F) & \longmapsto & \ell_{(y,F)} \end{array} \tag{17}$$

If, at a generic point (y,F), $T_{\mathcal{Y},(y,F)}$ contains the hyperplane $H_{\ell_{(y,F)}}$, but not the whole ideal $I_{\ell_{(y,F)}}(d)$, we get a well defined distribution $\mathcal{H} \subset T^{vert}_{\mathcal{Y}}$, whose fiber at a point (y,F) is given by $H_{\ell_{(y,F)}}$. We will call $\mathcal{H}$ the *vertical contact distribution.*

If at a generic point $T_{\mathcal{Y},(y,F)}$ contains $I_{\ell_{(y,F)}}(d)$, then one can consider the distribution $\mathcal{I} \subset T^{vert}_{\mathcal{Y}}$ fiberwise defined by $I_{\ell_{(y,F)}}(d)$. This case is easier and is actually the case considered in [V3]. It will be briefly treated at the end of this section.

In the former case, as in [V3] and by simply adapting the arguments given there to our situation, we now want to show the following natural fact: if we move infinitesimally in the directions parametrized by $H_{\ell_{(y,F)}} \subset I_{\ell_{(y,F)}}(d)$, then the line $\ell_{(y,F)}$ remains fixed. The integrability of $\mathcal{H}$ will then immediately follow.

Lemma 3.3. *(i) The differential ϕ_* at the point (y,F) vanishes on $H_{\ell_{(y,F)}}$.*

(ii) The vertical contact distribution $\mathcal{H} \subset T^{vert}_{\mathcal{Y}}$ is integrable.

Proof. Since the distribution $T^{vert}_{\mathcal{Y},(y,F)} = ker\ p_*$ is integrable, the brackets induce a map

$$\Psi : \bigwedge^2 \mathcal{H} \to T^{vert}_{\mathcal{Y}}/\mathcal{H} \subset T^{vert}_{\mathcal{X}}{}_{|\mathcal{Y}}/\mathcal{H},$$

which is given at the point (y,F) by

$$\psi : \wedge^2 H_{\ell_{(y,F)}} \to T^{vert}_{\mathcal{Y},(y,F)}\ mod\ H_{\ell_{(y,F)}} \subset S^d_y\ mod\ H_{\ell_{(y,F)}}.$$

Since we are supposing that $T^{vert}_{\mathcal{Y},(y,F)}$ contains $H_{(y,F)}$ but not the whole ideal $I_{\ell_{(y,F)}}(d)$, there is a canonical isomorphism

$$T^{vert}_{\mathcal{Y},(y,F)}\ mod\ H_{\ell_{(y,F)}} \cong T^{vert}_{\mathcal{Y},(y,F)}\ mod\ I_{\ell_{(y,F)}}(d),$$

and hence ψ identifies with a map

$$\wedge^2 H_{\ell_{(y,F)}} \to H^0(\mathcal{O}_\ell(d)(-y))$$

which we also denote by ψ. To prove the integrability of $\mathcal{H}$, by Frobenius' theorem it will suffice to show that Ψ is zero. In what follows we will denote $\ell_{(y,F)}$ and $H_{(y,F)}$ respectively by ℓ and H. Now, choose coordinates on $\mathbf{P}^n$ such that $\ell = \{X_2 = \ldots = X_n = 0\}$ and $y = [1, 0, \ldots, 0]$. Recall that $H^0(N_{\ell/\mathbf{P}^n}(-1))$ identifies naturally with the set of (-1)-graded homomorphisms from $\bigoplus_d I_\ell(d)/I_\ell^2(d)$ to $\bigoplus_d S^d/I_\ell(d)$. Hence there is a natural bilinear map, denoted by $(a,b)\mapsto a\cdot b$:

$$I_\ell(d)\otimes H^0(N_{\ell/\mathbf{P}^n}(-y))\to H^0(\mathcal{O}_\ell(d)(-y)),$$

which is explicitely given by

$$P\cdot(X_1\sum_{i=2}^n b_i\frac{\partial}{\partial X_i})=\sum_{i=2}^n b_iX_1(\frac{\partial P}{\partial X_i})_{|\ell}\in H^0(\mathcal{O}_\ell(d)(-y)).$$

