Lifting sections from a hypersurface, after Takayama

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Recall the theorem we want to prove.

Key Theorem. *Let X be a smooth complex projective variety, let $V \subset X$ be an irreducible subvariety and $\mu : X' \to X$ a modification with $V \not\subset \mu(\mathrm{Exc}(\mu))$. Let $V' \subset X'$ be the strict transform of V. Let D be a divisor on X and $\mu^* D = A' + E'$ a decomposition such that :*

a) A' is an ample $\mathbf{Q}-$divisor ;

*b) E' is an effective $\mathbf{Q}-$divisor such that V' is a maximal lc center * for the pair (X', E').*

Then : $$\mathrm{vol}_{X|V}(K_X + D) \geq \mathrm{vol}(K_V).$$

*That is an irreducible component of $\mathrm{Nklt}(X', E')$ such that at its general point the pair is log-canonique.

The proof of this result will make use of an extension theorem.

Extension Theorem. *Let X be a smooth complex projective variety and $H \subset X$ a smooth irreducible hypersurface. Let $D \sim A + E$ a big divisor on X with*

- *A a nef and big $\mathbf{Q}$-divisor such that $H \not\subset \mathbf{B}_+(A)$;*
- *E an effective $\mathbf{Q}$-divisor whose support does not contain H and such that the pair $(H, E|_H)$ is klt.*

Then the restriction

$$H^0(X, m(K_X + H + D)) \longrightarrow H^0(H, m(K_H + D|_H))$$

is surjective for all integer $m > 0$.

Idea of the proof of the Extension Thm. :

Take $s \in H^0(H, m(K_H + D|_H))$. Let $S \subset H$ be the associated divisor. Then construct a divisor $P := P_s$ on X such that

$$P|_H = \alpha S + \beta B + E|_H \leq S + \text{klt}$$

$$\text{with} \quad \alpha, \beta \in]0,1[, \quad \text{and} \quad B \quad \text{very ample.}$$

Hence

$$\mathscr{I}_{P|H} \supset \mathscr{O}_H(-S)$$

so that

$$s \in H^0(H, \mathscr{I}_{P|H}(m(K_H + D|_H))).$$

Then one uses Nadel vanishing plus the classical Calstelnuovo-Mumford argument to deduce that sections in the latter space may be lifted to X.

Details of the proof of the Ext. Thm. :

First notice that we may assume A ample. Indeed, write

$$A = A' + E' = \text{ample} + \text{effective}$$

with Supp$(E') \not\supset H$ (here we use $H \not\subset \mathbf{B}_+(A)$), and

$$D \sim [(1-\varepsilon)A + \varepsilon A'] + \varepsilon E' + E.$$

Then, for $0 < \varepsilon \leq 1$,

- $(1-\varepsilon)A + \varepsilon A'$ is ample ;
- the $\mathbf{Q}-$divisor $\varepsilon E' + E$ is effective and its support does not contain H ;
- the pair $(H, \varepsilon E'|_H + E|_H)$ is still klt (for $\varepsilon << 1$.)

Then choose an very ample divisor $B := V(s_B)$ on X : Supp$(B) \not\supset H$. Take $\ell \in \mathbf{N}$ large enough in order to have :

- $A - \frac{m-1}{\ell m} nB$ ample ;
- the pair $(H, \frac{m-1}{\ell m} nB|_H + E|_H)$ is still klt

Let s be an element of $H^0(H, m(K_H+D|_H))$, $S := V(s)$. For any integer $r > 0$, define an ideal sheaf on H as follows :

$$\mathscr{I}_r = \mathscr{I}_{\frac{r-1}{m}S+E|_H}$$

Since the pair $(H, E|_H)$ is klt, we have

$$\mathscr{I}_{\ell m} \supset \mathscr{I}_{\ell m+1} = \mathscr{I}_{\ell S+E|_H} = \mathscr{I}_{E|_H}(-\ell S) = \mathscr{O}_H(-\ell S)$$

So the section s^ℓ vanishes along $\mathscr{I}_{\ell m}$, and

$$s^\ell \cdot s_B^n \in H^0(H, \mathscr{I}_{\ell m}\ell m(K_H + D|_H) + nB|_H).$$

We have the

Lemma. *For any integer $r > 0$, the image of the restriction map :*

$$H^0(r(K_X+H+D)+nB) \longrightarrow H^0(r(K_H+D|_H)+nB|_H)$$

contains the sections vanishing along $\mathscr{I}_r$.

Assume the Lemma for the moment. Then we can lift the section $s^\ell \cdot s_B^n$ to X :

$$\exists t \in H^0(X, m\ell(K_X+H+D)+nB) \ : \ t|_H = s^\ell \cdot s_B^n.$$

Set

$$N := (m-1)(K_X + H + D) + D$$

and

$$P := \frac{m-1}{\ell m}\,\mathrm{div}(t) + E.$$

Then, by the choice of ℓ, their difference

$$N - P \sim D - \frac{m-1}{\ell m} nB - E \sim A - \frac{m-1}{\ell m} nB$$

is an ample $\mathbf{Q}$−divisor. So Nadel's vanishing gives

$$H^1(X, \mathscr{I}_P(K_X + N)) = 0.$$

This easily implies that

(†) the sections of $H^0(H, m(K_H + D|_H))$ vanishing along $\mathscr{I}_P$ lift to X.

