

ON FAMILIES OF RATIONAL CURVES IN THE HILBERT SQUARE OF A SURFACE

(WITH AN APPENDIX BY EDOARDO SERNESI)

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1. INTRODUCTION

For any smooth surface S , the Hilbert scheme $S^{[n]}$ parametrizing 0-dimensional length n subschemes of S is a smooth $2n$ -dimensional variety whose inner geometry is naturally related to that of S . For instance, if $\Delta \subset S^{[n]}$ is the exceptional divisor, that is, the exceptional locus of the Hilbert-Chow morphism $\mu : S^{[n]} \rightarrow \text{Sym}^n(S)$, then irreducible (possibly singular) rational curves not contained in Δ roughly correspond to irreducible (possibly singular) curves on S with a \mathfrak{g}_n^1 on their normalizations, for some $n' \leq n$ (see § 2.1 for the precise correspondence when $n = 2$). One of the features of this paper is to show how ideas and techniques from one of the two sides of the correspondence makes it possible to shed light on problems naturally arising on the other side.

If S is a $K3$ surface, $S^{[n]}$ is a *hyperkähler manifold* (cf. [7]) and rational curves play a fundamental rôle in the study of the (birational) geometry of $S^{[n]}$. Indeed a result due to Huybrechts and Boucksom [32, 11] implies in particular that these curves govern the ample cone of $S^{[n]}$. The presence of a $\mathbb{P}^n \subset S^{[n]}$ gives rise to a birational map (the so-called *Mukai flop* [42]) to another hyperkähler manifold and, for $n = 2$, all birational maps between hyperkähler fourfolds factor through a sequence of Mukai flops [12, 30, 61, 63]. Moreover, as shown by Huybrechts [32], uniruled divisors allow to describe the birational Kähler cone of $S^{[n]}$. For hyperkähler fourfolds deformation equivalent to the Hilbert square of a $K3$ surface a conjectural description of the Mori cone and of the numerical and geometric properties of the rational curves that are extremal in the Mori cone has been proposed by Hassett and Tschinkel [24] (and partly confirmed in [25]).

The scope of this paper, and the structure of it as well, is twofold: we first devise general methods and tools to study families of curves with hyperelliptic normalizations on a surface S in § 2-§ 4. Then we apply these to obtain concrete results in the case of $K3$ surfaces, in § 5-§ 7.

1.1. Families of singular curves with hyperelliptic normalizations. The main question we address in the first part of the paper is whether there exists an upper bound on the dimension of families of irreducible curves on a projective surface with hyperelliptic normalizations. One easily sees that, if the canonical system of the surface is birational, then no curve with hyperelliptic normalization can move, cf. e.g. [34]. On the other hand, any surface S admitting a (generically) $2 : 1$ map onto a rational surface R carries families of arbitrarily high dimensions of curves on S having hyperelliptic normalizations. Nevertheless, for a large class of surfaces, we derive a geometric consequence on the family when its dimension is greater than two:

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Theorem 4.6’. *Let S be a smooth, projective surface with $p_g(S) > 0$. Let V be a reduced and irreducible scheme parametrizing a flat family of irreducible curves on S with hyperelliptic normalizations (of genus ≥ 2) such that $\dim(V) \geq 3$. Then the algebraic equivalence class $[C]$ of the curves parametrized by V has a decomposition $[C] = [D_1] + [D_2]$ into algebraically moving classes such that the point parametrizing $D_1 + D_2$ lies in the closure \overline{V} of V in the component of the Hilbert scheme of S containing V . Moreover the rational curves in $S^{[2]}$ corresponding to the irreducible curves parametrized by V cover only a (rational) surface $R \subset S^{[2]}$.*

In fact we prove a stronger result, cf. Theorem 4.6, that in particular relates the decomposition $[C] = [D_1] + [D_2]$ to the \mathfrak{g}_2^1 s on the normalizations of the curves parametrized by V . This additional point will in fact be crucial in our application of this result below. An immediate corollary is a simple dimension bound under natural additional hypotheses on V , cf. Corollary 4.7.

The proof of Theorem 4.6 illustrates well the rich interplay between the properties of curves on S and those of subvarieties of $S^{[2]}$. It relies on the following two ingredients: first, by a suitable version of Mumford’s theorem on 0-cycles on surfaces with $p_g > 0$ (cf. Corollaries 3.2 and 3.4), the family of rational curves in $S^{[2]}$ associated to the irreducible curves on S with hyperelliptic normalizations can cover only a surface, if $\dim(V) \geq 3$. Then, by Mori’s bend-and-break technique (cf. Lemma 2.10), we produce a reducible member in $S^{[2]}$. From this, in Proposition 4.3, we produce a decomposition of the curves on S into algebraically moving classes.

One application of Theorem 4.6 is a Reidier-like result for families of singular curves with hyperelliptic normalizations obtained in [34], where also more examples of such families are given. In the rest of this paper, we focus on $K3$ surfaces and in particular we apply Theorem 4.6 to show:

Theorem 5.2. *Let (S, H) be a general, smooth, primitively polarized $K3$ surface of genus $p = p_a(H) \geq 4$. Then the family of nodal curves in $|H|$ of geometric genus 3 with hyperelliptic normalizations is nonempty, and each of its irreducible components is two-dimensional.*

Note that it is well-known that there exist finitely many (nodal) rational curves, a one-parameter family of (nodal) elliptic curves, and a two-dimensional family of (nodal) curves of geometric genus 2 in $|H|$ (see § 5). Every such family yields in a natural way a two-dimensional family of irreducible rational curves in $S^{[2]}$, cf. § 2. Therefore, Theorem 5.2 is the first nontrivial existence result about curves with hyperelliptic normalizations on general $K3$ surfaces of any polarization, and consequently about rational curves in $S^{[2]}$. Also note that, by a result of Ran [47], the *expected dimension* of a family of rational curves in a symplectic fourfold, whence a posteriori also of a family of curves with hyperelliptic normalizations lying on a $K3$, equals two (cf. Lemma 5.1).

The proof of Theorem 5.2 takes the whole § 5 and relies on a general principle of constructing curves with hyperelliptic normalizations on general $K3$ s outlined in Proposition 5.11: first construct a marked $K3$ surface (S_0, H_0) of genus p such that $|H_0|$ contains a family of dimension ≤ 2 of nodal (possibly reducible) curves with the property that a desingularization of some $\delta > 0$ of the nodes is a limit of a hyperelliptic curve in the moduli space $\overline{\mathcal{M}}_{p-\delta}$ of stable curves of genus $p - \delta$ and such that this family is not contained in a higher-dimensional such family. Then consider the parameter space $\mathcal{W}_{p,\delta}$ of pairs $((S, H), C)$, where (S, H) is a smooth, primitively marked $K3$ surface of genus p and $C \in |H|$ is a nodal curve with at least δ nodes. Now map (the local branches of) $\mathcal{W}_{p,\delta}$ into $\overline{\mathcal{M}}_{p-\delta}$ by partially normalizing the curves at δ of the nodes and mapping them to their respective classes. By construction, the image of this map intersects the hyperelliptic locus $\overline{\mathcal{H}}_{p-\delta} \subset \overline{\mathcal{M}}_{p-\delta}$. A dimension count then shows that the dimension of the parameter space $\mathcal{J} \subset \mathcal{W}_{p,\delta}$ consisting of $((S, H), C)$ such that a desingularization of some $\delta > 0$ of the nodes of C is a limit of a hyperelliptic curve is at least 21. Now the dominance on the 19-dimensional moduli space of primitively marked $K3$ surfaces of genus p follows as the dimension of the special family on S_0 was ≤ 2 .

The technical difficulties in the proof of Proposition 5.11 mostly arise because the curves in the special family on S_0 may be reducible. Therefore we need to partially desingularize families of nodal curves, and this tool is provided in Appendix A by E. Sernesi. Moreover, we need a careful study of the Severi varieties of *reducible* nodal curves on $K3$ s, and here we use results of Tannenbaum [56].

Given Proposition 5.11, the proof of Theorem 5.2 is then accomplished by constructing a suitable (S_0, H_0) in Proposition 5.19 with $|H_0|$ containing a desired two-dimensional family of special curves, with $\delta = p - 3$, and then showing that the curves in the special family on S_0 in fact deform to curves with *precisely* δ nodes on the general S in Lemma 5.20. Showing that the special family on S_0 is not contained in a family of higher dimension of curves with the same property, is quite delicate, and it is here that we use the full version of Theorem 4.6.

We also show that the associated rational curves in $S^{[2]}$ cover a threefold, cf. Corollary 5.3.

1.2. Results on the Mori cone of $S^{[2]}$. Let (S, H) be a general, smooth, primitively polarized $K3$ surface of genus $p = p_a(H) \geq 2$. Then $N_1(S^{[2]})_{\mathbb{R}} \simeq \mathbb{R}[Y] \oplus \mathbb{R}[\mathbb{P}_{\Delta}^1]$, where \mathbb{P}_{Δ}^1 is the class of a rational curve in the ruling of the exceptional divisor $\Delta \subset S^{[2]}$, and $Y := \{\xi \in S^{[2]} \mid \text{Supp}(\xi) = \{p_0, y\}, \text{ with } p_0 \in S \text{ and } y \in C \in |H|\}$, where p_0 and C are chosen. One has that \mathbb{P}_{Δ}^1 lies on the boundary of the Mori cone, and by the result of Huybrechts and Boucksom [32, 11] mentioned above, if the Mori cone is closed, then also the other boundary is generated by the class of a rational curve. Notice that the conjecture of Hassett and Tschinkel [24] on the properties of these extremal classes is still open even in the case of the Hilbert square of a general $K3$ surface. It seems therefore useful to obtain more information on the Mori cone and find examples where the particular classes pointed out by Hassett and Tschinkel appear.

If now $C \in |mH|$ is an irreducible curve with hyperelliptic normalization, let $g_0(C) \geq p_g(C)$ be the arithmetic genus of the *minimal* partial desingularization of C that carries the \mathfrak{g}_2^1 (see § 2.1 and § 6.2). By the unicity of the \mathfrak{g}_2^1 , C defines a unique irreducible rational curve $R_C \subset S^{[2]}$ with class $R_C \sim_{alg} mY - (\frac{g_0(C)+1}{2})\mathbb{P}_{\Delta}^1$, cf. (6.11). Thus, the higher $g_0(C)$ (or $p_g(C)$) is, and the lower m is, the closer R_C is to the boundary of the Mori cone. This motivates the search for curves on S with hyperelliptic normalizations of high geometric genus, thus “unexpected” from Brill-Noether theory.

If $X \sim_{alg} aY - b\mathbb{P}_{\Delta}^1$ is an irreducible curve in $S^{[2]}$, with $a, b \neq 0$, then we define a/b to be the *slope* of the curve. Describing the Mori cone $\text{NE}(S^{[2]})$ amounts to computing

$$\text{slope}(\text{NE}(S^{[2]})) := \inf \left\{ \text{slope}(X) \mid X \text{ is an irreducible curve in } S^{[2]} \right\},$$

and, if the Mori cone is closed, then $\text{slope}(\text{NE}(S^{[2]})) = \text{slope}_{rat}(\text{NE}(S^{[2]}))$, where

$$\text{slope}_{rat}(\text{NE}(S^{[2]})) := \inf \left\{ \text{slope}(X) \mid X \text{ is an irreducible } \textit{rational} \text{ curve in } S^{[2]} \right\}.$$

(See § 6.1, 6.2 and 6.3 for further details.) Combining various results, we obtain 5 bounds of a different nature on the slope of effective 1-cycles in the Hilbert square $S^{[2]}$ of a $K3$ with $\text{Pic}(S) = \mathbb{Z}[H]$:

- (1) If $X \in N_1(S^{[2]})_{\mathbb{Z}}$ with $X \sim_{alg} Y - k\mathbb{P}_{\Delta}^1$, then $k \leq \frac{p_a(H)+4}{4}$, i.e. $\text{slope}(X) \geq \frac{4}{p_a(H)+4}$ (cf. Theorem 6.18, related to the “singular Brill-Noether invariant” introduced in [21]);
- (2) $\text{slope}(\text{NE}(S^{[2]})) \leq \sqrt{\frac{2}{p_a-1}}$ (cf. Theorem 6.21, related to Seshadri constants).

In § 7 we give a couple of existence results of a different type than Theorem 5.2: in Propositions 7.2 and 7.7 we find *general* primitively polarized $K3$ surfaces (S, H) of infinitely many degrees such that $S^{[2]}$ contains either a \mathbb{P}^2 (shown to us by B. Hassett) or a threefold birational to a \mathbb{P}^1 -bundle over a $K3$. We also find the classes corresponding to the lines and fibres, respectively, and the geometric genus of the corresponding curves on S with hyperelliptic normalizations. The lines and the fibres

are interesting, since, according to the conjecture of Hassett and Tschinkel [24], they should generate an extremal ray of $\text{NE}(S^{[2]})$. As a by-product of these constructions we also obtain:

- (3) $\text{slope}_{\text{rat}}(\text{NE}(S^{[2]})) \leq \text{slope}(\text{line in a } \mathbb{P}^2) = \frac{2}{2n-9}$, if $p_a(H) = n^2 - 9n + 20$ for some $n \geq 6$;
- (4) $\text{slope}_{\text{rat}}(\text{NE}(S^{[2]})) \leq \text{slope}(\text{fiber of a } \mathbb{P}^1\text{-bundle}) = \frac{1}{d}$, if $p_a(H) = d^2$ for some $d \geq 2$.

Moreover, Proposition 7.2 also shows the sharpness of Theorem 4.6 even in the case of a surface with Picard number one: in fact, the $(3m-1)$ -dimensional family of rational curves in $\mathcal{O}_{\mathbb{P}^2}(m)$ gives rise to a $(3m-1)$ -dimensional family of curves with hyperelliptic normalizations in $|mH|$.

The idea of the proofs of Propositions 7.2 and 7.7 is to start with a special quartic surface $S_0 \subset \mathbb{P}^3$ such that $S_0^{[2]}$ contains a \mathbb{P}^2 or a threefold birational to a \mathbb{P}^1 -bundle over itself, perform the standard involution on $S_0^{[2]}$ to produce a new such and then deform $S_0^{[2]}$ keeping the new one by keeping a suitable polarization on the surface that is different from $\mathcal{O}_{S_0}(1)$. Here we use results on deformations of symplectic fourfolds by Hassett and Tschinkel [24] and Voisin [58].

Finally, we remark that, combining Theorem 4.6 with the deformation theoretic argument of Proposition 5.11 we obtain a general procedure to deform (even reducible) rational curves on the Hilbert square of a *special* $K3$ to the general one:

- (5) Let (S_0, H_0) be a primitively marked $K3$ surface. Suppose $|H_0|$ contains a maximal family of (possibly reducible) curves having the property that some partial desingularization is a limit of a smooth hyperelliptic curve of genus p_g . Suppose furthermore that this family is two-dimensional (apply Theorem 4.6). Then these curves deform to *irreducible* curves with hyperelliptic normalizations on the general $K3$ (S, H) . Hence also the associated rational curves R_0 in $S_0^{[2]}$ deform to $S^{[2]}$. In particular $\text{slope}_{\text{rat}}(\text{NE}(S^{[2]})) \leq \text{slope}(R_0) = \frac{2}{p_g+1}$.

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2. RATIONAL CURVES IN $S^{[2]}$

Let S be a smooth, projective surface. In this section we gather some basic results that will be needed in the rest of the paper. We first describe the natural correspondence between rational curves in $S^{[2]}$ and curves on S with rational, elliptic or hyperelliptic normalizations. Then, in § 2.2, we apply Mori's bend-and-break technique to rational curves in $\text{Sym}^2(S)$ covering a surface.

Recall that we have the natural Hilbert-Chow morphism $\mu : S^{[2]} \rightarrow \text{Sym}^2(S)$ that resolves $\text{Sing}(\text{Sym}^2(S)) \simeq S$. The μ -exceptional divisor $\Delta \subset S^{[2]}$ is a \mathbb{P}^1 -bundle over S . The Hilbert-Chow morphism gives an obvious one-to-one correspondence between irreducible curves in $S^{[2]}$ not contained in Δ and irreducible curves in $\text{Sym}^2(S)$ not contained in $\text{Sing}(\text{Sym}^2(S))$. We will therefore often switch back and forth between working on $S^{[2]}$ and $\text{Sym}^2(S)$.

2.1. Irreducible rational curves in $S^{[2]}$ and curves on S . Let $T \subset S \times S^{[2]}$ be the incidence variety, with projections $p_2 : T \rightarrow S^{[2]}$ and $p_S : T \rightarrow S$. Then p_2 is finite of degree two, branched along $\Delta \subset S^{[2]}$. (In particular, T is smooth as Δ is.)

Let $X \subset S^{[2]}$ be an irreducible rational curve not contained in Δ . We will now see how X is equivalent to one of three sets of data on S .

Let $\nu_X : \tilde{X} \simeq \mathbb{P}^1 \rightarrow X$ be the normalization and set $X' := p_2^{-1}(X) \subset T$. By the universal property of blowing up, we obtain a commutative square:

$$(2.1) \quad \begin{array}{ccc} \tilde{C}_X & \xrightarrow{f} & \tilde{X} \simeq \mathbb{P}^1 \\ \tilde{\nu}_X \downarrow & & \downarrow \nu_X \\ X' & \xrightarrow{p_2|_{X'}} & X, \end{array}$$

defining the curve \tilde{C}_X , $\tilde{\nu}_X$ and f . In particular, $\tilde{\nu}_X$ is birational and \tilde{C}_X admits a \mathfrak{g}_2^1 (i.e., a $2 : 1$ morphism onto \mathbb{P}^1 , given by f), but may be singular, or even reducible. Set $\tilde{\nu} := p_{S|X'} \circ \tilde{\nu}_X : \tilde{C}_X \rightarrow S$.

Assume first that \tilde{C}_X is irreducible.

We set $C_X := \tilde{\nu}(\tilde{C}_X) \subset S$. Since $X \not\subset \Delta$, C_X is a curve. As \tilde{C}_X carries a \mathfrak{g}_2^1 , it is easily seen that also the normalization of C_X does, that is, C_X has rational, elliptic or hyperelliptic normalization. Moreover, it is easily seen that $\tilde{\nu} : \tilde{C}_X \rightarrow C_X$ is generically of degree one. Indeed, for general $x \in C_X$, as $x \notin p_S(p_2^{-1}(\Delta))$, we can write $(p_{S|X'})^{-1}(x) = \{(x, x + y_1), \dots, (x, x + y_n)\}$, where $n := \deg \tilde{\nu}$. By definition of p_2 , and since $X' = p_2^{-1}(X)$, we must have that each $(y_i, x + y_i) \in X'$, for $i = 1, \dots, n$, and each couple $((x, x + y_i), (y_i, x + y_i))$ is the pushdown by $\tilde{\nu}_X$ of an element of the \mathfrak{g}_2^1 on \tilde{C}_X . Hence, each couple (x, y_i) is the pushdown by the normalization morphism of an element of the induced \mathfrak{g}_2^1 on the normalization of C_X . Since x has been chosen general, $x \notin \text{Sing}(C_X)$, so that we must have $n = 1$, as claimed.