Remark that, since $y\in\ell$ and $T^{vert}_{\mathcal{Y},(y,F)}\subset S^d_y$, any deformation of ℓ belonging to $\phi_*(T^{vert}_{\mathcal{Y},(y,F)})$ passes through y, i.e. $\phi_*(T^{vert}_{\mathcal{Y},(y,F)})\subset H^0(N_{\ell/\mathbf{P}^n}(-y))$. A verification in local coordinates shows that

$$\psi(A\wedge B)=A\cdot\phi_*(B)-B\cdot\phi_*(A),\ \ A,B\in H. \tag{18}$$

Note that

$$(QX_iX_j)\cdot(\sum_2^n b_l\frac{\partial}{\partial X_l})=\sum_2^n b_l(\frac{\partial QX_iX_j}{\partial X_l})_{|\ell}=0,$$

for every $Q\in S^{d-2}$ and $i,j\geq 2$, and therefore $\phi_*(A)\cdot B=0$, for every $A\in H\cap I_\ell^2(d)$, $B\in H$. We first show that ϕ_* vanishes on $I_\ell^2(d)$: if we had $\phi_*(A)\neq 0$ with $A\in H\cap I_\ell^2(d)$, then $T^{vert}_{\mathcal{Y},(y,F)}mod\ I_\ell(d)$ would contain the elements $B\cdot\phi_*(A)$ for any $B\in H$, hence at least a hyperplane of $H^0(\mathcal{O}_\ell(d)(-y))$. Thus ϕ_* vanishes on $H\cap I_\ell^2(d)$, giving a map $H/H\cap I_\ell^2(d)\to H^0(N_{\ell/\mathbf{P}^n}(-y))$, which we still call ϕ_*.
Identify $H^0(\mathcal{O}_\ell(d)(-y))$ with $H^0(\mathcal{O}_\ell(d-1))$, and recall again the natural isomorphism

$$I_\ell(d)/I_\ell^2(d)\cong H^0(\mathcal{O}_\ell(d-1))\otimes H^0(N_{\ell/\mathbf{P}^n}(-y))^*.$$

Then $H/H\cap I_\ell^2(d)$ corresponds to a subspace

$$\overline{H}\subset H^0(\mathcal{O}_\ell(d-1))\otimes H^0(N_{\ell/\mathbf{P}^n}(-y))^*,$$

with $codim\ \overline{H}\leq 1$, and the dot map is simply given by the contraction

$$<\cdot,\cdot>:\overline{H}\otimes H^0(N_{\ell/\mathbf{P}^n}(-y))\to H^0(\mathcal{O}_\ell(d)(-y))$$

Hence, by (18) the map $\overline{\psi} : \bigwedge^2 H/H \cap I_\ell^2(d) \to H^0(\mathcal{O}_\ell(d)(-y))$ identifies with

$$\begin{aligned} \bigwedge^2 \overline{H} &\longrightarrow H^0(\mathcal{O}_\ell(d)(-y)) \\ A \wedge B &\mapsto <A, \phi_*(B)> - <B, \phi_*(A)> . \end{aligned} \tag{19}$$

To conclude we need the following linear algebra result:

Lemma 3.4. *Let W and K be two vector spaces, $\overline{H}$ a codimension 1 subspace of in $W \otimes K^*$ and $\phi_* : \overline{H} \to K$ a linear map. If $\phi_* \neq 0$, then the image of the map*

$$\begin{aligned} \overline{\psi} \ : \ \bigwedge^2 \overline{H} &\longrightarrow W \\ A \wedge B &\mapsto < A, \phi_*(B) > - < B, \phi_*(A) > \end{aligned}$$

contains at least a codimension 2 subspace of W.

Proof. Let $J = \overline{\psi}(\wedge^2 \overline{H})$. Pick a complementary subspace $J^\perp$ to J in W and a basis $\{w_j\}$ for W which is compatible with the decomposition

$$W = J \oplus J^\perp.$$

Let $\{k_i\}$ be a basis of K and $\{k_i^*\}$ the dual one. Pick a complementary space $\overline{H}^\perp$ to $\overline{H}$ in $W \otimes K^*$ which will be generated by a monomial $w_{j_0} \otimes k_{i_0}^*$, and extend ϕ_* to the whole $W \otimes K^*$ by setting

$$\phi_*(w_{j_0} \otimes k_{i_0}^*) = 0.$$

The map $\overline{\psi}$ extends naturally to $\wedge^2(W \otimes K^*)$. Since

$$\dim \left(\psi \left(\overline{H}^\perp \otimes (W \otimes K^*) \right) \right) \leq 1,$$

we are reduced to proving that if the extended map $\phi_* : W \to K$ is not zero, then the codimension of the image of

$$\overline{\psi} \ : \ \bigwedge^2 W \otimes K^* \to W$$

is at least 1. This has already been checked in [V3], Lemma 3, and so we are done. ☐

Take $W := H^0(\mathcal{O}_\ell(d)(-y))$ and $K := H^0(N_{\ell/\mathbf{P}^n}(-y))$, and apply Lemma 3.4 to our situation. If we had $\phi_* \neq 0$, then the image of the map $\overline{\psi}$ would contain at least a codimension 2 subspace of $H^0(\mathcal{O}_\ell(d)(-y))$. But the image of $\overline{\psi}$ is contained in $T^{vert}_{\mathcal{Y},(y,F)}$ *mod* $I_\ell(d)$, and $T^{vert}_{\mathcal{Y},(y,F)}$ contains a hyperplane in $I_\ell(d)$. Hence the codimension of $T^{vert}_{\mathcal{Y}}$ in $T^{vert}_{\mathcal{X}}$ would be at most 3, which is in contradiction with Lemma 2.2. Thus $\phi_* = 0$, hence ψ is zero and so is Ψ. By Frobenius' theorem the distribution $\mathcal{H}$ is integrable. □

3.3 Proof of Proposition A

We can now prove Proposition A, i.e. we show that the line $\ell_{(y,F)}$ defined by the ideal $I_{\ell_{(y,F)}}(d)$ is such that

$$X_F \cap \ell_{(y,F)} = d \cdot y.$$

From (3.3) we know that $\mathcal{H}$ is integrable and ϕ is constant along the leaves of the corresponding foliation. Therefore the line $\ell_{(y,F)}$ is fixed along the leaf, and because its tangent space is contained at each point in $I_{\ell_{(y,F)}}(d)$, it follows that the restriction $G_{|\ell_{(y,F)}}$ is constant, for any polynomial G belonging to the leaf through (y, F). This means that the leaf is locally of the form $y \times F + W_{(y,F)}$, where $W_{(y,F)} \subset I_{\ell_{(y,F)}}(d)$ is a germ of complex hypersurface. Then consider the following diagram

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_{\ell_{(y,F)}} & \longrightarrow & I_{\ell_{(y,F)}}(d) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T^{vert}_{\mathcal{Y},(y,F)} & \longrightarrow & T^{vert}_{\mathcal{X},(y,F)} = S^d_y \\ & & \downarrow & & \downarrow \\ & & H^0(\mathcal{O}_\ell(d)(-y)) & = & H^0(\mathcal{O}_\ell(d)(-y)) \\ & & & & \downarrow \\ & & & & 0 \end{array} \tag{20}$$