In particular, by construction, $\tilde{\nu} : \tilde{C}_X \rightarrow C_X$ is a partial desingularization of C_X , in fact, it is the *minimal* partial desingularization of C_X carrying the \mathfrak{g}_2^1 in question (which is unique, if $p_g(C_X) \geq 2$).

We have therefore obtained:

- (I) the data of an irreducible curve $C_X \subset S$ together with a partial normalization $\tilde{\nu} : \tilde{C}_X \rightarrow C_X$ with a \mathfrak{g}_2^1 on \tilde{C}_X (unique, if $p_g(C_X) \geq 2$), such that $\tilde{\nu}$ is minimal with respect to the existence of the \mathfrak{g}_2^1 .

Next we treat the case where \tilde{C}_X is reducible. In this case, it must consist of two irreducible smooth rational components, $\tilde{C}_X = \tilde{C}_{X,1} \cup \tilde{C}_{X,2}$, that are identified by f .

If $\tilde{\nu}$ does not contract any of the components, set $C_{X,i} := \tilde{\nu}(\tilde{C}_{X,i}) \subset S$ and $n_{X,i} := \deg \tilde{\nu}|_{\tilde{C}_{X,i}}$, for $i = 1, 2$. We therefore obtain:

- (II) the data of a curve $C_X = n_{X,1}C_{X,1} + n_{X,2}C_{X,2} \subset S$, with $n_{X,i} \in \mathbb{N}$, $C_{X,i}$ an irreducible, rational curve, a morphism $\tilde{\nu} : \tilde{C}_X = \tilde{C}_{X,1} \cup \tilde{C}_{X,2} \rightarrow C_{X,1} \cup C_{X,2}$ (resp. $\tilde{\nu} : \tilde{C}_X \rightarrow C_{X,1}$ if $C_{X,1} = C_{X,2}$) that is $n_{X,i} : 1$ on each component and where $\tilde{C}_{X,i}$ is the normalization of $C_{X,i}$, and an identification morphism $f : \tilde{C}_{X,1} \cup \tilde{C}_{X,2} \simeq \mathbb{P}^1 \cup \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

If $\tilde{\nu}$ contracts one of the two components of \tilde{C}_X , say $\tilde{C}_{X,2}$, to a point $x_X \in S$ (it is easily seen that it cannot contract both), then $\mu(X) \subset \text{Sym}^2(S)$ is of the form $\{x_X + C_X\}$, for an irreducible curve $C_X \subset S$, which is necessarily rational. It is easily seen that $C_X = \tilde{\nu}(\tilde{C}_{X,1})$ and $\deg \tilde{\nu}|_{\tilde{C}_{X,1}} = 1$, so that we obtain:

- (III) the data of an irreducible rational curve $C_X \subset S$ together with a point $x_X \in S$.

Note that in all cases (I)-(III), the support of the curve C_X on S is simply

$$(2.2) \quad \text{Supp}(C_X) = \text{one-dimensional part of } \{x \in S \mid x \in \text{Supp}(\xi) \text{ for some } \xi \in X\}$$

and the set is already purely one-dimensional precisely unless we are in case (III) with $x_X \notin C$.

Conversely from the data (I), (II) or (III) one recovers an irreducible rational curve in $S^{[2]}$ not contained in Δ . Indeed, in case (I) (resp. (II)), the \mathfrak{g}_2^1 on \tilde{C}_X (respectively, the identification f)

induces a $\mathbb{P}^1 \subset \mathrm{Sym}^2(\tilde{C}_X)$ and this is mapped to an irreducible rational curve in $\mathrm{Sym}^2(S)$ by the natural composed morphism

$$\mathrm{Sym}^2(\tilde{C}_X) \xrightarrow{\tilde{\nu}^{(2)}} \mathrm{Sym}^2(C_X) \hookrightarrow \mathrm{Sym}^2(S).$$

The irreducible rational curve $X \subset S^{[2]}$ is the strict transform by μ of this curve. In case (III), $X \subset S^{[2]}$ is the strict transform by μ of $\{x_X + C_X\} \subset \mathrm{Sym}^2(S)$.

We see that the data (III) correspond precisely to rational curves of type $\{x_0 + C\} \subset \mathrm{Sym}^2(S)$, where $x_0 \in S$ is a point and $C \subset S$ is an irreducible rational curve. Moreover, it is easily seen that the data (II) correspond precisely to the images by

$$\alpha : \tilde{C}_1 \times \tilde{C}_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow C_1 + C_2 \subset \mathrm{Sym}^2(S),$$

resp.

$$\alpha : \mathrm{Sym}^2(\tilde{C}) \simeq \mathbb{P}^2 \longrightarrow \mathrm{Sym}^2(C) \subset \mathrm{Sym}^2(S),$$

of irreducible rational curves in $|n_1F_1 + n_2F_2|$ for $n_1, n_2 \in \mathbb{N}$, resp. $|nF|$ for an integer $n \geq 2$, where $\mathrm{Pic}(\tilde{C}_1 \times \tilde{C}_2) \simeq \mathbb{Z}[F_1] \oplus \mathbb{Z}[F_2]$, resp. $\mathrm{Pic}(\mathrm{Sym}^2(\tilde{C})) \simeq \mathbb{Z}[F]$, and C_1, C_2 , resp. C , are irreducible rational curves on S and “ \sim ” denotes normalizations. The data of type (II) will however not be studied more in this paper, where we will focus on the other two, mostly on (I).

Note that an irreducible rational curve $X \subset \mathrm{Sym}^2(S)$ arising from rational (resp. elliptic) curves C as in case (I) moves in $\mathrm{Sym}^2(C)$, which is a surface birational to \mathbb{P}^2 (resp. an elliptic ruled surface), and a curve $X \subset \mathrm{Sym}^2(S)$ of the form $\{x_X + C\}$ moves in the threefold $\{S + C\}$, which is birational to a \mathbb{P}^1 -bundle over S , and contains $\mathrm{Sym}^2(C)$.

At the same time, it is well-known that if $\mathrm{kod}(S) \geq 0$, then rational curves on S do not move and elliptic curves move in at most one-dimensional families. This follows for instance from the following general result (that we will later need in the case $p_g = 2$).

Lemma 2.3. *Let S be a smooth, projective surface with $\mathrm{kod}(S) \geq 0$ containing an n -dimensional irreducible family of irreducible curves of geometric genus p_g . Then $n \leq p_g$ and if equality occurs, then either the family consists of a single smooth rational curve; or $\mathrm{kod}(S) \leq 1$ and $n \leq 1$; or $\mathrm{kod}(S) = 0$.*

Proof. This is “folklore”. For a proof see [34]. □

As a consequence, if $\mathrm{kod}(S) \geq 0$, then rational curves in $\mathrm{Sym}^2(S)$ arising from rational or elliptic curves on S move in families of dimension at most two in $\mathrm{Sym}^2(S)$.

On the other hand, irreducible rational curves $X \subset \mathrm{Sym}^2(S)$ arising from curves on S with hyperelliptic normalizations of geometric genus $p_g \geq 2$ (necessarily of type (I)), move in a family whose dimension equals that of the family of curves with hyperelliptic normalizations in which $C \subset S$ moves (by unicity of the \mathfrak{g}_2^1). Apart from some special cases, it is easy to see that the converse is also true. The proof of this is straightforward and left to the reader:

Lemma 2.4. *Let $\{X_b\}_{b \in B}$ be a one-dimensional irreducible family of irreducible rational curves in $\mathrm{Sym}^2(S)$ covering a (dense subset of a) proper, reduced and irreducible surface $Y \subset \mathrm{Sym}^2(S)$ that does not coincide with $\mathrm{Sing}(\mathrm{Sym}^2(S)) \cong S$.*

Then $C = C_{X_b}$ in S for every $b \in B$ (notation as above) if and only if either $Y = \mathrm{Sym}^2(C_0)$, with either $C_0 \subset S$ an irreducible rational curve and $C \equiv nC_0$ for $n \geq 1$, or $C_0 = C \subset S$ an irreducible elliptic curve; or $Y = C + C' := \{p + p' \mid p \in C, p' \in C'\}$, with C an irreducible rational curve and $C' \subset S$ any irreducible curve; or $Y = C_1 + C_2$, with $C_1, C_2 \subset S$ irreducible rational curves and $C = n_1C_1 + n_2C_2$ for $n_1, n_2 \in \mathbb{N}$.

We note that by Lemma 2.3 also the rational curves in $\mathrm{Sym}^2(S)$ arising from singular curves of geometric genus 2 on S move in at most two-dimensional families. We will see below that this is a general phenomenon, under some additional hypotheses. We will focus our attention on curves with hyperelliptic normalizations (of genus $p_g \geq 2$) in Sections 4-7.

2.2. Bend-and-break in $\mathrm{Sym}^2(S)$. Let $V \subseteq \mathrm{Hom}(\mathbb{P}^1, \mathrm{Sym}^2(S))$ be a reduced and irreducible subscheme (not necessarily complete). We consider the universal map

$$(2.5) \quad \mathcal{P}_V := \mathbb{P}^1 \times V \xrightarrow{\Phi_V} \mathrm{Sym}^2(S)$$

and assume that the following two conditions hold:

$$(2.6) \quad \text{For any } v \in V, \Phi_V(\mathbb{P}^1 \times v) \not\subseteq \mathrm{Sing}(\mathrm{Sym}^2(S)) \simeq S; \text{ and}$$

$$(2.7) \quad \text{the natural map } \mathcal{P}_V \longrightarrow \mathrm{Rat}(\mathrm{Sym}^2(S)) \text{ defined by } \Phi_V \text{ is generically finite,}$$

where $\mathrm{Rat}(\mathrm{Sym}^2(S))$ is the union of the components of $\mathrm{Hilb}(\mathrm{Sym}^2(S))$ whose general points correspond to reduced connected curves with rational components, cf. [16, 5.6]. (The latter just means that V induces a flat family of rational curves in $\mathrm{Sym}^2(S)$ of dimension $\dim(V)$). Set

$$(2.8) \quad R_V := \overline{\mathrm{im}(\Phi_V)},$$

the Zariski closure of $\mathrm{im}(\Phi_V)$ in $\mathrm{Sym}^2(S)$. It is the (irreducible) uniruled subvariety of $\mathrm{Sym}^2(S)$ covered by the curves parametrized by V . In the language of [36, Def. 2.3], R_V is the closure of the locus of the family Φ_V . Note that $\dim(R_V) \geq 2$ if $\dim(V) \geq 1$ by (2.7). Moreover (cf. e.g. [23, Prop. 2.1]),

$$(2.9) \quad \dim(R_V) \leq 3 \text{ if } \mathrm{kod}(S) \geq 0.$$

When R_V is a surface, using Mori's bend-and-break technique we obtain the following result. In the statement we underline the fact that the breaking can be made in such a way that, for general $\xi, \eta \in R_V$, two components of the reducible (or non-reduced) member at the border of the family pass through ξ and η , respectively. This will be central in our applications (Proposition 4.3 and § 5, where we prove Theorem 5.2). We give the proof because we could not find in literature precisely the statement we will need.

Lemma 2.10. *Assume that $\dim(V) \geq 3$ and $\dim(R_V) = 2$.*

Let ξ and η be any two distinct general points of R_V . Then there is a curve $Y_{\xi, \eta}$ in R_V such that $Y_{\xi, \eta}$ is algebraically equivalent to $(\Phi_V)_(\mathbb{P}_v^1)$ and either*

- (a) *there is an irreducible nonreduced component of $Y_{\xi, \eta}$ containing ξ and η ; or*
- (b) *there are two distinct, irreducible components of $Y_{\xi, \eta}$ containing ξ and η , respectively.*

Proof. Since $\dim(V) \geq 3$ by assumption, by (2.7) we can pick a one-dimensional smooth subscheme $B = B_{\xi, \eta} \subset V$ parametrizing curves in V such that $(\Phi_V)_*(\mathbb{P}^1 \times v)$ contains both ξ and η , for every $v \in B$. We therefore have a family of rational curves:

$$(2.11) \quad \Phi_B := (\Phi_V)|_B : \mathbb{P}^1 \times B \longrightarrow R_V.$$

and two marked (distinct) points $x, y \in \mathbb{P}^1$ such that $\Phi_B(x \times B) = \xi$ and $\Phi_B(y \times B) = \eta$, such that each $\Phi_B(\mathbb{P}^1 \times v)$ is nonconstant, for any $v \in B$; in particular $\Phi_B(\mathbb{P}^1 \times B)$ is a surface.

As in the proofs of [37, Lemma 1.9] and [36, Cor. II.5.5], let \overline{B} be any smooth compactification of B . Consider the surface $\mathbb{P}^1 \times \overline{B}$. Let $0 \in \overline{B}$ denote a point at the boundary, \mathbb{P}_0^1 the fibre over 0 of the projection onto the second factor and $x_0, y_0 \in \mathbb{P}_0^1 \subset \mathbb{P}^1 \times \overline{B}$ the corresponding marked points. By the *Rigidity Lemma* [37, Lemma 1.6], Φ_B cannot be defined at the point x_0 , as in the proof of [37, Cor. 1.7], and the same argument works for y_0 .

Therefore, to resolve the indeterminacies of the rational map $\Phi_B : \mathbb{P}^1 \times \overline{B} \dashrightarrow R_V$, we must at least blow up $\mathbb{P}^1 \times \overline{B}$ at the points x_0 and y_0 . Now let W be the blow-up of $\mathbb{P}^1 \times \overline{B}$ such that $\overline{\Phi}_B : W \rightarrow R_V$ is an extension of Φ_B , that is, we have a commutative diagram

$$\begin{array}{ccc} W & & \\ \pi \downarrow & \searrow \overline{\Phi}_B & \\ \mathbb{P}^1 \times \overline{B} & \xrightarrow{\Phi_B} & R_V. \end{array}$$

Let $E_{x_0} := \pi^{-1}(x_0)$ and $E_{y_0} := \pi^{-1}(y_0)$. Note that neither of these can be contracted by $\overline{\Phi}_B$, for otherwise Φ_B itself would be defined at x_0 or y_0 .

Therefore the curve $\overline{\Phi}_B(E_{x_0})$ has an irreducible component Γ_ξ containing ξ and the curve $\overline{\Phi}_B(E_{y_0})$ has an irreducible component Γ_η containing η and by construction, $\Gamma_\xi + \Gamma_\eta \subseteq \overline{\Phi}_{B*}(\pi^{-1}(\mathbb{P}^1 \times 0))$ and the latter is the desired curve $Y_{\xi,\eta}$. The two cases (a) and (b) occur as $\Gamma_\xi = \Gamma_\eta$ or $\Gamma_\xi \neq \Gamma_\eta$, respectively. \square

3. RATIONALLY EQUIVALENT ZERO-CYCLES ON SURFACES WITH $p_g > 0$

In this section we extend to the singular case a consequence of Mumford's result on zero-cycles on surfaces with $p_g > 0$ (cf. [43, Corollary p. 203]) and reformulate the results in terms of rational quotients.

3.1. Mumford's Theorem. The main result of this subsection, which we prove in detail for the reader's convenience, relies on the following generalization of Mumford's result (cf. [59, Chapitre 22] and references therein, for a detailed account).

Theorem 3.1. (see [59, Prop. 22.24]) *Let T and Y be smooth projective varieties. Let $Z \subset Y \times T$ be a cycle of codimension equal to $\dim(T)$. Suppose there exists a subvariety $T' \subset T$ of dimension k_0 such that, for all $y \in Y$, the zero-cycle Z_y is rationally equivalent in T to a cycle supported on T' .*

Then, for all $k > k_0$ and for all $\eta \in H^0(T, \Omega_T^k)$, we have

$$[Z]^* \eta = 0 \text{ in } H^0(Y, \Omega_Y^k)$$

where, as customary, $[Z]^* \eta$ denotes the differential form induced on Y by the correspondence Z .

Combining this theorem with Mumford's original "symplectic" argument, we obtain

Corollary 3.2. *Let S be a smooth, irreducible projective surface with $p_g(S) > 0$ and $\Sigma \subset S^{[n]}$ a reduced, irreducible (possibly singular) complete subscheme such that $\mu(\Sigma) \not\subset \text{Sing}(\text{Sym}^n(S))$, where $\mu : S^{[n]} \rightarrow \text{Sym}^n(S)$ is the Hilbert-Chow morphism.*

If there exists a subvariety $\Gamma \subset \text{Sym}^n(S)$ such that $\dim(\Gamma) \leq 1$, $\Gamma \not\subset \text{Sing}(\text{Sym}^n(S))$ and all the zero-cycles parametrized by $\mu(\Sigma)$ are rationally equivalent to zero-cycles supported on Γ , then $\dim(\Sigma) \leq n$.

Proof. Let $\pi : \widetilde{\Sigma} \rightarrow \Sigma \subset S^{[n]}$ be the desingularization morphism of Σ . Let $Z = \Lambda_\pi \subset \widetilde{\Sigma} \times S^{[n]}$ be the graph of π . Then $Z \cong \widetilde{\Sigma}$, so that $\text{codim}(Z) = \dim(S^{[n]})$, as in Theorem 3.1. By assumption, $\mu(\Sigma)$ parametrizes zero-cycles of length n on S that are all rationally equivalent to zero-cycles supported on Γ , with $\dim(\Gamma) \leq 1$. Since $\mu(\Sigma)$ is not contained in $\text{Sing}(\text{Sym}^n(S))$ by assumption, $\mu|_\Sigma : \Sigma \rightarrow \mu(\Sigma)$ is birational. If Γ' denotes the strict transform of Γ under μ , we get that $\dim(\Gamma') \leq 1$.

We can apply Theorem 3.1 with $Z = Y = \widetilde{\Sigma}$, $T = S^{[n]}$ and $T' = \Gamma'$. Thus, for each $k > 1$ and for each $\eta \in H^0(\Omega_{S^{[n]}}^k)$, $[Z]^* \eta = 0$ in $H^0(\widetilde{\Sigma}, \Omega_{\widetilde{\Sigma}}^k)$.

Let $\omega \in H^0(S, K_S)$ be a non-zero 2-form on S . As in [43, Corollary], we define:

$$\omega^{(n)} := \sum_{i=1}^n p_i^*(\omega) \in H^0(S^n, \Omega_{S^n}^2)$$

where S^n is the n^{th} -cartesian product and p_i is the natural projection onto the i^{th} factor, $1 \leq i \leq n$. The form $\omega^{(n)}$ is $\text{Sym}(n)$ -invariant and, since we have that μ is surjective, this induces a canonical 2-form $\omega_\mu^{[n]} \in H^0(S^{[n]}, \Omega_{S^{[n]}}^2)$ (see [43, §1], where $\omega_\mu^{[n]} = \eta_\mu$ in the notation therein). From what we observed above, $[Z]^*(\omega_\mu^{[n]}) = 0$ as a form in $H^0(\tilde{\Sigma}, \Omega_{\tilde{\Sigma}}^2)$. Consider

$$(\text{Sym}^n(S))_0 := \left\{ \xi = \sum_{i=1}^n x_i \mid x_i \neq x_j, 1 \leq i \neq j \leq n \text{ and such that } \omega(x_i) \in \Omega_{S, x_i}^2 \text{ is not } 0 \right\}.$$

Then $(\text{Sym}^n(S))_0 \subset \text{Sym}^n(S)$ is an open dense subscheme that is isomorphic to its preimage via μ in $S^{[n]}$. For each $\xi \in (\text{Sym}^n(S))_0$, ξ is a smooth point and

$$\pi_n : S^n \rightarrow \text{Sym}^n(S)$$

is étale over ξ . Thus, the 2-form $\omega^{(n)} \in H^0(S^n, \Omega_{S^n}^2)$ is non-degenerate on the open subset $(S^n)_0$ of points in the preimage of $(\text{Sym}^n(S))_0$, i.e. it defines a non-degenerate skew-symmetric form on the tangent space of $(S^n)_0$.