By Lemma 2.1,

$$codim_{\ T^{vert}_{\mathcal{X},(y,F)}}\ T^{vert}_{\mathcal{Y},(y,F)} = codim_{\ \mathcal{X}}\ \mathcal{Y} = n - k - 1.$$

Then, by (20) the image

$$Im := \ Im \ (T^{vert}_{\mathcal{Y},(y,F)} \to H^0(\mathcal{O}_\ell(d)(-y)))$$

has codimension $(n-k-1)-1 = n-k-2$, and therefore, since $d = 2n-2-k$

$$dim \ Im \ = (2n-2-k) - (n-k-2) = n. \tag{21}$$

At the same time, again by Lemma 2.1, $T^{vert}_{\mathcal{Y},(y,F)}$ contains $S^1_y \cdot J^{d-1}_F$ and F itself. Take coordinates $X_0, \dots, X_n$ on $\mathbf{P}^n$ such that $y = [1, 0, \dots, 0]$, and $\ell := \ell_{(y,F)} = \{X_2 = \dots = X_n = 0\}$. Since ϕ is constant along the leaves of the foliation, we can generically choose a polynomial G in $F + W_{(y,F)}$, so that the $(n-1)$-elements $X_1 \frac{\partial G}{\partial X_i}$, $i \geq 2$, are generic in a hypersurface. Consider the subspace

$$K := < G_{|\ell}, X_1 (\frac{\partial G}{\partial X_0})_{|\ell}, X_1 (\frac{\partial G}{\partial X_1})_{|\ell} > \subset H^0(\ell, \mathcal{O}_\ell(d) \otimes \mathcal{I}_y),$$

which is uniquely determined by $F_{|\ell}$ and hence is constant along the leaf. Its codimension in $H^0(\ell, \mathcal{O}_\ell(d) \otimes \mathcal{I}_y)$ is at least $d-3 \geq n$ (since, by hypothesis, $k \leq n-5$ so that $d = 2n-2-k \geq n+3$). Since we know that along the leaf, G moves freely in the complex hypersurface $W_{(y,F)}$, the polynomials $X_1 (\frac{\partial G}{\partial X_i})_{|\ell}$ are generic in a codimension 1 subspace of $H^0(\ell, \mathcal{O}_\ell(d) \otimes \mathcal{I}_y)$, and it then follows that they will be generically independant modulo K. From (21) we thus get that

$$dim \ K \leq 1 \tag{22}$$

and so Proposition A is proved, since by (22) $F_{|\ell}$ has to be of the form αX_1^d.

If, for generic (y, F), $T^{vert}_{\mathcal{Y},(y,F)}$ contains the whole ideal $I_{\ell_{(y,F)}}(d)$, then consider the distribution $\mathcal{I} \subset T^{vert}_{\mathcal{X}}$ pointwise defined by $I_{\ell_{(y,F)}}(d)$. Arguing as we did before, we get

$$dim \ Im \ (T^{vert}_{\mathcal{Y},(y,F)} \to H^0(\mathcal{O}_\ell(d)(-y))) = n-1,$$

thus deducing

$$dim \ K = 0.$$

Then the polynomial F belongs to $I_{\ell_{(y,F)}}(d)$, and the theorem is true in this case, i.e. Y_F is a component of the subvariety of X_F covered by lines. $\square$

4 The geometry of $\Delta_{d,F}$

Let $X_F \subset \mathbf{P}^n$ be a general hypersurface of degree $d = 2n-2-k,\ 1 \leq k \leq n-5$, and $Y_F \subset X_F$ a k-dimensional subvariety whose desingularization $\tilde{Y}$ has $h^0(\tilde{Y}, K_{\tilde{Y}}) = 0$. Then, by Proposition A, we know that Y_F has to be contained in $\Delta_{d,F} \subset X_F$, the $(k+1)$-dimensional subvariety of points in X_F through which there is a d-osculating line. To prove Proposition B and hence our theorem, we have then to show that the only subvariety of dimension k of $\Delta_{d,F}$ with geometric genus zero is the subvariety covered by the lines in X_F.

4.1 A desingularization of $\Delta_{d,F}$

We start by giving an explicit description of a desingularization $\tilde{\Delta}_{d,F}$ of $\Delta_{d,F}$ in terms of the zero locus of a section of a vector bundle. This fact will allow us to calculate, by adjunction, the canonical bundle of $\tilde{\Delta}_{d,F}$ and see that it is very ample.

Let $G := Gr(1,n)$ be the Grassmannian of lines in $\mathbf{P}^n$. Let $\mathcal{O}_G(1)$ be the line bundle on G which gives the Plücker embedding, so that we have $H^0(\mathcal{O}_G(1)) = \bigwedge^2 S^1$. Let $\mathcal{P} \subset \mathbf{P}^n \times G$ be the incidence variety $\{(x,[\ell]) : x \in \ell\}$ with projections

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{p} & \mathbf{P}^n \\ \downarrow{\scriptstyle q} & & \\ G & & \end{array} \tag{23}$$

and $H := p^*\mathcal{O}_{\mathbf{P}^n}(1),\ L := q^*\mathcal{O}_G(1)$ the line bundles generating the Picard group of $\mathcal{P}$.

Define

$$\tilde{\Delta}_r := \{(x,[\ell],F) \ :\ \ell \cdot X_F \geq r \cdot x\} \subset \mathcal{P} \times S^r \subset \mathbf{P}^n \times G \times S^r,$$

(by $\ell \cdot X_F \geq r \cdot x$ we mean that the line has a contact of order at least r with X_F at x) and consider the various projections as illustrated in the following commutative diagram:

$$\begin{array}{ccccc} \tilde{\Delta}_r & \xrightarrow{\rho_r} & \mathcal{X} & \xrightarrow{s} & S^r \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle t} & & \\ \mathcal{P} & \xrightarrow{p} & \mathbf{P}^n & & \\ \downarrow{\scriptstyle q} & & & & \\ G & & & & \end{array} \tag{24}$$

Since the tangency of order at least r imposes r conditions, the fibres of the projection $\pi : \tilde{\Delta}_r \to \mathcal{P}$ are punctured vector spaces of dimension $N - r$. Hence $\tilde{\Delta}_r$ is smooth and irreducible of dimension

$$N - r + 2(n-1) + 1.$$

Lemma 4.1. *(i) The projection* $\rho_r : \tilde{\Delta}_r \to \mathcal{X}$ *is surjective for* $r \leq n$*, and generically injective for* $r > n$*.*

(ii) The composite projection $s \circ \rho_r \ : \tilde{\Delta}_r \to S^r$ *is surjective if* $r \leq 2(n-1)$*. In particular, in that case, its fiber* $\tilde{\Delta}_{r,F} := s \circ \rho_r^{-1}(F)$ *is smooth for generic* $F \in S^r$*, and the composite projection* $t \circ \rho_r : \tilde{\Delta}_{r,F} \to \mathbf{P}^n$*, gives a desingularization of*

$$\Delta_{r,F} := \{x \in X_F \ : \ \exists \ell \ s.t. \ \ell \cdot X_F \geq r \cdot x\}$$

Proof. (i) Assume $x = [1, 0, \ldots, 0] \in Proj\left(\mathbb{C}\left[X_0, \ldots, X_n\right]\right)$. Then the assertion follows from the fact that the contact condition $\ell \cdot X_F \geq r$ for a line ℓ through x with respect to

$$F = \sum\nolimits_{j=1}^{n} X_0^{d-j} F_j\left(X_1, \ldots, X_n\right)$$

becomes

$$\{F_1 = \ldots = F_{r-1} = 0\} \subseteq Proj\left(\mathbb{C}\left[X_1, \ldots, X_n\right]\right).$$