Let $\pi_n^0 := \pi_n|_{(S^n)_0}$; since $\pi_n^0 : (S^n)_0 \rightarrow (\text{Sym}^n(S))_0$ is étale, there exists a 2-form

$$\omega_0^{(n)} \in H^0((\text{Sym}^n(S))_0, \Omega_{(\text{Sym}^n(S))_0}^2)$$

such that $\omega^{(n)} = \pi_n^*(\omega_0^{(n)})$ and $\omega_0^{(n)}$ is also non-degenerate. Therefore, the maximal isotropic subspaces of $\omega_0^{(n)}(\xi)$ are n -dimensional.

Now $\Sigma \subset S^{[n]}$ and $\Sigma \cap \mu^{-1}((\text{Sym}^n(S))_0) \neq \emptyset$, since $\mu(\Sigma) \not\subset \text{Sing}(\text{Sym}^n(S))$ by assumption. Since Σ is reduced, let $\xi \in \Sigma \cap \mu^{-1}((\text{Sym}^n(S))_0)$ be a smooth point. Then, since $\Sigma_{\text{smooth}} = \pi^{-1}(\Sigma_{\text{smooth}})$, by abuse of notation we still denote by $\xi \in \tilde{\Sigma}$ the corresponding point. We know that $[Z]^*\omega_\mu^{[n]}(\xi) = 0$ in the tangent space $T_\xi(\tilde{\Sigma})$. Since

$$\xi \in \Sigma_{\text{smooth}} \cap \mu^{-1}((\text{Sym}^n(S))_0) \subset (\text{Sym}^n(S))_0,$$

then $[Z]^*(\omega_\mu^{[n]}) = \omega_0^{(n)}|_{\Sigma_{\text{smooth}} \cap \mu^{-1}((\text{Sym}^n(S))_0)}$. This implies $\dim(\Sigma) \leq n$. \square

3.2. The property *RCC* and rational quotients. Recall that a variety T (not necessarily proper or smooth) is said to be *rationally chain connected* (*RCC*, for brevity), if for each pair of very general points $t_1, t_2 \in T$ there exists a connected curve $\Lambda \subset T$ such that $t_1, t_2 \in \Lambda$ and each irreducible component of Λ is rational (see [36]). Furthermore, by [16, Remark 4.21(2)], if T is proper and *RCC*, then *each pair* of points can be joined by a connected chain of rational curves.

Also recall that, for any smooth variety T , there exists a variety Q , called the *rational quotient* of T , together with a rational map

$$(3.3) \quad f : T \dashrightarrow Q,$$

whose very general fibres are equivalence classes under the *RCC*-equivalence relation (see, for instance, [16, Theorem 5.13] or [36, IV, Thm. 5.4]).

In this language, an equivalent statement of Corollary 3.2 is:

Corollary 3.4. *Let S be a smooth, projective surface with $p_g(S) > 0$. If $Y \subset S^{[n]}$ is a complete subvariety of dimension $> n$ not contained in $\text{Exc}(\mu)$, then any desingularization of Y has a rational quotient of dimension at least two.*

Proof. Let \tilde{Y} be any desingularization of Y and Q its rational quotient. Up to resolving the indeterminacies of $f : \tilde{Y} \dashrightarrow Q$, we may assume that f is a proper morphism whose very general fibre is a *RCC*-equivalence class, so that in particular *each* fibre is *RCC* (see [36, Thm. 3.5.3]).

If $\dim(Q) = 0$, it follows that \tilde{Y} (so also Y) is *RCC*, contradicting Corollary 3.2.

If $\dim(Q) = 1$, then by cutting \tilde{Y} with $\dim(Y) - 1$ general very ample divisors, we get a curve Γ' that intersects every fibre of f . Every point of \tilde{Y} is connected by a chain of rational curves to some point on Γ' . We thus obtain a contradiction by Corollary 3.2 (with Γ the image of Γ' in $\text{Sym}^2(S)$). \square

Let now R_V be the variety covered by a family of rational curves in $\text{Sym}^2(S)$ parametrized by V , as defined in (2.8), \tilde{R}_V be any desingularization of R_V and Q_V be the rational quotient of \tilde{R}_V . Of course $\dim(Q_V) \leq \dim(R_V) - 1$, as R_V is uniruled by construction.

Lemma 3.5. *If $\dim(V) \geq \dim(R_V)$, then $\dim(Q_V) \leq \dim(R_V) - 2$ (for any desingularization \tilde{R}_V of R_V). In particular, if $\dim(V) \geq 2$ and $\dim(R_V) = 2$, then any desingularization of R_V is a rational surface.*

Proof. With notation as in § 2.2, we have $\dim(\mathcal{P}_V) \geq \dim(R_V) + 1$, so that the general fibre of Φ_V is at least one-dimensional, cf. (2.5). This means that, if ξ is a general point of R_V , there exists a family of rational curves in R_V passing through ξ , of dimension ≥ 1 . Of course the same is true for a general point of \tilde{R}_V . Thus, the very general fibre of f in (3.3) has dimension at least two, whence $\dim(Q_V) \leq \dim(R_V) - 2$. The last statement follows from the fact that any smooth surface that is *RCC* is rational (cf. [36, IV.3.3.5]). \square

Combining Corollary 3.4 and Lemma 3.5, we then get:

Proposition 3.6. *If $p_g(S) > 0$ and $\dim(V) \geq 2$, then either*

- (i) *R_V is a surface with rational desingularization; or*
- (ii) *$\dim(V) = 2$, R_V is a threefold and any desingularization of R_V has a two-dimensional rational quotient.*

Proof. By (2.9), $\dim(R_V) = 2$ or 3 . If $\dim(R_V) = 2$, then (i) holds by Lemma 3.5. If $\dim(R_V) = 3$, then $\dim(Q_V) = 2$ by Corollary 3.4. Hence $\dim(V) = 2$ by Lemma 3.5 and (ii) holds. \square

Remark 3.7. Let S be a smooth, projective surface with $p_g(S) > 0$ and let $Y \subset S^{[2]}$ be a uniruled threefold different from $\text{Exc}(\mu)$, where $\mu : S^{[2]} \rightarrow \text{Sym}^2(S)$ is the Hilbert-Chow morphism.

Take a covering family $\{C_v\}_{v \in V}$ of rational curves on Y . By Corollary 3.4 the family must be two-dimensional (see Lemma 3.5). Then the curves in the covering family yield, via the correspondence described in § 2.1, curves on S with rational, elliptic or hyperelliptic normalizations, and the correspondence is one-to-one in the hyperelliptic case. We therefore see that we must be in one of the following cases:

- (a) S contains an irreducible rational curve Γ and

$$Y = \{\xi \in S^{[2]} \mid \text{Supp}(\xi) \cap \Gamma \neq \emptyset\};$$

- (b) S contains a one-dimensional irreducible family $\{E\}_{v \in V}$ of irreducible elliptic curves and

$$Y = \overline{\{\xi \in E_v^{[2]}\}_{v \in V}};$$

- (c) S contains a two-dimensional, irreducible family of irreducible curves with hyperelliptic normalizations, not contained in a higher dimensional irreducible family, and Y is the locus covered by the corresponding rational curves in $S^{[2]}$.

(Note that in fact case (b) can only occur for $\text{kod}(S) \leq 1$ by Lemma 2.3 and case (c) only when $|K_S|$ is not birational. The latter fact is easy to see, cf. e.g. [34].)

In the case of $K3$ surfaces, uniruled divisors play a particularly important rôle [32, §5]. Now all cases (a)-(c) above occur on a general, projective $K3$ surface with a polarization of genus ≥ 6 . In fact, cases (a) and (b) occur on any projective $K3$ surface since it necessarily contains a one-dimensional family of irreducible, elliptic curves and a zero-dimensional family of rational curves, by a well-known theorem of Mumford (see the proof in [39, pp. 351-352] or [2, pp. 365-367]). Case (c) occurs on a general primitively polarized $K3$ surface of genus $p \geq 6$ by Corollary 5.3 below with a family of curves of geometric genus 3. In addition to this, in Proposition 7.7 we will see that there is another threefold as in (c) arising from curves of geometric genus > 3 in the hyperplane linear system on general projective $K3$ surfaces of infinitely many degrees.

Moreover, there is not a one-to-one correspondence between families as in (a)-(c) above and uniruled threefolds in $S^{[2]}$. In fact, in Proposition 7.2 we will see that there is a two-dimensional family of curves with hyperelliptic normalizations, as in (c), in the hyperplane linear systems on general $K3$ surfaces of infinitely many degrees whose associated rational curves cover only a \mathbb{P}^2 in $S^{[2]}$.

4. FAMILIES OF CURVES WITH HYPERELLIPTIC NORMALIZATIONS

The purpose of this section is to study the dimension of families of curves on a smooth projective surface S with hyperelliptic normalizations.

It is not difficult to see that if $|K_S|$ is birational, then the dimension of such a family is forced to be zero (see e.g. [34]). At the same time it is easy to find obvious examples of surfaces, even with $p_g(S) > 0$, with large families of curves with hyperelliptic normalizations, namely surfaces admitting a finite $2 : 1$ map onto a rational surface. (For examples of such cases, see e.g. [26, 27, 28, 29, 49, 52, 54, 10] to mention a few.) In these cases one can pull back the families of rational curves on the rational surface to obtain families of curves on S with hyperelliptic normalizations of arbitrarily high dimensions. Moreover, in Proposition 7.2 below we will see that even a general, primitively polarized $K3$ surface (S, H) , for infinitely many degrees, contains a \mathbb{P}^2 in its Hilbert square, which is not contained in Δ (but the surface is not a double cover of a \mathbb{P}^2 , by generality). Therefore, by the correspondence in § 2.1, S contains large families of curves with hyperelliptic normalizations. One can see that in all these examples of large families the algebraic equivalence class of the members breaks into nontrivial effective decompositions. For example, in the mentioned $K3$ case of Proposition 7.2, we will see that the curves in $|\mathcal{O}_{\mathbb{P}^2}(n)|$ in $\mathbb{P}^2 \subset S^{[2]}$ correspond to curves in $|nH|$. In this section we will see that this is a general phenomenon, with the help of Lemma 2.10.

To this end, let V be a reduced and irreducible scheme parametrizing a flat family of curves on S all having constant geometric genus $p_g \geq 2$ and hyperelliptic normalizations. Let $\varphi : \mathcal{C} \rightarrow V$ be the universal family. Normalizing \mathcal{C} we obtain, possibly restricting to an open dense subscheme of V , a flat family $\tilde{\varphi} : \tilde{\mathcal{C}} \rightarrow V$ of smooth hyperelliptic curves of genus $p_g \geq 2$ (cf. [57, Thm. 1.3.2]). Let $\omega_{\tilde{\mathcal{C}}/V}$ be the relative dualizing sheaf. As in [38, Thm. 5.5 (iv)], consider the morphism $\gamma : \tilde{\mathcal{C}} \rightarrow \mathbb{P}(\tilde{\varphi}_*(\omega_{\tilde{\mathcal{C}}/V}))$ over V . This morphism is finite and of relative degree two onto its image, which we denote by \mathcal{P}_V . We thus obtain a universal family $\psi : \mathcal{P}_V \rightarrow V$ of rational curves mapping to $\text{Sym}^2(S)$, as in (2.5), satisfying (2.6) and (2.7). (Strictly speaking, (2.5) denoted a universal family of *maps*, whereas it now denotes a universal family of *curves*.) To summarize, recalling (2.8), we have

$$(4.1) \quad \begin{array}{ccccc} & & \tilde{\mathcal{C}} & \xrightarrow{\gamma} & \mathcal{P}_V & \xrightarrow{\Phi_V} & R_V \\ & \swarrow \pi & & \searrow \tilde{\varphi} & \downarrow \psi & & \\ & S & & & V & & \end{array}$$

Also note that (4.1) is compatible with the correspondence of case (I) in § 2.1, in the sense that, for general $v \in V$, we have (using the same notation as in § 2.1)

$$(4.2) \quad \pi(\tilde{\varphi}^{-1}(v)) = p_S(p_2^{-1}X_v) = (p_S)_*(p_2^{-1}X_v) = C_{X_v}, \text{ with } X_v = \mu_*^{-1}(\Phi_V(\psi^{-1}(v))) \subset S^{[2]},$$

where μ is the Hilbert-Chow morphism (in particular, p_S and p_2 are the first and second projections, respectively, from the incidence variety $T \subset S \times S^{[2]}$). Note that the second equality in (4.2) follows as p_S is generically one-to-one on the curves in question, as we saw in § 2.1. This will be central in the proof of the next result. We now apply Lemma 2.10 to “break” the curves on S .

Proposition 4.3. *Let S be a smooth, projective surface and V and R_V as above. Assume that $\dim(V) \geq 3$ and $\dim(R_V) = 2$ and let $[C]$ be the algebraic equivalence class of the members parametrized by V .*

Then there is a decomposition into two effective, algebraically moving classes

$$[C] = [D_1] + [D_2]$$

such that, for general $\xi, \eta \in R_V$, there are effective divisors $D_1' \sim_{alg} D_1$ and $D_2' \sim_{alg} D_2$ such that $\xi \subset D_1'$ and $\eta \subset D_2'$ and $[D_1' + D_2'] \in \overline{V}$, where \overline{V} is the closure of V in the component of the Hilbert scheme of S containing V .

Proof. For general $\xi, \eta \in R_V$, both being supported at two distinct points on S , let $B = B_{\xi, \eta} \subset V$ be as in the proof of Lemma 2.10 and \overline{B} be any smooth compactification of B . By abuse of notation, we will consider ξ and η as being points in $S^{[2]}$. By (the proof of) Lemma 2.10, using the Hilbert-Chow morphism, there is a flat family $\{X_b\}_{b \in \overline{B}}$ of curves in the surface $\mu_*^{-1}(R_V) \subset S^{[2]}$ (where μ is the Hilbert-Chow morphism as usual) parametrized by \overline{B} , such that, for general $b \in B$, X_b is an irreducible rational curve and

$$(4.4) \quad C_{X_b} = (p_S)_*(p_2^{-1}(X_b)) = \pi(\tilde{\varphi}^{-1}(b)),$$

with notation as in § 2.1 (cf. (4.2)). In particular, $\{C_{X_b}\}_{b \in B}$ is a one-dimensional nontrivial subfamily of the family $\{C_{X_v}\}_{v \in V}$ given by V . Moreover, for some $b_0 \in \overline{B} \setminus B$, we have $X_{b_0} \supseteq Y_\xi + Y_\eta$, where Y_ξ and Y_η are irreducible rational curves (possibly coinciding) such that $\xi \in Y_\xi$ and $\eta \in Y_\eta$. Also note that $Y_\xi, Y_\eta \not\subset \Delta \subset S^{[2]}$.

Pulling back to the incidence variety $T \subset S \times S^{[2]}$, we obtain a flat family $\{X_b' := p_2^{-1}(X_b)\}_{b \in \overline{B}}$ of curves in T , such that

$$(4.5) \quad X_{b_0}' := p_2^{-1}(X_{b_0}) \supseteq p_2^{-1}(Y_\xi) + p_2^{-1}(Y_\eta) =: Y_\xi' + Y_\eta'.$$

Note that the family $\{X_b'\}_{b \in \overline{B}}$ is in fact a family of curves in the incidence variety $T_0 \subset S \times \mu_*^{-1}(R_V)$, which is a surface contained in T . Since p_S maps this family to a family of curves covering (an open dense subset of) S , by (4.4), we see that $(p_S)|_{T_0}$ is surjective, in particular generically finite. Thus, choosing ξ and η general enough, we can make sure they lie outside of the images by p_2 of the finitely many curves contracted by $(p_S)|_{T_0}$. Hence $p_2^{-1}(Y_\xi)$ and $p_2^{-1}(Y_\eta)$ are not contracted by p_S .

Therefore, recalling (4.4) and (4.5) and letting $b' \in B$ be a general point, we get

$$C \sim_{alg} (p_S)_* X_{b'}' \sim_{alg} (p_S)_* X_{b_0}' \supseteq (p_S)_* Y_\xi' + (p_S)_* Y_\eta' \supseteq D_\xi + D_\eta,$$

where $D_\xi := p_S(p_2^{-1}Y_\xi)$ and $D_\eta := p_S(p_2^{-1}Y_\eta)$.

By construction we have $D_\xi \supset \xi$ and $D_\eta \supset \eta$, viewing ξ and η as length-two subschemes of S . (Note that D_ξ and D_η are not necessarily distinct.) Possibly after adding additional components to D_ξ and D_η , we can in fact assume that

$$C \sim_{alg} (p_S)_* X_{b'}' = D_\xi + D_\eta,$$

with D_ξ and D_η not necessarily reduced and irreducible. Since this construction can be repeated for general $\xi, \eta \in R_V$ and the set $\{x \in S \mid x \in \text{Supp}(\xi) \text{ for some } \xi \in R_V\}$ is dense in S , as the curves parametrized by V cover the whole surface S , the obtained curves D_ξ and D_η must move in an algebraic system of dimension at least one.

By construction, $D_\xi + D_\eta$ lies in the border of the family $\varphi : \mathcal{C} \rightarrow V$ of curves on S , and as such, $[D_\xi + D_\eta]$ lies in the closure of V in the component of the Hilbert scheme of S containing V . Moreover, as the number of such decompositions is finite (as S is projective and the divisors are effective), we can find one decomposition $[C] = [D_1] + [D_2]$ holding for general $\xi, \eta \in R_V$. \square

The next two results are immediate consequences:

Theorem 4.6. *Let S be a smooth, projective surface with $p_g(S) > 0$. Then the following conditions are equivalent:*

- (i) $S^{[2]}$ contains an irreducible surface R with rational desingularization, such that $R \neq \mu_*^{-1}(C_1 + C_2)$, $\mu_*^{-1}(\text{Sym}^2(C))$ for rational curves $C, C_1, C_2 \subset S$ and $R \not\subset \text{Exc}(\mu)$, where $\mu : S^{[2]} \rightarrow \text{Sym}^2(S)$ is the Hilbert-Chow morphism;
- (ii) S contains a flat family of irreducible curves with hyperelliptic normalizations of geometric genus $p_g \geq 3$, parametrized by a reduced and irreducible scheme V such that $\dim(V) \geq 3$.