(ii) A dimension count shows that all hypersurfaces X_F in $\mathbf{P}^n$ of degree $d \leq 2n - 2$ admit a point through which passes a line having contact with X_F of maximal order. □

In what follows, by abuse of notation, we will identify $\tilde{\Delta}_{r,F}$ to its image, $\pi(\tilde{\Delta}_{r,F})$, in $\mathcal{P}$. We will show that $\tilde{\Delta}_{r,F}$ can be seen as the zero locus of a global section of a vector bundle over $\mathcal{P}$. This will enable us to compute its canonical bundle.

Let $\mathcal{E}_d$ be the d^{th}-symmetric power of the dual of the tautological subbundle on G, and recall that, by definition, its fibre at a point $[\ell]$ is then given by $H^0(\ell, \mathcal{O}_\ell(d))$, and its first Chern class is

$$c_1(\mathcal{E}_d) = \mathcal{O}_G(\frac{d(d+1)}{2}).$$

Let $\mathcal{L}_d := dL - dH$ be the rank 1 subbundle of $q^*\mathcal{E}_d$. Note that its fibre $\mathcal{L}_{d,(x,[\ell])}$ is equal to the space of degree d homogeneous polynomials on ℓ vanishing to the order d at x. Finally, let $\mathcal{F}_d$ be the quotient

$$0 \to \mathcal{L}_d \to q^*\mathcal{E}_d \to \mathcal{F}_d \to 0. \tag{25}$$

It is possible to associate to every $F \in S^d$ a section $\sigma_F \in H^0(G, \mathcal{E}_d)$, whose value at a point $[\ell]$ is exactly the polynomial $F_{|\ell}$. We will denote by $\overline{\sigma}_F$ the induced section in $H^0(\mathcal{P}, \mathcal{F}_d)$. Then $V(\overline{\sigma}_F)$ is equal to $\tilde{\Delta}_{r,F}$, and, as checked in Lemma 4.1, for generic F, $\tilde{\Delta}_{r,F} = V(\overline{\sigma}_F) \xrightarrow{p} \Delta_{r,F}$ is a desingularization. It is computed in [V3] that

$$K_{\mathcal{P}} = -2H - nL, \tag{26}$$

and then

$$det\ \mathcal{F}_d = det\ q^*\mathcal{E}_d - dL + dH = \frac{d(d-1)}{2}L + dH. \tag{27}$$

Hence, by adjunction,

$$K_{\tilde{\Delta}_{d,F}} = (K_{\mathcal{P}} \otimes det\ \mathcal{F}_d)_{|\tilde{\Delta}_{d,F}} = [(d-2)H + (\frac{d(d-1)}{2} - n)L]_{|\tilde{\Delta}_{d,F}}, \tag{28}$$

so $K_{\tilde{\Delta}_{d,F}}$ is very ample under our numerical hypothesis $d = 2n-2-k,\ 1 \leq k \leq n-5$.

4.2 Proof of Proposition B

Let $\Delta_d \subset \mathbf{P}^n \times S^d$ be the family of the $\Delta_{d,F}$'s, and $\tilde{\Delta}_d \subset \mathcal{P} \times S^d$ the family of the desingularizations. Let $\mathcal{Y} \subset \tilde{\Delta}_d$ be a subscheme of relative dimension k, invariant under the action of $GL(n+1)$, and $\tilde{\mathcal{Y}} \to \mathcal{Y}$ a desingularization. Assume $h^0(\tilde{Y}_F, K_{\tilde{Y}_F}) = 0$. Recall the isomorphisms

$$T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}} \cong \Omega^{N+k}_{\tilde{\Delta}_d\ |\tilde{\Delta}_{d,F}} \tag{29}$$

$$\Omega^{N+k}_{\tilde{\mathcal{Y}}\ |\tilde{Y}_F} \cong K_{\tilde{Y}_F} \tag{30}$$

and consider the natural map

$$T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}} \cong \Omega^{N+k}_{\tilde{\Delta}_d}{}_{|\tilde{\Delta}_{d,F}} \to \Omega^{N+k}_{\tilde{\mathcal{Y}}}{}_{|\tilde{Y}_F} \quad \cong \quad K_{\tilde{Y}_F}. \tag{31}$$