Furthermore, if any of the above conditions holds, then

- (a) the rational curves in $S^{[2]}$ that correspond to the irreducible curves parametrized by V , cover only the surface R in $S^{[2]}$; and
- (b) the algebraic equivalence class $[C]$ of the curves parametrized by V has an effective decomposition $[C] = [D_1] + [D_2]$ into algebraically moving classes such that, for general $\xi, \eta \in R$, there are effective divisors $D'_1 \sim_{\text{alg}} D_1$ and $D'_2 \sim_{\text{alg}} D_2$ such that $\xi \subset D'_1$, $\eta \subset D'_2$ and such that the point parametrizing $D'_1 + D'_2$ lies in the closure \bar{V} of V in the component of the Hilbert scheme of S containing V .

Proof. Assume (ii) holds. By Proposition 3.6 we have that $R_V \subset \text{Sym}^2(S)$ is a surface with rational desingularization, so that (i) holds.

Assume now that (i) holds. Then R carries a family of rational curves of dimension $n \geq 3$. By Lemma 2.4 and the assumptions in (i), this yields an n -dimensional family of curves on S that have rational, elliptic or hyperelliptic normalizations. From Lemma 2.3, we get (ii).

Finally, assume that these conditions hold. Then (a) follows from Proposition 3.6 again, where R is the proper transform via μ of the surface R_V therein; finally, (b) follows from Proposition 4.3. \square

Corollary 4.7. *Let S be a smooth, projective surface with $p_g(S) > 0$ and V be a reduced, irreducible scheme parametrizing a flat family of irreducible curves with hyperelliptic normalizations (of geometric genus ≥ 2). Denote by $[C]$ the algebraic equivalence class of the members of V .*

If $[C]$ has no decomposition into effective, algebraically moving classes, then $\dim(V) \leq 2$.

In particular, Corollary 4.7 holds when e.g. $NS(S) = \mathbb{Z}[C]$.

The examples with the double covers of smooth rational surfaces and the result in Proposition 7.2 mentioned above, show that the results above are natural.

The statement in Theorem 4.6(b) shows that in fact the length-two zero-dimensional schemes on the curves in the family corresponding to the elements of the \mathfrak{g}_2^1 s on their normalization, are in fact “generically cut out” by moving divisors in a fixed algebraic decomposition of the class of the members in the family. This reminds of the nowadays well-known results of Reider and their generalizations [48, 8, 9]. In fact, Theorem 4.6(b) can be used to prove a Reider-like result involving the arithmetic and geometric genera of the curves in the family, cf. [34]. Moreover, the precise statement in Theorem 4.6(b) will be crucial in the next section, where we will prove existence of curves with hyperelliptic normalizations by degeneration methods.

5. NODAL CURVES OF GEOMETRIC GENUS 3 WITH HYPERELLIPTIC NORMALIZATIONS ON $K3$ SURFACES

In the rest of the paper we will focus on the existence of curves with “Brill-Noether special” hyperelliptic normalizations (i.e. of geometric genera > 2) and in this section we will see that Theorem 4.6(b) is particularly suitable to prove existence results by degeneration arguments.

To do this and to discuss some consequences on $S^{[2]}$, we will in the rest of the paper focus on $K3$ surfaces, which in fact were one of our original motivations for this work.

We start with the following observation combining a result of Ran, already mentioned in the Introduction, with the results from the previous section.

Lemma 5.1. *Let S be a smooth, projective $K3$ surface and L be a globally generated line bundle of sectional genus $p \geq 2$ on S . Let $|L|^{hyper} \subseteq |L|$ be the subscheme parametrizing irreducible curves in $|L|$ with hyperelliptic normalizations.*

Then, any irreducible component of $|L|^{hyper}$ has dimension ≥ 2 , with equality holding if L has no decomposition into moving classes.

Proof. Any n -dimensional component of $|L|^{hyper}$ yields an n -dimensional family of irreducible rational curves in $S^{[2]}$. By [47, Cor. 5.1], we have $n \geq 2$. The last statement follows from Corollary 4.7. \square

The main aim of this section is to apply Theorem 4.6(b) to prove:

Theorem 5.2. *Let (S, H) be a general, smooth, primitively polarized $K3$ surface of genus $p = p_a(H) \geq 4$. Then the family of nodal curves in $|H|$ of geometric genus 3 with hyperelliptic normalizations is nonempty, and each of its irreducible components is two-dimensional.*

In [21] we studied which linear series may appear on normalizations of irreducible curves on $K3$ surfaces. To do so, we introduced a singular Brill-Noether number $\rho_{sing}(p_a, r, d, p_g)$ whose negativity, when $\text{Pic}(S) \simeq \mathbb{Z}[H]$, ensures non-existence of curves in $|H|$, with $p_a = p_a(H)$ and of geometric genus p_g , having normalizations admitting a \mathfrak{g}_d^r (we will return to this in § 6.3 below). Moreover, in [21, Examples 2.8 and 2.10], we already gave examples of nodal curves with hyperelliptic normalizations with geometric genus 3 and arithmetic genus 4 or 5. Theorem 5.2 shows that this is a general phenomenon. The proof will be given in the remainders of this section. Moreover, we will also determine the dimension of the locus covered in $S^{[2]}$ by the rational curves associated to curves in a component of the family:

Corollary 5.3. *Let (S, H) be a general, smooth, primitively polarized $K3$ surface of genus $p = p_a(H) \geq 6$. Then the subscheme of $|H|$ parametrizing nodal curves of geometric genus 3 with hyperelliptic normalizations contains a two-dimensional component V such that $\dim(R_V) = 3$.*

This corollary in particular shows that all three cases in Remark 3.7 occur on a general $K3$ surface. In § 6.2-6.3 we will both compute the classes of the corresponding rational curves in $S^{[2]}$ (see (6.25)) and discuss some of the consequences of Theorem 5.2 on the Mori cone of $S^{[2]}$.

Before starting on the proof of Theorem 5.2, we recall that, for any smooth surface S and any line bundle L on S , such that $|L|$ contains smooth, irreducible curves of genus $p := p_a(L)$, and any positive integer $\delta \leq p$, one denotes by $V_{|L|, \delta}$ the locally closed and functorially defined subscheme of $|L|$ parametrizing the universal family of irreducible curves in $|L|$ having δ nodes as the only singularities and, consequently, geometric genus $p_g := p - \delta$. These are classically called *Severi varieties* of irreducible, δ -nodal curves on S in $|L|$.

It is nowadays well-known, as a direct consequence of Mumford’s theorem on the existence of nodal rational curves on $K3$ surfaces (see the proof in [39, pp. 351-352] or [2, pp. 365-367]) and standard results on Severi varieties, that if (S, H) is a general, primitively polarized $K3$ surface of genus $p \geq 3$,

then the Severi variety $V_{|H|,\delta}$ is nonempty and *regular*, i.e. it is smooth and of the expected dimension $p - \delta$, for each $\delta \leq p$ (cf. [56, Lemma 2.4 and Theorem 2.6]; see also e.g. [15, 20]).

The regularity property follows from the fact that, since by definition $V_{|L|,\delta}$ parametrizes irreducible curves, the nodes of these curves impose independent conditions on $|L|$ (cf. [15, 20] and [56, Remark 2.7]). From equisingular deformation theory, this implies that suitable obstructions to some locally trivial deformations are zero. In other words, it implies first that, for any $\delta' > \delta$, $V_{|L|,\delta'} \subset \overline{V}_{|L|,\delta}$ (see [53, Anhang F], [60] and [51, Thm. 4.7.18] for \mathbb{P}^2 and [56, § 3] for $K3$ s). Furthermore, if $[C] \in V_{|L|,\delta+k}$, $k > 0$, is a general point of an irreducible component, the fact that the nodes impose independent conditions allows to clearly describe what $\overline{V}_{|L|,\delta}$ looks like locally around the point $[C]$: it is the union of $\binom{\delta+k}{\delta}$ smooth branches through $[C]$, each branch corresponding to a choice of δ "marked" (or "assigned") nodes among the $\delta+k$ nodes of C , and these branches intersect transversally at $[C]$; moreover, the other k "unassigned" nodes of C disappear when one deforms $[C]$ in the corresponding branch of $\overline{V}_{|L|,\delta}$ (see [53, Anhang F], [60] and [50, § 1] for \mathbb{P}^2 and [56, § 3] for $K3$ s).

The situation is slightly different for *reducible*, nodal curves in $|L|$. Since they appear in the proof of Theorem 5.2, we also have to take care of this case. To this end, we define the "degenerated" version of $V_{|L|,\delta}$ by

$$(5.4) \quad W_{|L|,\delta} := \left\{ C \in |L| \mid C, \text{ not necessarily irreducible, has only nodes as singularities and at least } \delta \text{ nodes} \right\}.$$

For the same reasons as above, $W_{|L|,\delta}$ is a locally closed subscheme of $|L|$. Note that

$$(5.5) \quad W_{|L|,\delta} = \cup_{\delta' \geq \delta} V_{|L|,\delta'}$$

which is a partial compactification of $V_{|L|,\delta}$.

Let $[C] \in W_{|L|,\delta}$. Choosing any subset $\{p_1, \dots, p_\delta\}$ of δ of its nodes, one obtains a *pointed curve* $(C; p_1, \dots, p_\delta)$, where p_1, \dots, p_δ are also called the *marked* (or *assigned*) nodes of C (cf. [56, Definitions 3.1-(ii) and 3.6-(i)]).

Recall that there exists an algebraic scheme, which we denote by

$$(5.6) \quad \mathcal{B}(C; p_1, p_2, \dots, p_\delta),$$

locally closed in $|L|$, representing the functor of infinitesimal deformations of C in $|L|$ that preserve the marked nodes, i.e. the functor of locally trivial infinitesimal deformations of the pointed curve $(C; p_1, \dots, p_\delta)$ (cf. [56, Proposition 3.3], where we have identified the schemes therein with their projections into the linear system $|L|$). In other words, $\mathcal{B}(C; p_1, p_2, \dots, p_\delta)$ is the *local branch* of $W_{|L|,\delta}$ around $[C] \in W_{|L|,\delta}$, corresponding to the choice of the δ marked nodes. We have:

Theorem 5.7. (cf. [56, Theorem 3.8]) *Let $(C; p_1, \dots, p_\delta)$ be as above. Assume that the general element of $|L|$ is a smooth, irreducible curve and that the partial normalization of C at the δ marked nodes p_1, \dots, p_δ is a connected curve.*

Then $\mathcal{B}(C; p_1, p_2, \dots, p_\delta)$ is smooth at the point $[(C; p_1, p_2, \dots, p_\delta)]$ of dimension $\dim(|L|) - \delta$.

Proof. This follows from [56, Theorem 3.8] since, by our assumptions, the pointed curve $(C; p_1, \dots, p_\delta)$ is *virtually connected* in the language of [56, Definition 3.6]. \square

For the proof of Theorem 5.2 we need to recall other fundamental facts. We first define, for any globally generated line bundle L of sectional genus $p := p_a(L) \geq 2$, on a $K3$ surface S , and any integer δ such that $0 < \delta \leq p - 2$, the locus in the Severi variety $V_{|L|,\delta}$,

$$(5.8) \quad V_{|L|,\delta}^{hyper} := \left\{ C \in V_{|L|,\delta} \mid \text{its normalization is hyperelliptic} \right\}.$$

Observe that in particular, for any $p \geq 3$, one always has $V_{|L|,p-2}^{hyper} = V_{|L|,p-2} \neq \emptyset$ and, by regularity of $V_{|L|,p-2}$, this is smooth and of dimension two.

Let \mathcal{M}_g be the moduli space of smooth curves of genus g , which is quasi-projective of dimension $3g - 3$ for $g \geq 2$. Denote by $\overline{\mathcal{M}}_g$ its Deligne-Mumford compactification. Then $\overline{\mathcal{M}}_g$ is the moduli space of stable, genus g curves. Let $\mathcal{H}_g \subset \mathcal{M}_g$ denote the locus of hyperelliptic curves, which is known to be an irreducible variety of dimension $2g - 1$ (see e.g. [1]) and $\overline{\mathcal{H}}_g \subset \overline{\mathcal{M}}_g$ be its compactification.

Moreover, recall from [22, Def.(3.158)] that a nodal curve C (not necessarily irreducible) is *stably equivalent* to a stable curve C' if C' is obtained from C by contracting to a point all smooth rational components of C meeting the other components in only one or two points.

As above, we define the degenerated version of $V_{|L|,\delta}^{hyper}$ by

$$(5.9) \quad W_{|L|,\delta}^{hyper} := \left\{ C \in W_{|L|,\delta} \mid \text{there exists a desingularization } \tilde{C} \text{ of } \delta \text{ of the} \right. \\ \left. \text{nodes of } C, \text{ such that } \tilde{C} \text{ is stably equivalent to a} \right. \\ \left. \text{(stable) curve } C' \text{ with } [C'] \in \overline{\mathcal{H}}_{p_a(L)-\delta} \right\}.$$

Note that, by definition, any such \tilde{C} is connected. Similarly as in (5.5), we have:

$$(5.10) \quad W_{|L|,\delta}^{hyper} = \cup_{\delta' \geq \delta} V_{|L|,\delta'}^{hyper} \text{ if all the curves in } |L| \text{ are irreducible.}$$

Theorem 5.2 will be a direct consequence of the next three results, Propositions 5.11 and 5.19 and Lemma 5.20. The central degeneration argument is given by the following:

Proposition 5.11. *Let $p \geq 3$ and $\delta \leq p - 2$ be positive integers. Assume there exists a smooth K3 surface S_0 with a globally generated, primitive line bundle H_0 on S_0 with $p_a(H_0) = p$ and such that $W_{|H_0|,\delta}^{hyper}(S_0) \neq \emptyset$ and $\dim(W_{|H_0|,\delta}^{hyper}(S_0)) \leq 2$.*

Then, on the general, primitively marked K3 surface (S, H) of genus p , $W_{|H|,\delta}^{hyper}(S)$ is nonempty and equidimensional of dimension two.

Proof. Let \mathcal{B}_p be the moduli space of primitively marked K3 surfaces of genus p . It is well-known that \mathcal{B}_p is smooth and irreducible of dimension 19, cf. e.g. [2, Thm.VIII 7.3 and p. 366]. We let $b_0 = [(S_0, H_0)] \in \mathcal{B}_p$. Similarly as in [5], consider the *scheme of pairs*

$$(5.12) \quad \mathcal{W}_{p,\delta} := \left\{ (S, C) \mid [(S, H)] \in \mathcal{B}_p \text{ and } [C] \in W_{|H|,\delta}(S) \right\},$$

and the natural projection

$$(5.13) \quad \pi : \mathcal{W}_{p,\delta} \longrightarrow \mathcal{B}_p.$$

(The fact that $\mathcal{W}_{p,\delta}$ is a scheme, in fact a locally closed scheme, follows from the already mentioned proof of Mumford's theorem on the existence of nodal rational curves as in [39, pp. 351-352] or [2, pp. 365-367].)

Note that for general $[(S_b, H_b)] = b \in \mathcal{B}_p$ we have

$$\pi^{-1}(b) = \cup_{\delta' \geq \delta} V_{|H_b|,\delta'}(S_b)$$

by (5.5) (as $\text{Pic}(S_b) \simeq \mathbb{Z}[H_b]$), so that $\pi^{-1}(b)$ is nonempty, equidimensional and of dimension $g := p - \delta$, by the regularity property recalled above. In particular, π is dominant. Observe that $\mathcal{W}_{p,\delta}$ is singular in codimension one, so in particular it is not normal.

For brevity, let $\mathcal{W} := \mathcal{W}_{p,\delta}$ and let $\mathcal{C} \xrightarrow{f} \mathcal{W}$ be the universal curve. As in Theorem A.1, (i) and (ii), in Appendix A, there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ f' \downarrow & & \downarrow f \\ \mathcal{W}_{(\delta)} & \xrightarrow{\alpha} & \mathcal{W}, \end{array}$$

where α is a finite, unramified morphism defining a marking of all the δ -tuples of nodes of the fibres of f (cf. Theorem A.1, with $V = \mathcal{W}$, $E_{(\delta)} = \mathcal{W}_{(\delta)}$). Precisely, by using notation as in Theorem A.1, if for $w \in \mathcal{W}$ the curve $\mathcal{C}(w)$ has $\delta + \tau$ nodes, $\tau \in \mathbb{Z}^+$, $\alpha^{-1}(w)$ consists of $\binom{\delta + \tau}{\delta}$ elements, since any $\eta_w \in \alpha^{-1}(w)$ parametrizes an unordered, marked δ -tuple of the $\delta + \tau$ nodes of $\mathcal{C}(w)$.

Let $\eta_w \in \mathcal{W}_{(\delta)}$. Then η_w is represented by a pointed curve $(C; p_1, p_2, \dots, p_\delta)$, where $(S, C) \in \mathcal{B}_p$ and where $p_1, p_2, \dots, p_\delta$ are δ marked nodes on C .

Let $\mathcal{W}(S, H)$ (resp. $\mathcal{W}_{(\delta)}(S, H)$) be the fibre of π (resp. of $\alpha \circ \pi$) over $[(S, H)] \in \mathcal{B}_p$, and let

$$\alpha(S, H) : \mathcal{W}_{(\delta)}(S, H) \longrightarrow \mathcal{W}(S, H)$$

be the induced morphism. For $\eta_w \in \mathcal{W}_{(\delta)}(S, H)$ as above, we have

$$(5.14) \quad T_{[\eta_w]}(\mathcal{W}_{(\delta)}(S, H)) \cong T_{[(C; p_1, p_2, \dots, p_\delta)]}(\mathcal{B}(C; p_1, p_2, \dots, p_\delta)),$$

where $\mathcal{B}(C; p_1, p_2, \dots, p_\delta)$ is as in (5.6). Indeed, since α is finite and unramified, then also $\alpha(S, H)$ is. Therefore, it suffices to consider the image of the differential $d\alpha(S, H)_{[\eta_w]}$. The latter is given by first-order deformations of C in S (equivalently in $|H|$) that are locally trivial at the δ marked nodes; these are precisely given by $T_{[(C; p_1, p_2, \dots, p_\delta)]}(\mathcal{B}(C; p_1, p_2, \dots, p_\delta))$ (cf. [56, Remark 3.5]).

Let $\tilde{\mathcal{W}}_{(\delta)}$ be the smooth locus of $\mathcal{W}_{(\delta)}$. By Theorem 5.7 and by (5.14), together with the fact that \mathcal{B}_p is smooth, $\tilde{\mathcal{W}}_{(\delta)}$ contains all the pairs (S, C) with δ marked nodes on C , such that $|C|$ is globally generated (i.e. its general element is a smooth, irreducible curve) and the partial normalization of C at these marked nodes is a connected curve. More precisely, by the proof of Mumford's theorem on the existence of nodal rational curves on $K3$ surfaces, as in [39, pp. 351-352] or [2, pp. 365-367]), any irreducible component of $\mathcal{W}_{(\delta)}$ has dimension $\geq 19 + p - \delta = 19 + g$; furthermore, by (5.14), $\dim(T_{[\eta_w]}(\mathcal{W}_{(\delta)}(S, H))) = g$, where η_w represents (S, C) with C with the δ marked nodes. It also follows that $\tilde{\mathcal{W}}_{(\delta)}$ is smooth, of dimension $19 + g$ at these points.