Then, by assumption, the induced map in cohomology

$$H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}) \to H^0(K_{\tilde{Y}_F}) \tag{32}$$

is zero. Let $T^{vert}_{\tilde{\Delta}_d}$ be the sheaf defined by

$$0 \to T^{vert}_{\tilde{\Delta}_d} \to T\tilde{\Delta}_d \xrightarrow{\pi_*} T\mathcal{P} \to 0.$$

Using the positivity result proved in Lemma 2.2, (ii), we will construct a sub-bundle of $(T^{vert}_{\tilde{\Delta}_d})_{|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}$, generated by its global sections. This will allow us to show that any point $(y,[\ell],F) \in \mathcal{Y}$ is such that $y \in \ell \subset X_F$. Comparing the dimension, we will thus obtain that Y_F has to be a component of the subvariety of lines in X_F.

From (32) we see that, at a smooth point $(y,[\ell],F) \in \mathcal{Y} \subset \mathcal{P} \times S^d$, the tangent space $T_{\mathcal{Y},(y,[\ell],F)}$ is in the base locus of $H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}})$, considered as the space of sections of a line bundle on the Grassmannian of hyperplanes in $T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}}$. Consider the vector bundle M^d_G on $G := Gr(1,n)$ defined by the short exact sequence:

$$0 \to M^d_G \to S^d \otimes \mathcal{O}_G \to \mathcal{E}_d \to 0.$$

Notice that the fiber of M^d_G at a point $[\ell]$ is equal to $I_\ell(d)$, and recall that, by Proposition 2.2, (ii), $M^d_G \otimes \mathcal{O}_G(1)$ is generated by its global sections.

Then it follows that the vector bundle $q^*M^d_G \otimes det\mathcal{F}_d \otimes K_{\mathcal{P}}$, that, by (26), (27) and (28) can be written as

$$\begin{aligned} q^*M^d_G \otimes det\mathcal{F}_d \otimes K_{\mathcal{P}} &= q^*(M^d_G) \otimes \mathcal{O}_{\mathcal{P}}((d-2)H + (\frac{d(d-1)}{2} - n)L) \\ &= q^*(M^d_G(1)) \otimes \mathcal{O}_{\mathcal{P}}((d-2)H + (\frac{d(d-1)}{2} - n - 1)L), \end{aligned}$$

is generated by its global sections, and so is its restriction to $\tilde{\Delta}_{d,F}$, i.e. the sheaf

$$q^*M^d_{G|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}$$

is generated by its global sections.

Let $\mathcal{N}_d$ be the vector bundle over $\mathcal{P}$ defined by the exact sequence

$$0 \to \mathcal{N}_d \to S^d \otimes \mathcal{O}_{\mathcal{P}} \to \mathcal{F}_d \to 0. \tag{33}$$

We have

$$0 \to \mathcal{N}_{d|\tilde{\Delta}_{d,F}} \to T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \xrightarrow{\pi_*} T\mathcal{P}_{|\tilde{\Delta}_{d,F}} \to 0,$$

where $\mathcal{S} \subset \mathcal{P} \times S^d \xrightarrow{\pi} \mathcal{P}$ is the projection on the first component, i.e. $\mathcal{N}_{d|\tilde{\Delta}_{d,F}}$ is the vertical component of $T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}}$ w.r.t. π. Now consider the vector bundle $\mathcal{M}_G^d$ defined by the exact sequence

$$0 \to \mathcal{M}_G^d \to S^d \otimes \mathcal{O}_{\mathcal{P}} \xrightarrow{ev} q^*\mathcal{E}_d \to 0,$$