If we restrict \mathcal{C}' to $\tilde{\mathcal{W}}_{(\delta)}$, from Theorem A.1, (iv) and (v), we have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \longrightarrow & \mathcal{C} \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{\mathcal{W}}_{(\delta)} & \xrightarrow{\tilde{\alpha}} & \mathcal{W}, \end{array}$$

where $\tilde{\alpha} = \alpha|_{\tilde{\mathcal{W}}_{(\delta)}}$ and where \tilde{f} is the flat family of partial normalizations at δ nodes of the curves parametrized by $\alpha(\tilde{\mathcal{W}}_{(\delta)})$ (in the notation of Theorem A.1 in Appendix A, $\tilde{f} = \bar{f}$ in (v) and $\tilde{\mathcal{C}} = \bar{\mathcal{C}}$ in (iii) and (iv)).

There is an obvious rational map

$$\tilde{\mathcal{W}}_{(\delta)} \dashrightarrow \overline{\mathcal{M}}_g,$$

defined on the open dense subscheme $\tilde{\mathcal{W}}_{(\delta)}^0 \subset \tilde{\mathcal{W}}_{(\delta)}$ such that, for $\eta_w \in \tilde{\mathcal{W}}_{(\delta)}^0$, $\tilde{\mathcal{C}}(\eta_w)$ is stably equivalent to a stable curve of genus g .

Set $\psi := c|_{\widetilde{\mathcal{W}}_{(\delta)}^0}$. By definition, for any $\eta_w \in \widetilde{\mathcal{W}}_{(\delta)}^0$, the map ψ contracts all possible smooth rational components of $\mathcal{C}(\eta_w)$ meeting the other components in only one or two points and maps the resulting stable curve into its equivalence class in $\overline{\mathcal{M}}_g$.

Pick any $C_0 \in W_{|H_0|, \delta}^{hyper}(S_0)$ and let $w_0 = [(S_0, C_0)] \in \mathcal{W}$ be the corresponding point. Now $|H_0|$ is globally generated and the normalization of C_0 at some δ nodes satisfying the conditions in (5.9) is a connected curve. Therefore, letting $\eta_{w_0} \in \alpha^{-1}(w_0)$ be the point corresponding to marking these δ nodes, we have that $\eta_{w_0} \in \widetilde{\mathcal{W}}_{(\delta)}^0$ and the map c is defined at η_{w_0} .

Let $\widetilde{\mathcal{V}} \subseteq \widetilde{\mathcal{W}}_{(\delta)}^0$ be the irreducible component containing η_{w_0} ; then, as proved above, $\dim(\widetilde{\mathcal{V}}) = 19 + g$.

By assumption, $\psi(\widetilde{\mathcal{V}}) \cap \overline{\mathcal{H}}_g \neq \emptyset$. Hence, for any irreducible component $\mathcal{K} \subseteq \psi(\widetilde{\mathcal{V}}) \cap \overline{\mathcal{H}}_g$, we have

$$(5.15) \quad \dim(\mathcal{K}) \geq \dim(\psi(\widetilde{\mathcal{V}})) + \dim(\overline{\mathcal{H}}_g) - \dim(\overline{\mathcal{M}}_g) = \dim(\psi(\widetilde{\mathcal{V}})) + 2 - g.$$

Pick any \mathcal{K} containing $\psi(\eta_{w_0})$ and let $\mathcal{J} \subseteq \psi|_{\widetilde{\mathcal{V}}}^{-1}(\mathcal{K})$ be any irreducible component containing η_{w_0} . Since the general fibre of $\psi|_{\widetilde{\mathcal{V}}}$ has dimension $\dim(\widetilde{\mathcal{V}}) - \dim(\psi(\widetilde{\mathcal{V}})) = 19 + g - \dim(\psi(\widetilde{\mathcal{V}}))$, from (5.15) we have

$$(5.16) \quad \begin{aligned} \dim(\mathcal{J}) &= \dim(\mathcal{K}) + 19 + g - \dim(\psi(\widetilde{\mathcal{V}})) \\ &\geq \dim(\psi(\widetilde{\mathcal{V}})) + 2 - g + 19 + g - \dim(\psi(\widetilde{\mathcal{V}})) = 21. \end{aligned}$$

Consider now

$$(5.17) \quad \pi \circ (\tilde{\alpha}|_{\mathcal{J}}) : \mathcal{J} \longrightarrow \mathcal{B}_p.$$

Since, by assumption, the fibre over $b_0 = [(S_0, H_0)]$ is at most two-dimensional, we conclude from (5.16) that $\pi \circ (\tilde{\alpha}|_{\mathcal{J}})$ is dominant, that all the fibres are precisely two-dimensional and that $\dim(\mathcal{J}) = 21$. This shows that $W_{|H|, \delta}^{hyper} \neq \emptyset$ for general $[(S, H)] \in \mathcal{B}_p$ and Lemma 5.1 implies that in fact any irreducible component of $W_{|H|, \delta}^{hyper}(S)$ has dimension two. \square

Remark 5.18. In particular, Lemma 5.1, Proposition 5.11 and [21, Examples 2.8 and 2.10] prove Theorem 5.2 for $p = 4$ and 5.

We next construct the desired special primitively marked $K3$ surface:

Proposition 5.19. *Let $d \geq 2$ and $k \geq 1$ be integers. There exists a $K3$ surface S_0 with*

$$\text{Pic}(S_0) = \mathbb{Z}[E] \oplus \mathbb{Z}[F] \oplus \mathbb{Z}[R]$$

and intersection matrix

$$\begin{bmatrix} E^2 & E.F & E.R \\ F.E & F^2 & F.R \\ R.E & R.F & R^2 \end{bmatrix} = \begin{bmatrix} 0 & d & k \\ d & 0 & k \\ k & k & -2 \end{bmatrix},$$

and such that the following conditions are satisfied:

- (a) $|E|$ and $|F|$ are elliptic pencils;
- (b) R is a smooth, irreducible rational curve.
- (c) $H_0 := E + F + R$ is globally generated, in particular the general member of $|H_0|$ is a smooth, irreducible curve of arithmetic genus $p := 2k + d$;
- (d) the only effective decompositions of H_0 are

$$H_0 \sim E + F + R \sim (E + F) + R \sim (E + R) + F \sim (F + R) + E.$$

Proof. Since the lattice has signature $(1, 2)$, then, by a result of Nikulin [44] (see also [40, Cor. 2.9(i)]), there is a $K3$ surface S_0 with that as Picard lattice. Performing Picard-Lefschetz reflections on the lattice, we can assume that H_0 is nef, by [2, VIII, Prop. 3.9]. Straightforward calculations on the Picard lattice rules out the existence of effective divisors Γ satisfying $\Gamma^2 = -2$ and $\Gamma.E < 0$ or $\Gamma.F < 0$, or $\Gamma^2 = 0$ and $\Gamma.H_0 = 1$. Hence (a) and (c) follow from [49, Prop. 2.6 and (2.7)]. Similarly one computes that if $\Gamma > 0$, $\Gamma^2 = -2$ and $\Gamma.R < 0$, then $\Gamma = R$, proving (b).

Similarly, (d) is proved by direct calculations using the nefness of E , F and H_0 and recalling that by Riemann-Roch and Serre duality a divisor D on a $K3$ surface is effective and irreducible only if $D^2 \geq -2$ and $D.N > 0$ for some nef divisor N . \square

The following result, together with (5.10) and Proposition 5.11, now concludes the proof of Theorem 5.2 and Corollary 5.3. From Remark 5.18, we need only consider $p \geq 6$.

Lemma 5.20. *Let $p \geq 6$ be an integer. There exists a smooth $K3$ surface S_0 with a globally generated, primitive line bundle H_0 on S_0 with $p = p_a(H_0)$ such that*

- (a) $W_{|H_0|, p-3}^{hyper}(S_0) \neq \emptyset$;
- (b) $\dim(W_{|H_0|, p-3}^{hyper}(S_0)) = 2$;
- (c) *there exists a component of $W_{|H_0|, p-3}^{hyper}(S_0)$ whose general member deforms to a curve $[C_t] \in V_{|H_t|, p-3}^{hyper}(S_t)$, for general $[(S_t, H_t)] \in \mathcal{B}_p$;*
- (d) *for general $[(S_t, H_t)] \in \mathcal{B}_p$, the two-dimensional irreducible component $V_t \subseteq V_{|H_t|, p-3}^{hyper}(S_t)$ given by (c), satisfies $\dim(R_{V_t}) = 3$ (with notation as in § 2.2).*

Proof. Set $k = 1$ if p is even and $k = 2$ if p is odd and let $d := p - 2k \geq 2$. Consider the marked $K3$ surface (S_0, H_0) in Proposition 5.19.

We will consider two general smooth elliptic curves $E_0 \in |E|$ and $F_0 \in |F|$ and curves of the form

$$C_0 := E_0 \cup F_0 \cup R,$$

with transversal intersections and a desingularization

$$(5.21) \quad \tilde{C}_0 = \tilde{E}_0 \cup \tilde{F}_0 \cup \tilde{R} \rightarrow C_0$$

of the $\delta := p - 3 = d + 2k - 3$ nodes marked in Figure 1 below, that is, all but one of each of the intersection points $E_0 \cap F_0$, $E_0 \cap R$ and $F_0 \cap R$.

Then $[C_0] \in W_{|H_0|, p-3}^{hyper}$, as \tilde{C}_0 is stably equivalent to a union of two smooth elliptic curves intersecting in two points (cf. [22, Exercise (3.162)]), proving (a). Clearly the closure of the family we have constructed is isomorphic to $|E| \times |F| \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and is therefore two-dimensional. Denote by $W_0 \subset W_{|H_0|, p-3}^{hyper}$ this two-dimensional subscheme.

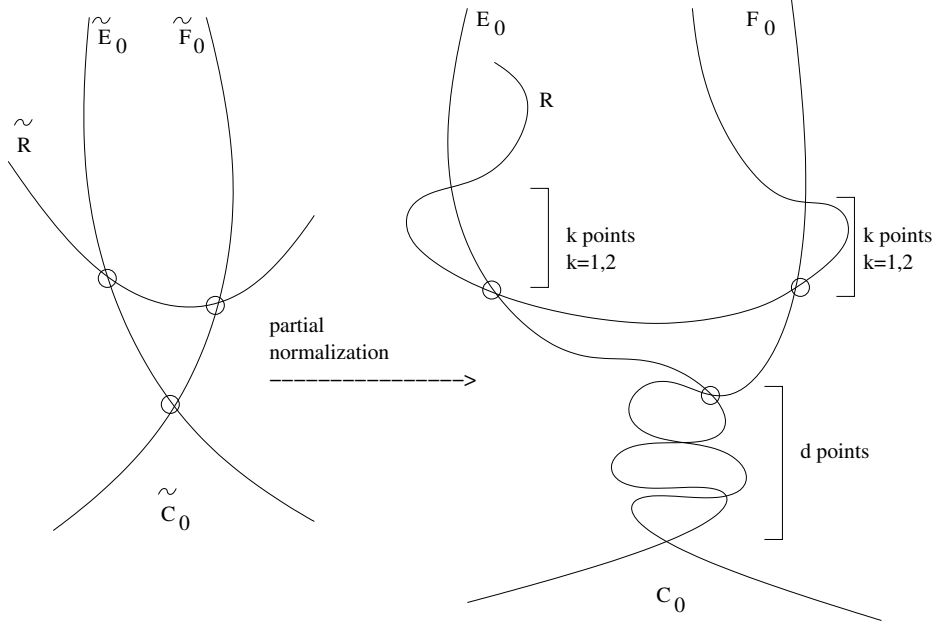
We will now show that any irreducible component W of $W_{|H_0|, p-3}^{hyper}$ has dimension ≤ 2 .

A central observation, which will be used together with Theorem 4.6(b), will be that, with the above choices of k , we have

$$(5.22) \quad E.H_0 = F.H_0 = d + k = p - k \text{ is odd.}$$

We start by considering families of reducible curves. These are all classified in Proposition 5.19(d).

If the general element in W is of the form $D \cup R$, for $D \in |E + F|$, then in order to have a partial desingularization $\tilde{D} \cup \tilde{R}$ to be (degenerated) hyperelliptic, we must have $\deg(\tilde{D} \cap \tilde{R}) = 2$, so that we must desingularize $2(k - 1)$ of the intersection points of $D \cap R$. Finally, as $p_a(\tilde{D} \cup \tilde{R}) = 3$, we must have $p_a(\tilde{D}) = 2$. Therefore $W \subseteq W_D \times \{R\} \simeq W_D$, where $W_D \subset |D|$ is a subfamily of irreducible curves of geometric genus ≤ 2 . It follows that $\dim(W) \leq \dim(W_D) \leq 2$, by Lemma 2.3.

FIGURE 1. The curves C_0 and \tilde{C}_0

If the general element in W is of the form $D \cup E$, for $D \in |F + R|$, then in order to have a partial desingularization $\tilde{D} \cup \tilde{R}$ that is (degenerated) hyperelliptic, we must have $\deg(\tilde{D} \cap \tilde{E}) = 2$. If the projection $W \rightarrow |E|$ is dominant, this means that $\mathfrak{g}_2^1(\tilde{D}) \subseteq |f^*E|_{|\tilde{D}}$, where $f : \tilde{S} \rightarrow S$ denotes the composition of blow-ups of S that induces the partial desingularization $\tilde{D} \cup \tilde{R} \rightarrow D \cup R$. But this would mean that $|f^*E|_{|\tilde{D}}$, which is base point free on \tilde{D} , is composed with the $\mathfrak{g}_2^1(\tilde{D})$, a contradiction, as $\deg(\mathcal{O}_{\tilde{D}}(f^*E)) = E \cdot D = E \cdot H_0$ is odd by (5.22). Therefore, the projection $W \rightarrow |E|$ is not dominant, whence $\dim(W) \leq \dim(|D|) = \frac{1}{2}D^2 + 1 = k \leq 2$, as desired. By symmetry, the case where the general element in W is of the form $D \cup F$, for $D \in |E + R|$ is treated in the same way.

Finally, we have to consider the case of a family $W \subseteq |H_0|$ of irreducible curves.

In this case assume $\dim(W) \geq 3$, and let C be a general curve parametrized by W . Then by Theorem 4.6 (b), there exists an effective decomposition into moving classes $H_0 \sim M + N$ such that

$$\mathfrak{g}_2^1(\tilde{C}) \subseteq |f^*M|_{|\tilde{C}}, |f^*N|_{|\tilde{C}},$$

where $f : \tilde{S} \rightarrow S$ denotes the succession of blow ups of S that induces the normalization $\tilde{C} \rightarrow C$. From Proposition 5.19(d) we see that we must have

$$\mathfrak{g}_2^1(\tilde{C}) \subseteq |f^*E|_{|\tilde{C}}, \text{ or } |f^*F|_{|\tilde{C}},$$

which means that either $|f^*E|_{|\tilde{C}}$ or $|f^*F|_{|\tilde{C}}$ is composed with the $\mathfrak{g}_2^1(\tilde{C})$, again a contradiction, as both have odd degree by (5.22). We have therefore proved (b).

To prove (c) we will show that any $[C_0] \in W_{|H_0|, p-3}^{hyper}$ in the two-dimensional, irreducible component W_0 considered above in fact deforms to a curve $[C_t] \in W_{|H_t|, p-3}^{hyper}(S_t)$, for general $[(S_t, H_t)] \in \mathcal{B}_p$, that has precisely $\delta = p - 3$ nodes (cf. (5.10)).

To this end, denote by $\mathcal{S} \rightarrow \mathcal{B}_p$ the universal family of $K3$ surfaces, $\tilde{f} : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{W}}_{(\delta)}$ and $\mathcal{J} \subset \tilde{\mathcal{W}}_{(\delta)}$ as in the proof of Proposition 5.11, and let $\varphi : \tilde{\mathcal{C}}_{\mathcal{J}} \rightarrow \mathcal{J}$ be the restriction of \tilde{f} .

Since the fiber over $[(S_0, H_0)]$ of $\mathcal{J} \rightarrow \mathcal{B}_p$ as in (5.17) contains an open, dense subset of $\mathbb{P}^1 \times \mathbb{P}^1$, we can find a smooth, irreducible curve $B \subset \mathcal{J}$ satisfying: for $x \in B$ general, $\varphi^{-1}(x)$ is a (partial) desingularization of $\delta = p - 3$ of the nodes of a curve in $W_{|H_t|, \delta}(S_t)$ (cf. (5.4)), for general $[(S_t, H_t)] \in \mathcal{B}_p$, and $\varphi^{-1}(x) \in \overline{\mathcal{H}}_3 \subset \overline{\mathcal{M}}_3$; moreover B contains a point $x_0 \in \mathcal{J}$ such that $\varphi^{-1}(x_0)$ is \tilde{C}_0 as in (5.21), for C_0 general in W_0 .

Let $\varphi_B : \tilde{\mathcal{C}}_B \rightarrow B$ be the induced universal curve. Since the dualizing sheaf of $\varphi_B^{-1}(x_0) = \tilde{C}_0$ is globally generated (as each component intersects the others in two points), we in fact have, possibly after substituting B with an open neighbourhood of x_0 , a morphism $\gamma_B : \tilde{\mathcal{C}}_B \rightarrow \mathbb{P}(\tilde{\varphi}_*(\omega_{\tilde{\mathcal{C}}_B/B}))$ over B that is $2 : 1$ on the general fibre $\varphi_B^{-1}(x)$ and contracts the rational component \tilde{R} of $\varphi_B^{-1}(x_0)$ and maps the two elliptic curves \tilde{E}_0 and \tilde{F}_0 each $2 : 1$ onto (different) \mathbb{P}^1 s (cf. (5.21) and Figure 1).

Let $\nu : \tilde{\mathcal{C}}'_B \rightarrow \tilde{\mathcal{C}}_B$ be the normalization and

$$\tilde{\mathcal{C}}'_B \xrightarrow{\gamma_1} \tilde{\mathcal{C}}''_B \xrightarrow{\gamma_2} \mathbb{P}(\tilde{\varphi}_*(\omega_{\tilde{\mathcal{C}}_B/B}))$$

the Stein factorization of $\gamma_B \circ \nu$. In particular, γ_2 is finite of degree two onto its image. Moreover, $\nu \circ \varphi_B : \tilde{\mathcal{C}}'_B \rightarrow B$ is a flat family whose general fiber $(\nu \circ \varphi_B)^{-1}(x)$ is a desingularization of $\varphi_B^{-1}(x) \in \tilde{\mathcal{C}}_B$. Let p_g be the geometric genus of this general fibre.

Let $\mathcal{D} \subset \tilde{\mathcal{C}}'_B$ be the strict transform via γ_1 of the closure of the branch divisor of γ_2 on the smooth locus of $\tilde{\mathcal{C}}''_B$. By Riemann-Hurwitz, for general $x \in B$, we have $\mathcal{D} \cdot \varphi_B^{-1}(x) = 2p_g + 2$, whereas $\mathcal{D} \cdot \varphi_B^{-1}(x_0) \geq 8$, as the curve $\gamma_1(\varphi_B^{-1}(x_0))$ contains two smooth elliptic curves, each being mapped $2 : 1$ by γ_2 onto (different) \mathbb{P}^1 s. This implies $p_g = 3$. Since, for general $x \in B$, we have $p_g \leq p_a(\varphi_B^{-1}(x)) = p - \delta = 3$, we find that $\varphi_B^{-1}(x)$ is smooth. This means that the general curve in $W_{|H_t|, \delta}(S_t)$, for $(S_t, H_t) \in \mathcal{B}_p$ general, has precisely $\delta = p - 3$ nodes, proving (c).