whose fiber at a point $(y, F, [\ell])$ is equal to $I_\ell(d)$. From (25) and the definition of $\mathcal{N}_d$ we also obtain that

$$0 \to \mathcal{M}_G^d \to \mathcal{N}_d \to \mathcal{L}_d \to 0.$$

So, $\mathcal{M}_G^d \otimes K_{\tilde{\Delta}_{d,F}}$ is a subbundle of $T\tilde{\Delta}^{vert}_{d|\tilde{\Delta}_{d,F}}$. Finally note that $\mathcal{M}_G^d = q^* M_G^d$, hence

$$\mathcal{M}_G^d \otimes K_{\tilde{\Delta}_{d,F}}$$

is generated by its global sections. Using this property of the bundle $\mathcal{M}_G^d \otimes K_{\tilde{\Delta}_{d,F}} \subset T\tilde{\Delta}^{vert}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}$ we are now able to conclude our proof.

Proof of Proposition B. Let $H \subset T_{\tilde{\Delta}_{d,(x,\Delta,F)}}$ be a hyperplane contained in the base locus of $H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}})$, considered as the space of sections of a line bundle on the Grassmannian of codimension 1 subspaces of $T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}}$. Then we must have

$$H^{vert} := H \cap \mathcal{N}_{d,(x,[\ell])} = \mathcal{M}^d_{G,(x,\ell)}. \tag{34}$$

Indeed, if $\bar{H} := H \cap \mathcal{M}^d_{G,(x,[\ell])}$ was strictly contained in $\mathcal{M}^d_{G,(x,[\ell])}$, then consider the following, well defined, commutative diagram:

$$\begin{array}{ccccc} H^0(\mathcal{M}_G^d \otimes K_{\tilde{\Delta}_{d,F}}) & \xrightarrow{ev} & (\mathcal{M}_G^d \otimes K_{\tilde{\Delta}_{d,F}})_{(x,[\ell])} & \xrightarrow{<\cdot,\bar{H}>} & \mathbb{C} \\ \downarrow & & \downarrow & & \| \\ H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}) & \xrightarrow{ev} & (T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}})_{(x,[\ell])} & \xrightarrow{<\cdot,H>} & \mathbb{C} \end{array} \tag{35}$$

(ev is the evaluation of the sections at the point $(x, [\ell])$, and $< \cdot, H >$ is the contraction defined by the hyperplane H). Since H belongs to the base locus of $H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}})$, then the composite map $< \cdot, H > \circ\, ev$ is zero, and so would be $< \cdot, \bar{H} > \circ\, ev$. But this is absurd, because $\mathcal{M}_G^d \otimes K_{\tilde{\Delta}_{d,F}}$ is generated by its global sections.

Let then $\mathcal{Y} \subset \Delta_d$ be a subvariety, which is stable under the action of $GL(n+1)$ and of relative codimension 1. Assume moreover that the restriction map (32)

$$H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}) \to H^0(K_{\tilde{Y}_F})$$

is zero. By (34), $T^{vert}_{\mathcal{Y},(y,[\ell],F)}$ is equal to

$$\mathcal{M}_{d,(y,[\ell])} = \{G \in S^d : G_{|\ell} = 0\}. \tag{36}$$

On the other hand, by Lemma 2.1, (ii), $T^{vert}_{\mathcal{Y},(y,[\ell],F)}$ contains F itself. So by (36) we have that $F_{|\ell} = 0$ for every point $(y,[\ell]) \in Y_F$, i.e. Y_F is a component of the subvariety covered by the lines contained in X_F. □

Remark 4.2. *If $k > 1$, the k-dimensional subvariety covered by the lines of the general hypersurface of degree $d = 2n - 2 - k$ is irreducible (see [DM]), so in this case Y_F has to coincide with it.*

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