To prove (d), again we consider the morphism (up to possibly restricting \mathcal{J} as above)

$$\gamma_{\mathcal{J}} : \mathcal{C}_{\mathcal{J}} \rightarrow \mathbb{P}(\varphi_*(\omega_{\mathcal{C}_{\mathcal{J}}/\mathcal{J}}))$$

over \mathcal{J} which, apart some possible contractions of rational components in special fibres over \mathcal{J} , is relatively $2 : 1$ onto its image. We have a natural morphism $h : \mathcal{C}_{\mathcal{J}} \rightarrow \mathcal{S}$, inducing a natural map

$$\Phi : \text{im}(\gamma_{\mathcal{J}}) \dashrightarrow \text{Sym}^2(\mathcal{S}),$$

whose domain has nonempty intersection with every fibre over \mathcal{B}_p .

Let $\mathcal{R} := \overline{\text{im}(\Phi)}$. Then $\mathcal{R} \cap \text{Sym}^2(S_t) = R_{V_t}$, for general $[(S_t, H_t)] \in \mathcal{B}_p$. One easily sees that

$$\{\text{Sym}^2(E')\}_{E' \in |E|} \cup \{\text{Sym}^2(F')\}_{F' \in |F|} \subseteq \overline{\mathcal{R} \cap \text{Sym}^2(S_0)}.$$

Since the two varieties on the left are threefolds, we have $\dim(\Phi^{-1}(\xi_0)) = 0$ for general $\xi_0 \in \mathcal{R} \cap \text{Sym}^2(S_0) \subset \mathcal{R}$. Therefore, for general $\xi \in \mathcal{R}$, we have $\dim(\Phi^{-1}(\xi)) = 0$, so that $\dim(\mathcal{R}) = \dim(\mathcal{C}_{\mathcal{J}}) = \dim(\mathcal{J}) + 1 = 22$, whence $\dim(R_{V_t}) = 22 - \dim(\mathcal{B}_p) = 3$. \square

Remark 5.23. For general $[(S_t, H_t)] \in \mathcal{B}_p$ the obtained curves in the last proof have in fact $\delta = p - 3$ *non-neutral* nodes (cf. [21, §3]). In fact a desingularization of less than $p - 3$ nodes of C_t admits no \mathfrak{g}_{2S}^1 s, as clearly a desingularization of less than $p - 3$ nodes of C_0 is not stably equivalent to a curve in the hyperelliptic locus $\overline{\mathcal{H}}_3 \subset \overline{\mathcal{M}}_3$.

6. ON THE MORI CONE OF THE HILBERT SQUARE OF A $K3$ SURFACE

In this section we first summarize central results on the Hilbert square of a $K3$ surface and show how to compute the class of a rational curve in $S^{[2]}$. Then we discuss the relations between the existence of curves on S and the slope of the Mori cone of $S^{[2]}$, that is, the cone of effective classes in $N_1(S^{[2]}) = N_1(S^{[2]})_{\mathbb{R}}$. In particular, we show how to deduce the different bounds in §1.2. Finally, we

discuss the relation between the existence of a curve on S with given singular Brill-Noether number and the slope of the Mori cone of $S^{[2]}$.

6.1. Preliminaries on $S^{[2]}$ for a $K3$ surface. Recall that for any smooth surface S we have

$$(6.1) \quad H^2(S^{[2]}, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\epsilon,$$

where $\Delta := 2\epsilon$ is the class of the divisor parametrizing 0-dimensional subschemes supported on a single point (see [7]). So we may identify a class in $H^2(S, \mathbb{Z})$ with its image in $H^2(S^{[2]}, \mathbb{Z})$. When S is a $K3$ surface the cohomology group $H^2(S^{[2]}, \mathbb{Z})$ is endowed with a quadratic form q , called the *Beauville-Bogomolov form*, such that its restriction to $H^2(S, \mathbb{Z})$ is simply the cup product on S , the two factors $H^2(S, \mathbb{Z})$ and $\mathbb{Z}\epsilon$ are orthogonal with respect to this form and $q(\epsilon) = -2$. The decomposition (6.1) induces an isomorphism

$$(6.2) \quad \text{Pic}(S^{[2]}) \simeq \text{Pic}(S) \oplus \mathbb{Z}[\epsilon],$$

and each divisor D on S corresponds to the divisor on $S^{[2]}$, by abuse of notation also denoted by D , consisting of length-two subschemes with some support on D .

Given a primitive class $\alpha \in H_2(S^{[2]}, \mathbb{Z})$, there exists a unique class $w_\alpha \in H^2(S^{[2]}, \mathbb{Q})$ such that $\alpha \cdot v = q(w_\alpha, v)$, for all $v \in H^2(S^{[2]}, \mathbb{Z})$, and one sets

$$(6.3) \quad q(\alpha) := q(w_\alpha).$$

We denote also by $\rho_\alpha \in H^2(S^{[2]}, \mathbb{Z})$ the corresponding primitive $(1, 1)$ -class such that $\rho_\alpha = cw_\alpha$, for some $c > 0$ (for further details, we refer the reader to [24]).

If now $\text{Pic}(S) = \mathbb{Z}[H]$, then the Néron-Severi group of $S^{[2]}$ has rank two. We may take as generators of $N_1(S^{[2]})_{\mathbb{R}}$ the class \mathbb{P}_Δ^1 of a rational curve in the ruling of the exceptional divisor $\Delta \subset S^{[2]}$, and the class of the curve in $S^{[2]}$ defined as follows

$$\{\xi \in S^{[2]} \mid \text{Supp}(\xi) = \{p_0, y\} \mid y \in Y\},$$

where Y is a curve in $|H|$ and p_0 is a fixed point on S . By abuse of notation, we still denote the class of the curve in $S^{[2]}$ by Y . Note that we always have that

$$(6.4) \quad \mathbb{P}_\Delta^1 \text{ lies on the boundary of the Mori cone.}$$

Indeed, the curve \mathbb{P}_Δ^1 is contracted by the Hilbert-Chow morphism $S^{[2]} \rightarrow \text{Sym}^2(S)$, so that the pull-back of an ample divisor on $\text{Sym}^2(S)$ is nef, but zero along \mathbb{P}_Δ^1 .

Therefore, describing the Mori cone $\text{NE}(S^{[2]})$ amounts, by (6.4), to computing

$$(6.5) \quad \text{slope}(\text{NE}(S^{[2]})) := \inf \left\{ \frac{a}{b} \mid aY - b\mathbb{P}_\Delta^1 \in N_1(S^{[2]}) \text{ is effective, } a, b \in \mathbb{Q}^+ \right\}.$$

We will also call the (possibly infinite) number a/b associated to an irreducible curve $X \sim_{alg} aY - b\mathbb{P}_\Delta^1$ with $a > 0$ and $b \geq 0$, the *slope of the curve X* and denote it by $\text{slope}(X)$. Thus, the smaller $\text{slope}(X)$ is, the nearer is X to the boundary of $\text{NE}(S^{[2]})$.

By a general result due to Huybrechts [32, Prop. 3.2] and Boucksom [11], a divisor D on $S^{[2]}$ is ample if and only if $q(D) > 0$ and $D \cdot R > 0$ for any (possibly singular) rational curve $R \subset S^{[2]}$. As a consequence, if the Mori cone is closed then the boundary (which remains to be determined) is generated by the class of a rational curve (the other boundary is generated by \mathbb{P}_Δ^1 , by (6.4)). This means that one would have $\text{slope}(\text{NE}(S^{[2]})) = \text{slope}_{rat}(\text{NE}(S^{[2]}))$, where

$$(6.6) \quad \text{slope}_{rat}(\text{NE}(S^{[2]})) := \inf \left\{ \frac{a}{b} \mid aY - b\mathbb{P}_\Delta^1 \in N_1(S^{[2]}) \text{ is the class of a rational curve, } a, b \in \mathbb{Q}^+ \right\}.$$

(A priori, one only has $\text{slope}(\text{NE}(S^{[2]})) \leq \text{slope}_{rat}(\text{NE}(S^{[2]}))$.)

Hassett and Tschinkel proposed in [24] a conjectural description of the Mori cone of hyperkähler fourfolds deformation equivalent to the Hilbert square of a $K3$ surface (cf. [24, Conjecture 3.1]) and of the numerical and geometric properties of the rational curves that are extremal in the Mori cone (cf. [24, Conjecture 3.6]). We refer the reader to their recent preprint [25] where one implication of [24, Conjecture 3.1] has been proved.

6.2. The classes of rational curves in $S^{[2]}$. Assume that $\text{Pic}(S) = \mathbb{Z}[H]$ with $p_a(H) = p_a \geq 2$. Let $X \subset S^{[2]}$ be an irreducible rational curve. Let $C_X \subset S$ be the corresponding curve as in § 2.1 and assume that $C_X \in |mH|$ with $m \geq 1$. (In particular, $m \geq 2$ if we are in case (II)). We can write

$$X \sim_{alg} a_1 Y + a_2 \mathbb{P}_\Delta^1.$$

Since $X.H = m(2p_a - 2)$, $Y.H = 2p_a - 2$ and $\mathbb{P}_\Delta^1.H = 0$ by the very definition of H as a divisor in $S^{[2]}$, and $Y.\epsilon = 0$ and $\mathbb{P}_\Delta^1.\epsilon = -2$, we obtain, defining $g_0(X) := X.\epsilon - 1$,

$$(6.7) \quad X \sim_{alg} mY - \left(\frac{g_0(X) + 1}{2} \right) \mathbb{P}_\Delta^1.$$

To compute $g_0(X)$, consider the diagram (2.1). Since $\nu_X^* \mathcal{O}_X(\Delta) \simeq (\nu_X^* \mathcal{O}_X(\epsilon))^{\otimes 2}$, the double cover f is defined by $\nu_X^* \mathcal{O}_X(\Delta)$. By Riemann-Hurwitz we therefore get

$$(6.8) \quad g_0(X) = p_a(\tilde{C}_X).$$

Note that in the cases (II) and (III) in the correspondence in § 2.1, $X.\epsilon = g_0(X) + 1$ is precisely the length of the intersection scheme $\tilde{C}_{X,1} \cap \tilde{C}_{X,2}$, where $\tilde{C}_X = \tilde{C}_{X,1} \cup \tilde{C}_{X,2}$. In case (III), since $\tilde{\nu} : \tilde{C}_X \rightarrow S$ contracts one of the two components of \tilde{C}_X to a point $x_X \in S$, we obtain that

$$(6.9) \quad g_0(X) = \text{mult}_{x_X}(C_X) - 1 \text{ (if } C_X \text{ is of type (III))}.$$

One can check that for all divisors D in $S^{[2]}$ one has $X.D = q(w_X, D)$ with

$$(6.10) \quad w_X := mH - \left(\frac{g_0(X) + 1}{2} \right) \epsilon \in H^2(S^{[2]}, \mathbb{Q}).$$

In particular, $2w_X \in H^2(S^{[2]}, \mathbb{Z})$.

From (6.5) and (6.7) we see that searching for irreducible rational curves in (or at least “near”) the boundary of the Mori cone of $S^{[2]}$, or with negative square $q(X)$, amounts to searching for irreducible curves in $|mH|$ with (partial) hyperelliptic normalizations of high genus (case (I)), or to irreducible rational curves in $|mH|$ with high multiplicity at a point (case (III)), or to irreducible rational curves on S with some correspondence between some coverings of their normalizations (case (II)). Moreover, we should search for curves with as low m as possible. Now $m \geq 2$ in case (II), as remarked above. Moreover, any rational curve in $|H|$ on a general S is nodal, by a result of Chen [13, Thm. 1.1] (the same is also conjectured for rational curves in $|mH|$ for $m > 1$, see [14, Conj. 1.2]), so that $g_0(X) \leq 1$ if C_X is of type (III) in these cases, by (6.9). Hence, we see that the most natural candidates are irreducible curves in $|H|$ with hyperelliptic normalizations.

By the above, an irreducible curve $C \in |mH|$ with hyperelliptic normalization defines, by the unicity of the \mathfrak{g}_2^1 , a unique irreducible rational curve $X = R_C \subset S^{[2]}$ with class

$$(6.11) \quad R_C \sim_{alg} mY - \left(\frac{g_0(C) + 1}{2} \right) \mathbb{P}_\Delta^1,$$

where $g_0(C) := g_0(R_C)$ is well-defined as

$$(6.12) \quad g_0(C) := \text{the arithmetic genus of a minimal partial desingularization of } C \text{ admitting a } \mathfrak{g}_2^1.$$

(For example, if C is nodal, then we simply take the desingularization of the *non-neutral nodes* of C , cf. [21, §3]). From (6.5) we then get

$$(6.13) \quad \text{slope}(\text{NE}(S^{[2]})) \leq \frac{2m}{g_0(C) + 1} \leq \frac{2m}{p_g(C) + 1}, \text{ if there exists a } C \in |mH| \text{ with hyp. norm.}$$

and, by (6.3) and (6.10),

$$(6.14) \quad q(R_C) = 2m^2(p_a - 1) - \frac{(g_0(C) + 1)^2}{2} \leq 2m^2(p_a - 1) - \frac{(p_g(C) + 1)^2}{2}.$$

In particular, the higher $g_0(C)$ (or $p_g(C)$) is - thus the more “unexpected” the curve on S is from a Brill-Noether theory point of view - the lower is the bound on the slope of $\text{NE}(S^{[2]})$ and the more negative is the square $q(R_C)$ in $S^{[2]}$.

6.3. The invariant ρ_{sing} , Seshadri constants, curves with hyperelliptic normalizations and the slope of the Mori cone. In [21] we introduced a singular Brill-Noether invariant

$$(6.15) \quad \rho_{\text{sing}}(p_a, r, d, g) := \rho(g, r, d) + p_a - g,$$

in order to study linear series on the normalization of singular curves. Precisely, we proved

Theorem 6.16. *Let S be a K3 surface such that $\text{Pic}(S) \simeq \mathbb{Z}[H]$ with $p_a := p_a(H) \geq 2$. Let $C \in |H|$ and $\tilde{C} \rightarrow C$ be a partial normalization of C , such that $g := p_a(\tilde{C})$.*

If $\rho_{\text{sing}}(p_a, r, d, g) < 0$, then \tilde{C} carries no \mathfrak{g}_d^r .

Proof. One easily sees that the proof of [21, Thm. 1] also holds for a *partial* normalization of C . \square

For $r = 1$ and $d = 2$, we have

$$(6.17) \quad \rho_{\text{sing}}(p_a, 1, 2, g) < 0 \Leftrightarrow g > \frac{p_a + 2}{2}.$$

In particular, a consequence of Theorem 6.16 is the following:

Theorem 6.18. *Let S be a smooth, projective K3 surface with $\text{Pic}(S) \simeq \mathbb{Z}[H]$ and $p_a := p_a(H) \geq 2$. Let Y and \mathbb{P}_Δ^1 be the generators of $N_1(S^{[2]})$ with notation as in § 6.1.*

If $X \subset S^{[2]}$ is an effective 1-cycle such that $X \sim_{\text{alg}} Y - k\mathbb{P}_\Delta^1$, then $k \leq \frac{p_a + 4}{4}$.

Proof. We can assume that X is an irreducible curve. Then, precisely as in the case of a rational curve, X corresponds either to the data of an irreducible curve $C \in |H|$ on S , with a partial normalization \tilde{C} admitting a $2 : 1$ morphism onto the normalization \tilde{X} of X , or to the data of an irreducible curve $C \in |H|$ on S together with a point $x_0 := x_X \in S$. (The case corresponding to case (II) in § 2.1 does not occur, since the coefficient of Y is one, precisely as in the case of a rational X explained above.)

In the latter case $\mu(X) = \{x_0 + C\} \subset \text{Sym}^2(S)$, where $\mu : S^{[2]} \rightarrow \text{Sym}^2(S)$ is the Hilbert-Chow morphism as usual, and one easily computes $k = (1/2) \text{mult}_{x_0}(C)$ as in the rational case above. Since clearly $\text{mult}_{x_0}(C) \leq 2$ if $p_a = 2$ and $\text{mult}_{x_0}(C) \leq 3$ if $p_a = 3$, we have $k \leq \frac{p_a + 4}{4}$ in these two cases. If $p_a \geq 4$, then from $\dim |H| - 3 - (p_a - 4) = 1$ and the fact that being singular at a given point imposes at most three independent conditions on $|H|$, we can find an irreducible curve $C' \in |H|$, different from C , singular at x_0 , and passing through at least $p_a - 4$ points of C . Therefore

$$2p_a - 2 = H^2 = C' \cdot C \geq \text{mult}_{x_0}(C') \cdot \text{mult}_{x_0}(C) + p_a - 4 \geq 2 \text{mult}_{x_0}(C) + p_a - 4,$$

whence $\text{mult}_{x_0}(C) \leq (p_a + 2)/2$, so that $k \leq (p_a + 2)/4$.

In the first case, then, precisely as in the rational case above,

$$(6.19) \quad k = \frac{p_a(\tilde{C}) + 1}{2} - p_g(X)$$

from Riemann-Hurwitz. By Brill-Noether theory on \tilde{X} , it follows that \tilde{C} carries a \mathfrak{g}_d^1 , with

$$d \leq 2 \lfloor \frac{p_g(X) + 3}{2} \rfloor.$$

By Theorem 6.16 we have $\rho_{\text{sing}}(p_a(C), 1, d, p_a(\tilde{C})) \geq 0$, whence $p_a(\tilde{C}) \leq d - 1 + p_a(C)/2$. The desired result now follows. \square

By the proof of Theorem 6.18 we see that if $C \in |mH|$ is an irreducible curve and $x_0 \in C$, then the class of the corresponding curve $\mu_*^{-1}\{x_0 + C\} \subset S^{[2]}$ is given by $mY - (1/2)\text{mult}_{x_0}(C)\mathbb{P}_\Delta^1$. Hence

$$\text{slope}(\text{NE}(S^{[2]})) \leq \inf_{m \in \mathbb{N}} \left(\inf_{C \in |mH|} \left(\inf_{x \in C} \frac{2m}{\text{mult}_x(C)} \right) \right) = \inf_{m \in \mathbb{N}} \frac{2}{H^2} \left(\inf_{C \in |mH|} \left(\inf_{x \in C} \frac{C.H}{\text{mult}_x(C)} \right) \right).$$

It follows that

$$(6.20) \quad \text{slope}(\text{NE}(S^{[2]})) \leq \frac{\varepsilon(H)}{p_a - 1},$$

where

$$\varepsilon(H) := \inf_{x \in S} \left(\inf_{C \ni x} \frac{C.H}{\text{mult}_x(C)} \right)$$

(and the infimum is taken over all irreducible curves $C \subset S$ passing through x) is the (*global*) *Seshadri constant* of H (cf. [17, § 6], [18] or [4]). These constants are very difficult to compute; for instance on general primitively polarized $K3$ s the only cases where they have been computed are when $H^2 = \alpha^2$, where one has $\varepsilon(H) = \alpha$, cf. [3, 33]. However, it is well-known that $\varepsilon(H) \leq \sqrt{H^2}$ on any surface, see e.g. [55, Rem. 1]. Hence, by (6.20) we obtain

Theorem 6.21. *Let (S, H) be a primitively polarized $K3$ surface of genus $p_a := p_a(H) \geq 2$ such that $\text{Pic}(S) \simeq \mathbb{Z}[H]$. Then (cf. (6.5))*

$$(6.22) \quad \text{slope}(\text{NE}(S^{[2]})) \leq \frac{\varepsilon(H)}{p_a - 1} \leq \sqrt{\frac{2}{p_a - 1}}.$$

In particular, (6.22) shows that there is no lower bound on the slope of the Mori cone of $S^{[2]}$ of $K3$ surfaces, as the degree of the polarization tends to infinity, that is,

$$(6.23) \quad \inf \left\{ \text{slope}(\text{NE}(S^{[2]})) \mid S \text{ is a projective } K3 \text{ surface} \right\} = 0,$$

The same fact about $\text{slope}_{\text{rat}}(\text{NE}(S^{[2]}))$ will follow from (7.4) and (7.9) below.

Note that by [33] one always has $\varepsilon(H) \geq \lfloor \sqrt{H^2} \rfloor - \frac{2}{\lfloor \sqrt{H^2} \rfloor + 1}$ under the hypotheses of Theorem 6.21. It follows that

$$(6.24) \quad \frac{\varepsilon(H)}{p_a - 1} \geq \frac{\lfloor \sqrt{2p_a - 2} \rfloor - 2/3}{p_a - 1}, \text{ for } (S, H) \text{ as in Theorem 6.21,}$$

showing that there is a natural limit to how good a bound one can get on $\text{slope}(\text{NE}(S^{[2]}))$ by using Seshadri constants.

The bound in (6.22) is not (necessarily) obtained by rational curves in $S^{[2]}$. However, the presence of $p_g(X)$ in (6.19) above tends to indicate that the better bounds will be obtained by rational curves in $S^{[2]}$. (Of course, if the Mori cone is closed, then the bound will indeed be obtained by rational curves, as explained at the end of § 6.1.) In fact, the bound (6.22) above will be improved, for infinitely many values of H^2 , in Propositions 7.2 and 7.7 below by rational curves.

We now return to the study of irreducible *rational* curves in $S^{[2]}$ and to $\text{slope}_{\text{rat}}(\text{NE}(S^{[2]}))$.

Given Theorem 6.16 and (6.17), a natural question to ask is whether there may exist singular curves in $|H|$ with hyperelliptic normalizations of geometric genus p_g , for $3 \leq p_g \leq \frac{p_a + 2}{2}$. It is

natural to try to construct such curves with at most nodes as singularities, as one has better control of their deformations and their parameter spaces (the Severi varieties considered in § 5). After the positive answer to the above existence problem (with nodal curves) for the specific values $p_g = 3$ and $p_a = 4, 5$ in [21, Examples 2.8 and 2.10], Theorem 5.2 gives the first examples, at least as far as we know, of positive answers for primitively polarized $K3$ surfaces of *any* degree.

In Remark 5.23 we showed that $p_g(C) = g_0(C) = 3$ for these constructed curves $C \in |H|$ (cf. (6.12)), so that the classes of the associated rational curves $R_C \subset S^{[2]}$ are, using (6.10),

$$(6.25) \quad w_{R_C} = H - 2\epsilon,$$

with

$$q(w_{R_C}) = q(R_C) = 2p - 10 \geq -2.$$

Moreover, using (6.13), Theorem 5.2 yields (cf. (6.6)):

Corollary 6.26. *Let (S, H) be a general, primitively polarized $K3$ surface of genus $p_a(H) \geq 4$. Then*

$$(6.27) \quad \text{slope}_{\text{rat}}(\text{NE}(S^{[2]})) \leq \frac{1}{2}.$$

Note that the existence of nodal curves of geometric genus 2 in $|H|$, which was already known and followed from the nonemptiness of the Severi varieties on general $K3$ surfaces, as explained in the beginning of § 5, leads to the less good bound of $\frac{2}{3}$. Therefore, again as far as we know, (6.27) is the first “nontrivial” bound on the slope of rational curves holding for all degrees of the polarization. As already mentioned, for infinitely many degrees of the polarization we will in fact improve this bound in Propositions 7.2 and 7.7 below.

Remark 6.28. We do not know whether there will always be components in $|H|^{\text{hyper}}$ (whenever nonempty) of singular curves with hyperelliptic normalizations such that the singularities of the general member are as nice as possible, that is, all nodes and all non-neutral [21, §3].

7. \mathbb{P}^2 S AND THREEFOLDS BIRATIONAL TO \mathbb{P}^1 -BUNDLES IN THE HILBERT SQUARES OF $K3$ S

We will in this section give an infinite series of examples of *general*, primitively polarized $K3$ surfaces (S, H) , of infinitely many degrees such that $S^{[2]}$ contains either a \mathbb{P}^2 or a threefold birational to a \mathbb{P}^1 -bundle, thus showing both possibilities occurring in Proposition 3.6.

Both series of examples are similar to Voisin’s constructions in [58, § 3]. The idea is to start with a smooth quartic surface S_0 such that $S_0^{[2]}$ contains an “obvious” \mathbb{P}^2 or threefold birational to a \mathbb{P}^1 -bundle over S_0 , use the involution on the quartic to produce another such \mathbb{P}^2 or uniruled threefold, and then deform S_0 keeping the latter one and loosing the first one in the Hilbert square.

We remark that the question of existence of \mathbb{P}^2 s in $S^{[2]}$ when S is $K3$ is a very interesting problem because of the following fact: a \mathbb{P}^2 in $S^{[2]}$ gives rise to a birational map from $S^{[2]}$ onto another hyperkähler fourfold, and conversely any birational transformation $X \dashrightarrow X'$ between projective, symplectic fourfolds can be factorized into a finite sequence of Mukai flops (cf. [42, Thm. 0.7]), by [61, Thm. 2], see also [12, 30, 63]. Therefore, in the case of a $K3$ surface, if $S^{[2]}$ contains no \mathbb{P}^2 s, then $S^{[2]}$ admits no other birational model than itself. Also uniruled divisors govern the birational Kähler cone of a hyperkähler manifold X , cf. [32].

7.1. \mathbb{P}^2 s in $S^{[2]}$. The first nontrivial case, the case of degree 10, is particularly easy:

Example 7.1. (Hassett) Let $S \subset \mathbb{P}^6$ be a general $K3$ surface of degree 10. By [41] the surface S is a complete intersection $S = G \cap T \cap Q$, where $G := \text{Grass}(2, 5)$ is the Grassmannian of lines in \mathbb{P}^4 embedded in \mathbb{P}^9 by its Plücker embedding, T is a general 6-dimensional linear subspace of \mathbb{P}^9 , and Q is a hyperquadric in \mathbb{P}^9 . Set $Y := G \cap T$. Then Y is a Fano 3-fold of index 2. Let $F(Y)$ be its variety

of lines. It is classically known (see e.g. [19] for a modern proof) that $F(Y) \cong \mathbb{P}^2$. Then we may embed this plane in $S^{[2]}$ by mapping the point corresponding to a line $[\ell]$ to $\ell \cap Q$. By generality, S does not contain any line, so that this map is a morphism.

The construction behind the following result, generalizing the previous example, was shown to us by B. Hassett.

Proposition 7.2. *Let (S, H) be a general primitively polarized K3 surface of degree $H^2 = 2(n^2 - 9n + 19)$, for $n \geq 6$. Then $S^{[2]}$ contains a \mathbb{P}^2 .*

The class $w_\ell \in H^2(S^{[2]}, \mathbb{Q})$ corresponding to a line $\ell \subset \mathbb{P}^2$ is

$$(7.3) \quad w_\ell = H - \frac{2n-9}{2}\epsilon,$$

In particular

$$(7.4) \quad \text{slope}_{\text{rat}}(\text{NE}(S^{[2]})) \leq \frac{2}{2n-9}.$$

Moreover the curves $C \subset S$ with hyperelliptic normalizations associated to the lines $\ell \subset \mathbb{P}^2 \subset S^{[2]}$ lie in $|H|$, have geometric genus $p_g = 2n - 10$, and $\rho_{\text{sing}}(p_a(C), 1, 2, p_g) = n(n - 13) + 42 \geq 0$.

Proof. Consider the lattice $\mathbb{Z}F \oplus \mathbb{Z}G$ with intersection matrix

$$\begin{bmatrix} F^2 & F.G \\ G.F & G^2 \end{bmatrix} = \begin{bmatrix} 2 & n \\ n & 4 \end{bmatrix}, \quad n \geq 6.$$

Since it has signature $(1, 1)$, then, by a result of Nikulin [44] (see also [40, Cor. 2.9(i)]), there is an algebraic K3 surface S_0 with the given Picard lattice. Performing Picard-Lefschetz reflections on the lattice, we can assume that G is nef, by [2, VIII, Prop. 3.9]. By Riemann-Roch and Serre duality, we have $G > 0$ and $F > 0$. Straightforward computations on the Picard lattice rules out the existence of divisors Γ satisfying $\Gamma^2 = -2$ and $\Gamma.F \leq 0$ or $\Gamma.G \leq 1$; or $\Gamma^2 = 0$ and $\Gamma.F = 1$ or $\Gamma.G = 1, 2$. By [49] it follows that both $|F|$ and $|G|$ are base point free, $\varphi_{|F|} : S_0 \rightarrow \mathbb{P}^2$ is a double cover and $\varphi_{|G|} : S_0 \rightarrow \mathbb{P}^3$ is an embedding onto a smooth quartic not containing lines. As explained in § 4, $S_0^{[2]}$ contains a \mathbb{P}^2 arising from the double cover.

If ℓ_0 is a line on the \mathbb{P}^2 , the corresponding class in $H^2(S_0^{[2]}, \mathbb{Q})$ is $w_{\ell_0} = 2F - 3\epsilon$, which coincides with the corresponding integral class ρ_{ℓ_0} (cf. [24, Example 5.1]).

As S_0 is a quartic surface not containing lines, $S_0^{[2]}$ admits an involution

$$\iota : S_0^{[2]} \rightarrow S_0^{[2]}; \quad \xi \mapsto (\ell_\xi \cap S_0) \setminus \xi,$$

by [6, Prop. 11], where ℓ_ξ is the line determined by ξ , and the sign \setminus means that we take the residual subscheme. The corresponding involution on cohomology is given by (cf. e.g. [46, (4.1.6)-(4.1.7)])

$$v \mapsto q(G - \epsilon, v) \cdot (G - \epsilon) - v.$$

The involution sends the \mathbb{P}^2 into another \mathbb{P}^2 , and the corresponding class associated to a line on it is

$$(7.5) \quad q(G - \epsilon, 2F - 3\epsilon) \cdot (G - \epsilon) - (2F - 3\epsilon) = 2((n-3)G - F) - (2n-9)\epsilon.$$

In order to obtain a general K3 with the desired property we now deform $S_0^{[2]}$. Precisely, we consider a general deformation of $S_0^{[2]}$ such that (i) ϵ remains algebraic and (ii) $\iota(\mathbb{P}^2)$ is preserved. Deformations satisfying (i) form a countable union of hyperplanes in the deformation space of $S_0^{[2]}$, which is smooth and of dimension 21, and may be characterized as those of the form $S^{[2]}$, where S is a K3 surface (see [7, Thm. 6 and Rem. 2]). Deformations preserving $\iota(\mathbb{P}^2)$ can be characterized as those preserving the image in $H^2(S^{[2]}, \mathbb{Z})$ of the class of the line in $\iota(\mathbb{P}^2)$ as an algebraic class (see [24, Thm. 4.1 and

Cor. 4.2] or [58]), that is, using (7.5), those deformations keeping $H := (n-3)G - F \in \text{Pic}(S_0^{[2]})$, or, equivalently, $H \in \text{Pic}(S)$, by (6.2). As $H^2 = [(n-3)G - F]^2 = 2(n^2 - 9n + 19) \geq 2$ for $n \geq 6$ and H is primitive, those deformations form a divisor in the 20-dimensional space of deformations keeping ϵ algebraic, by [35, Thm. 14].

We therefore obtain a 19-dimensional space of deformations of $S_0^{[2]}$, whose general member is $S^{[2]}$, where (S, H) is a general primitively polarized (algebraic) $K3$ surface of degree $H^2 = 2(n^2 - 9n + 19)$, $n \geq 6$, and $S^{[2]}$ contains a plane.

The class $w_\ell \in H^2(S^{[2]}, \mathbb{Q})$ corresponding to the line ℓ is as in (7.3), yielding (7.4).

As S is general, it does not contain smooth rational curves, so that the \mathbb{P}^2 is not of the form $C^{[2]}$, for a smooth rational curve C on S . By Lemma 2.4, the lines in the \mathbb{P}^2 in $S^{[2]}$ give rise to a two-dimensional family V of curves on S with hyperelliptic normalizations, so that $R_V = \mu(\mathbb{P}^2)$, where $\mu : S^{[2]} \rightarrow \text{Sym}^2(S)$ is the Hilbert-Chow morphism. By (7.3) we have $\ell.H = H^2$, so that, by the very definition of the divisor H in $H^2(S^{[2]}, \mathbb{Z})$, the lines in the \mathbb{P}^2 correspond to curves $C \in |H|$. Comparing (6.10) and (7.3), we see that $g_0(C) = 2n - 10$, cf. (6.12). Now we note that the general line in the \mathbb{P}^2 is not tangent to $\Delta = 2\epsilon$. (Indeed, this follows by deformation since in $S_0^{[2]}$ we have that $\iota(\mathbb{P}^2) \cap \Delta$ is a smooth plane sextic, since we have a composite map $S_0 \rightarrow \mathbb{P}^2 \rightarrow \iota(\mathbb{P}^2)$ that is finite of degree two, whence ramified along a smooth sextic, as S_0 is a smooth $K3$.) Therefore we have $p_g(C) = 2n - 10$. We compute $\rho_{\text{sing}} = n(n - 13) + 42 \geq 0$ (recall that $n \geq 6$). \square

The examples contained in the above proposition is interesting in several regards.

Notice first that $q(\ell) = -5/2$, cf. (6.3), in accordance with the prediction in [24, Conj. 3.6].

The proposition shows in particular that the correspondence in Remark 3.7 is not one-to-one and also shows that the case $\dim(V) = \dim(R_V) = 2$ of Proposition 3.6 actually occurs.

The result also gives nontrivial examples of curves in $|H|$ with hyperelliptic normalizations and positively answers the hyperelliptic existence problem for $p_a = n^2 - 9n + 20$ and $p_g = 2n - 10$, $n \geq 6$.

Moreover (7.4) shows that there is no lower bound on $\text{slope}_{\text{rat}}(\text{NE}(S^{[2]}))$ as the degree of the polarization tends to infinity. The same follows from (7.9) in Proposition 7.7 below. Both the bounds (7.4) and (7.9) below in fact yield better bounds on $\text{slope}(\text{NE}(S^{[2]}))$ than (6.22).

Finally, the conics on the \mathbb{P}^2 give a five-dimensional family $V(2)$ of irreducible curves with hyperelliptic normalizations on S . Of course this family has obvious non-integral members, corresponding to non-integral conics. More generally, for any $m \geq 3$, the $(3m - 1)$ -dimensional family of nodal rational curves in $|\mathcal{O}_{\mathbb{P}^2}(m)|$ (cf. [15, Thm. 1.1]) yields corresponding families $V(m)$ of curves in $|mH|$ with hyperelliptic normalizations with $\dim V(m) = 3m - 1 \geq 5$ and $\dim(R_V) = 2$, showing in particular that the case $\dim(V) > \dim(R_V) = 2$ of Proposition 3.6 actually occurs.

In the case of the conics, we compute $p_g = 4n - 19$ as above and as $p_a(2H) = 4n^2 - 36n + 77$, we get $\rho_{\text{sing}} = 4n(n - 11) + 117 \geq -3$ in these cases. This does not contradict [21, Thm. 1].

7.2. Threefolds birational to \mathbb{P}^1 -bundles in $S^{[2]}$. We start with an explicit example in the special case of a quartic surface.

Example 7.6. In the case of a general quartic S in \mathbb{P}^3 we can find a \mathbb{P}^1 -bundle over S in $S^{[2]}$, arising from the two-dimensional family of hyperplane sections of geometric genus two. In fact, taking the tangent plane through the general point of S we get a nodal curve of geometric genus 2. We obtain in this way a family V of nodal curves with hyperelliptic normalizations in the hyperplane linear system. This family is parametrized by an open subset of S , and the locus in $S^{[2]}$ covered by the associated rational curves is birational to a \mathbb{P}^1 -bundle over this open subset. To see this, set $C_p := (S \cap T_p S)$, and let \tilde{C}_p be the normalization of C_p . Note that the \mathfrak{g}_2^1 on \tilde{C}_p , viewed on C_p , is given by the pencil of lines in $T_p S$ through the node p . If, for two distinct points $p, q \in S$, the \mathfrak{g}_2^1 s on \tilde{C}_p and \tilde{C}_q had two common points, say x and y (so that the map Φ_V in (2.5) sends $(p, x + y)$ and $(q, x + y)$ to the same

point $x + y$ in $\text{Sym}^2(S)$, then the line $T_p S \cap T_q S$, which is bitangent to S , would also pass through x and y . This is absurd, as $\deg(S) = 4$.

By (6.10), the class $w \in H^2(S^{[2]}, \mathbb{Q})$ corresponding to the curves of geometric genus 2 is $w = H - \frac{3}{2}\epsilon$, whence $q(w) = -1/2$, as predicted by [24, Conj. 3.6]. Moreover, performing the usual involution on the quartic, we send the constructed uniruled threefold to another one, with corresponding fibre class given by ϵ , so that it simply is the \mathbb{P}^1 -bundle Δ over S . This shows that also our original threefold was smooth, so in fact a \mathbb{P}^1 -bundle over S .

We now give an infinite series of examples of general $K3$ s whose Hilbert squares contain threefolds birational to \mathbb{P}^1 -bundles.

Proposition 7.7. *Let (S, H) be a general primitively polarized $K3$ surface of degree $H^2 = 2(d^2 - 1)$, for $d \geq 2$. Then $S^{[2]}$ contains a threefold birational to a \mathbb{P}^1 -bundle over a $K3$ surface.*

The class $w_f \in H^2(S^{[2]}, \mathbb{Q})$ corresponding to a fibre is

$$(7.8) \quad w_f = H - d\epsilon \in H^2(S^{[2]}, \mathbb{Z}).$$

In particular

$$(7.9) \quad \text{slope}_{\text{rat}}(\text{NE}(S^{[2]})) \leq \frac{1}{d}.$$

Moreover the curves $C \subset S$ with hyperelliptic normalizations associated to the fibres of the threefold lie in $|H|$, have geometric genus $p_g = 2d - 1$, and $\rho_{\text{sing}}(p_a(C), 1, 2, p_g) = d(d - 4) + 4 \geq 0$.

Proof. This time we start with the lattice $\mathbb{Z}F \oplus \mathbb{Z}G$ with intersection matrix

$$\begin{bmatrix} F^2 & F.G \\ G.F & G^2 \end{bmatrix} = \begin{bmatrix} -2 & d \\ d & 4 \end{bmatrix}, \quad d \geq 2.$$

As in Proposition 7.2 one easily shows that there is an algebraic $K3$ surface S_0 with $\text{Pic}(S_0) = \mathbb{Z}F \oplus \mathbb{Z}G$ and that $\varphi_{|G|} : S_0 \rightarrow \mathbb{P}^3$ is an embedding onto a smooth quartic not containing lines and F is a smooth, irreducible rational curve.

We now consider the divisor $F \subset S_0^{[2]}$, defined as the length-two schemes with some support along F . One easily sees that this is a threefold birational to a \mathbb{P}^1 -bundle over S_0 and that the class in $H^2(S_0^{[2]}, \mathbb{Z})$ corresponding to the fibres f is $\rho_f = F$, cf. [24, Example 4.6].

The involution on the quartic sends this threefold to another threefold birational to a \mathbb{P}^1 -bundle over S_0 and the corresponding class of the fibres is $dG - F - d\epsilon$. This threefold satisfies the conditions in [24, Thm. 4.1] by [24, Example 4.6], so that, as in the previous proposition, we can deform $S_0^{[2]}$, keeping ϵ algebraic and $H := dG - F$. The rest now follows as in the proof of Proposition 7.2. \square

The square of the class of the fibres of the uniruled threefolds constructed above is $q(f) = -2$, as predicted in [24, Conj. 3.6].

The obtained family V of curves on S with hyperelliptic normalizations has $\dim(V) = 2$ and $\dim(R_V) = 3$, showing that also this case of Proposition 3.6 actually occurs. This family gives nontrivial examples of curves in $|H|$ with hyperelliptic normalizations and positively answers the hyperelliptic existence problem for $p_a = 2(d^2 - 1)$ and $p_g = 2d - 1$ for every $d \geq 2$. Note that the case $d = 2$ is the case described in [21, Example 2.8].

REFERENCES

- [1] E. Arbarello, M. Cornalba, *Su una congettura di Petri*, Comment. Math. Helvetici **56** (1981), 1–38.
- [2] W. Barth, K. Hulek, C. Peters, A. Van de Ven, *Compact complex surfaces*, Second edition, Springer-Verlag, Berlin, 2004.
- [3] T. Bauer, *Seshadri constants of quartic surfaces*, Math. Ann. **309** (1997), 475–481.

- [4] T. Bauer, *Seshadri constants on algebraic surfaces*, Math. Ann. **313** (1999), 547–583.
- [5] A. Beauville, *Fano threefolds and K3 surfaces*, The Fano Conference, 175–184, Univ. Torino, Turin, 2004.
- [6] A. Beauville, *Some remarks on Kähler manifolds with $c_1 = 0$* , Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math. **39**, Birkhäuser, 1983, pp. 1–26.
- [7] A. Beauville, *Variétés Kählériennes dont la première classe de Chern est nulle*, J. Diff. Geom. **18** (1983), 755–782.
- [8] M. Beltrametti, P. Francia, A. J. Sommese, *On Reider’s method and higher order embeddings*, Duke Math. J. **58** (1989), 425–439.
- [9] M. Beltrametti, A. J. Sommese, *Zero cycles and k th order embeddings of smooth projective surfaces. With an appendix by Lothar Göttsche*, Sympos. Math., XXXII, Problems in the theory of surfaces and their classification (Cortona, 1988), 33–48, Academic Press, London, 1991.
- [10] A. Biancofiore, M. L. Fania, A. Lanteri, *Polarized surfaces with hyperelliptic sections*, Pacific J. Math. **143** (1990), 9–24.
- [11] S. Boucksom, *Le cône kählérien d’une variété hyperkählérienne*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), 935–938.
- [12] D. Burns, Y. Hu, T. Luo, *HyperKähler manifolds and birational transformations in dimension 4*, Contemp. Math. **322** (2003), 141–149.
- [13] X. Chen, *A simple proof that rational curves on K3 are nodal*, Math. Ann. **324** (2002), 71–104.
- [14] X. Chen, *Rational curves on K3 surfaces*, J. Alg. Geom. **8** (1999), 245–278.
- [15] L. Chiantini, E. Sernesi, *Nodal curves on surfaces of general type*, Math. Ann. **307** (1997), 41–56.
- [16] O. Debarre, *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.
- [17] J.-P. Demailly, *Singular Hermitian metrics on positive line bundles*, Complex algebraic varieties (Bayreuth, 1990), Lecture Notes in Math. **1507**, 87–104. Springer, Berlin, 1992.
- [18] L. Ein, O. Küchle, R. Lazarsfeld, *Local positivity of ample line bundles*, J. Diff. Geom. **42** (1995), 193–219.
- [19] D. Faenzi, *Bundles over the Fano threefold V_5* , Comm. Alg. **33** (9) (2005), 3061–3080.
- [20] F. Flamini, *Some results of regularity for Severi varieties of projective surfaces*, Comm. Alg. **29** (6) (2001), 2297–2311.
- [21] F. Flamini, A. L. Knutsen, G. Pacienza, *Singular curves on a K3 surface and linear series on their normalizations*, Internat. J. Math. **18** (2007), 1–23.
- [22] J. Harris, I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics **187**, Springer-Verlag, New York, 1998.
- [23] B. Hassett, Y. Tschinkel, *Abelian fibrations and rational points on symmetric products*, Internat. J. Math. **11** (2000), 1163–1176.
- [24] B. Hassett, Y. Tschinkel, *Rational curves on holomorphic symplectic fourfolds*, Geom. Func. Anal. **11** (2001), 1201–1228.
- [25] B. Hassett, Y. Tschinkel, *Moving and ample cones of holomorphic symplectic fourfolds*, preprint arXiv:0710.0390v1 [math.AG].
- [26] E. Horikawa, *Algebraic surfaces of general type with small c_1^2 . I.*, Ann. of Math. (2) **104** (1976), 357–387.
- [27] E. Horikawa, *Algebraic surfaces of general type with small c_1^2 . II.*, Invent. Math. **37** (1976), 121–155.
- [28] E. Horikawa, *Algebraic surfaces of general type with small c_1^2 . III.*, Invent. Math. **47** (1978), 209–248.
- [29] E. Horikawa, *Algebraic surfaces of general type with small c_1^2 . IV.*, Invent. Math. **50** (1978/79), 103–128.
- [30] Y. Hu, S.-T. Yau, *HyperKähler manifolds and birational transformations*, Adv. Theor. Math. Phys. **6** (2002), 557–574.
- [31] D. Huybrechts, *Compact hyperkähler manifolds: basic results*, Invent. Math. **135** (1999), 63–113, *Erratum: “Compact hyperkähler manifolds: basic results”*, Invent. Math. **152** (2003), 209–212.
- [32] D. Huybrechts, *The Kähler cone of a compact hyperkähler manifold*, Math. Ann. **326** (2003), 499–513.
- [33] A. L. Knutsen, *A note on Seshadri constants on general K3 surfaces*, arXiv:0704.2377.
- [34] A. L. Knutsen, *Remarks on families of curves with hyperelliptic normalizations*, arXiv:0705.0906.
- [35] K. Kodaira, *On the structure of compact complex analytic surfaces, I*, Amer. J. Math. **86** (1964) 751–798.
- [36] J. Kollár, *Rational curves on algebraic varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3 Folge. A Series of Modern Surveys in Mathematics* **32**. Springer-Verlag, Berlin, 1996.
- [37] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics **134**, Cambridge University Press, Cambridge, 1998.
- [38] K. Lønsted, S. L. Kleiman, *Basics on families of hyperelliptic curves*, Comp. Math. **38**, 1 (1979), 83–111.
- [39] S. Mori, S. Mukai, *The uniruledness of the moduli space of curves of genus 11*, Algebraic Geometry, Proc. Tokyo/Kyoto, Lecture Notes in Math. **1016**, 334–353. Springer, Berlin, 1983.
- [40] D. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. **75** (1984), 105–121.

- [41] S. Mukai, *Curves, K3 surfaces and Fano 3-folds of genus ≤ 10* , Algebraic geometry and commutative algebra, Vol. I, 357–377, Kinokuniya, Tokyo, 1988.
- [42] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984), 101–116.
- [43] D. Mumford, *Rational equivalence of 0-cycles on surfaces*, J. Math. Kyoto Univ. **9** (1968), 195–204.
- [44] V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR-Izv. **14** (1980), 103–167.
- [45] K. Oguiso, *Seshadri constants in a family of surfaces*, Math. Ann. **323** (2002), 625–631.
- [46] K. O’Grady, *Involutions and linear systems on holomorphic symplectic manifolds*, Geom. Funct. Anal. **15** (2005), 1223–1274.
- [47] Z. Ran, *Hodge theory and deformations of maps*, Comp. Math. **97** (1995), 309–328.
- [48] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. **127** (1988), 309–316.
- [49] B. Saint-Donat, *Projective models of $K - 3$ surfaces*, Amer. J. Math. **96** (1974), 602–639.
- [50] E. Sernesi, *On the existence of certain families of curves*, Invent. Math. **75** (1984), 25–57.
- [51] E. Sernesi, *Deformations of Algebraic Schemes*, Grundlehren der mathematischen Wissenschaften **334**, Springer-Verlag, Berlin, 2006.
- [52] F. Serrano, *The adjunction mapping and hyperelliptic divisors on surfaces*, J. reine und Angw. Math. **381** (1987), 90–109.
- [53] F. Severi, *Vorlesungen über algebraische Geometrie*, Teubner, Leipzig, 1921.
- [54] A. J. Sommese, A. Van de Ven, *On the adjunction mapping*, Math. Ann. **278** (1987), 593–603.
- [55] A. Steffens, *Remarks on Seshadri constants*, Math. Z. **227** (1998), 505–510.
- [56] A. Tannenbaum, *Families of curves with nodes on $K - 3$ surfaces*, Math. Ann. **260** (1982), 239–253.
- [57] B. Tesser, *Résolution simultanée I, II*. Lecture Notes in Math. **777**, 71–146. Springer, Berlin, 1980.
- [58] C. Voisin, *Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes*, Complex projective geometry (Trieste, 1989/Bergen, 1989), 294–303, London Math. Soc. Lecture Note Ser. **179**, Cambridge Univ. Press, Cambridge, 1992.
- [59] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe. Cours Spécialisés [Specialized Courses] 10*. Société Mathématique de France, Paris, 2002. See also *Hodge theory and complex algebraic geometry. I and II*, Cambridge Studies in Advanced Mathematics, 76 & 77. Cambridge University Press, Cambridge.
- [60] J. Wahl, *Deformations of plane curves with nodes and cusps*, Amer. J. Math. **96** (1974), 529–577.
- [61] J. Wierzba, *Birational geometry of symplectic 4-folds*, unpublished preprint.
- [62] J. Wierzba, *Contractions of symplectic varieties*, J. Alg. Geom. **12** (2003), 507–534.
- [63] J. Wierzba, J. Wisniewski, *Contractions of symplectic varieties*, Duke Math. J. **120** (2003), 65–94.

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APPENDIX A

PARTIAL DESINGULARIZATIONS OF FAMILIES OF NODAL CURVES

 EDOARDO SERNESI¹

In this Appendix we show how to construct simultaneous partial desingularizations of families of nodal curves, generalizing a well known procedure of simultaneous total desingularization, as described in [4].

We work over an algebraically closed field \mathbf{k} of characteristic 0. For every morphism $X \rightarrow Y$, and for every $y \in Y$, we denote by $X(y)$ the scheme-theoretic fibre of y .

Theorem A.1. *Let*

$$f : \mathcal{C} \longrightarrow V$$

be a flat projective family of curves, with \mathcal{C} and V algebraic schemes, such that all fibres have at most ordinary double points (nodes) as singularities. Let $\delta \geq 1$ be an integer. Then there is a commutative diagram:

$$\begin{array}{ccccc} D_\delta & \hookrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} \\ & \searrow q & \downarrow f' & & \downarrow f \\ & & E_{(\delta)} & \xrightarrow{\alpha} & V \end{array}$$

with the following properties:

- (i) α is finite and unramified, the square is cartesian, and q is an étale cover of degree δ .
- (ii) The left triangle defines a marking of all δ -tuples of nodes of fibres of f . In particular f' parametrizes all curves of the family f having $\geq \delta$ nodes and, for each $\eta \in E_{(\delta)}$, $D_\delta(\eta) \subset \mathcal{C}'(\eta)$ is a set of δ nodes of the curve $\mathcal{C}'(\eta)$.
- (iii) The diagram is universal with respect to properties (i) and (ii). Precisely, if

$$\begin{array}{ccccc} \tilde{D} & \hookrightarrow & \tilde{E} \times_V \mathcal{C} & \longrightarrow & \mathcal{C} \\ & \searrow \tilde{q} & \downarrow \tilde{f} & & \downarrow f \\ & & \tilde{E} & \longrightarrow & V \end{array}$$

is a diagram having the properties analogous to (i) and (ii), then there is a unique factorization

$$\tilde{E} \xrightarrow{\varphi} E_{(\delta)} \xrightarrow{\alpha} V$$

such that \tilde{q} and \tilde{f} are obtained by pulling back q and f' by φ .

If moreover $E_{(\delta)}$ is normal, then the above diagram can be enlarged as follows:

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$$\begin{array}{ccccc}
& & \bar{\mathcal{C}} & & \\
& & \downarrow \beta & & \\
D_\delta & \hookrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} \\
& \searrow q & \downarrow f' & & \downarrow f \\
& & E_{(\delta)} & \xrightarrow{\alpha} & V
\end{array}$$

where:

(iv) β is a birational morphism such that, for each $\eta \in E_{(\delta)}$, the restriction:

$$\beta(\eta) : \bar{\mathcal{C}}(\eta) \longrightarrow \mathcal{C}'(\eta)$$

is the partial normalization at the nodes $D_\delta(\eta)$.

(v) The composition $\bar{f} := f' \circ \beta$ is flat.

Proof. Consider the first relative cotangent sheaf $\mathcal{T}_{\mathcal{C}/V}^1$. Since all fibres of f are nodal, $\mathcal{T}_{\mathcal{C}/V}^1$ commutes with base change ([3, Lemma 4.7.5] or [5]), thus on every fibre $\mathcal{C}(v)$, $v \in V$, it restricts to $\mathcal{T}_{\mathcal{C}(v)}^1$, which is the structure sheaf of the scheme of nodes of $\mathcal{C}(v)$. It follows that we have

$$\mathcal{T}_{\mathcal{C}/V}^1 = \mathcal{O}_E$$

for a closed subscheme $E \subset \mathcal{C}$ supported on the nodes of the fibres of f . Consider the composition

$$f_E : E \subset \mathcal{C} \xrightarrow{f} V$$

By construction it follows that f_E is finite and unramified. Now fix $\delta \geq 1$ and consider the fibre product:

$$\underbrace{E \times_V \cdots \times_V E}_\delta$$

Since f_E is finite and unramified, it follows from [1, Exp.1, Prop. 3.1], and by induction on δ (see [3, Lemma 4.7.11(i)]), that we have a disjoint union decomposition:

$$E \times_V \cdots \times_V E = \Delta \coprod E_\delta$$

where Δ is the union of all the diagonals, and E_δ consists of all the ordered δ -tuples of distinct points of E mapping to the same point of V ; moreover the natural projection morphism

$$E_\delta \longrightarrow V$$

is finite and unramified.

There is a natural action of the symmetric group Σ_δ on E_δ that commutes with the projection to V . We denote the quotient E_δ/Σ_δ by $E_{(\delta)}$. Since the composition

$$E_\delta \longrightarrow E_{(\delta)} \longrightarrow V$$

is finite and unramified and the first morphism is an étale cover, the morphism $\alpha : E_{(\delta)} \rightarrow V$ is finite and unramified. Note that if, for a closed point $v \in V$, $\mathcal{C}(v)$ has $\delta + t$ nodes as the only singularities, with $t > 0$, then $\alpha^{-1}(v)$ has degree $\binom{\delta+t}{t}$. Now let

$$D_\delta = \{(\eta, e) : e \in \text{Supp}(\eta)\} \subset E_{(\delta)} \times_V E$$

Then the first projection defines the tautological family:

$$(A.2) \quad \begin{array}{c} D_\delta \subset E_{(\delta)} \times_V E \subset E_{(\delta)} \times_V \mathcal{C} \\ \downarrow q \\ E_{(\delta)} \end{array}$$

which is an étale cover of degree δ . The fibre $D_\delta(\eta)$ is the δ -tuple parametrized by η , for each $\eta \in E_{(\delta)}^2$. We therefore have the following diagram:

$$\begin{array}{ccccc} D_\delta & \hookrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} \\ & \searrow q & \downarrow f' & & \downarrow f \\ & & E_{(\delta)} & \xrightarrow{\alpha} & V \end{array}$$

where we have denoted by $\mathcal{C}' = E_{(\delta)} \times_V \mathcal{C}$. The fibres of f' are all the curves of the family f having $\geq \delta$ nodes. For each $\eta \in E_{(\delta)}$ the divisor $D_\delta(\eta) \subset \mathcal{C}'(\eta)$ marks the set of δ nodes parametrized by η . This proves (i) and (ii).

(iii) follows from the fact that $\alpha : E_{(\delta)} \rightarrow V$ is the relative Hilbert scheme of degree δ of $f_E : E \rightarrow V$, and (A.2) is the universal family.

Assume that $E_{(\delta)}$ is normal. Then we can normalize \mathcal{C}' locally around D_δ as in [4, Theorem 1.3.2], to obtain a birational morphism β having the required properties (iv) and (v). \square

A typical example of the situation considered in the theorem is when V parametrizes a complete linear system of curves on an algebraic surface. If the morphism f_E is *self-transverse of codimension 1* (see [3, Definition 4.7.13]) then the Severi variety of irreducible δ -nodal curves is nonsingular and of codimension δ , and $E_{(\delta)}$ is nonsingular (see [3, Lemma 4.7.14]), so that the theorem applies and the simultaneous partial desingularization exists. This happens for example for the linear systems of plane curves [3, Proposition 4.7.17].

REFERENCES

- [1] *Revêtements étales et groupe fondamentale*, Séminaire de Géométrie Algébrique du Bois Marie 1960-61 (SGA1), Lecture Notes in Math. **224**. Springer, Berlin, 1971.
- [2] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**. Springer-Verlag, New York-Heidelberg, 1977.
- [3] E. Sernesi, *Deformations of Algebraic Schemes*, Grundlehren der Mathematischen Wissenschaften **334**. Springer-Verlag, Berlin, 2006.
- [4] B. Tesser, *Résolution simultanée I, II*. Lecture Notes in Math. **777**, 71–146. Springer, Berlin, 1980.
- [5] J. Wahl, *Deformations of plane curves with nodes and cusps*, Amer. J. Math. **96** (1974), 529–577.

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²If $\delta = 1$ then $E_{(1)} = E$ and $D_1 \subset E \times_V E$ is the diagonal.