Indeed we have $K_X + N = m(K_X + H + D) - H$. On the other hand, there exists an ideal sheaf

$$\mathrm{adj}_{H,P} \subset \mathscr{I}_P \subset \mathscr{O}_X$$

such that the following sequence is exact :

$$0 \longrightarrow \mathscr{I}_P(-H) \longrightarrow \mathrm{adj}_{H,P} \longrightarrow \mathscr{I}_{P|_H} \longrightarrow 0$$

Tensoring this exact sequence with $m(K_X + N + H)$, and using the vanishing above we get a surjection from

$$H^0(X, \mathrm{adj}_{H,P}(m(K_X + H + D)))$$

to

$$H^0(H, \mathscr{I}_{P|_H}(m(K_X + H + D))),$$

and (†) is proved.

So we are reduced to prove that our section s vanishes along $\mathscr{I}_{P|_H}$. For this notice that

$$\begin{aligned} P|_H &= (\frac{m-1}{\ell m}\mathrm{div}(t) + E)|_H \\ &= \frac{m-1}{\ell m}\mathrm{div}(s^\ell \cdot s_B^n) + E|_H \\ &= \frac{m-1}{\ell m}(\ell S + nB|_H) + E|_H \\ &\leqslant S + [\frac{m-1}{\ell m}nB|_H + E|_H] = S + \mathrm{klt} \end{aligned}$$

Hence

$$\mathscr{I}_{P|_H} \supset \mathscr{O}_H(-S)$$

and, by (†), the proof of the Extension Theorem is proved (modulo the Lemma !).

Proof of the Lemma : By induction on r.

Base of the induction. (Nadel vanishing).

Notice that, since $D + nB - E \sim A + nB$ is ample, we have

$$H^1(X, \mathscr{I}_E(K_X + D + nB)) = 0 \quad \text{(Nadel.)}$$

Hence, twisting the exact sequence

$$0 \longrightarrow \mathscr{I}_E(-H) \longrightarrow \mathrm{adj}_{H,E} \longrightarrow \mathscr{I}_{E|_H} \longrightarrow 0$$

by $K_X + H + D + nB$, we get a surjection

$$H^0(X, \mathrm{adj}_{H,E}(K_X + H + D + nB)) \\ \twoheadrightarrow H^0(H, \mathscr{I}_{E|_H}(K_H + D|_H + nB|_H))$$

Among the hypotheses of the Extension Theorem we have that $(H, E|_H)$ is klt, so

$$\mathscr{O}_H = \mathscr{I}_{E|_H} \quad (= \mathscr{I}_1),$$

and we are done.

Inductive step. (Nadel vanishing+CM).

We want to pass from r to $r+1$. Let $|V| \subset |r(K_X+H+D)+nB|$ be the inverse image of the linear system of sections of $\mathscr{O}_H(r(K_H + D|_H) + nB|_H)$ vanishing along de $\mathscr{I}_r$. By the inductive hypothesis, the latter is the restriction $|V||_H$.

We start with a

Claim. *The sections of*

$$H^0(H, (r+1)(K_H + D|_H) + nB|_H)$$

vanishing along $\mathscr{I}_{E|_H;|V||_H}$[†] *may be lifted to* X.

Indeed, using again the adjoint sequence :

$$0 \longrightarrow \mathscr{I}_{E;|V|}(-H) \longrightarrow \mathrm{adj}_{E,H;|V|} \longrightarrow \mathscr{I}_{E|_H;|V||_H} \longrightarrow 0$$

and a Nadel vanishing, we get a surjection

$$H^0(X, \mathrm{adj}_{E,H;|V|}((r+1)(K_X + H + D) + nB))$$
$$\twoheadrightarrow H^0(H, \mathscr{I}_{E|_H;|V||_H}((r+1)(K_H + D|_H) + nB|_H))$$

and the Claim is proved.

[†]Recall that

$$\mathscr{I}_{D;|V|} = \mu_* \mathscr{O}_{X'}(K_{X'/X} - [\mu^* D] - F)$$

where $F \subset X'$ is the fixed part of $\mu^*|V|$ in a log-resolution of $|V|$. In particular, if (X, D) is klt, then $K_{X'/X} - [\mu^* D]$ is effective, thus

$$\mathscr{I}_{D;|V|} = \mu_* \mathscr{O}_{X'}(K_{X'/X} - [\mu^* D] - F) \supset \mu_* \mathscr{O}_{X'}(-F) = \mathfrak{b}(|V|)$$

By the Claim, it is sufficient to prove that

$$(\clubsuit) \qquad \mathscr{I}_{r+1} \subset \mathscr{I}_{E|_H;|V||_H}$$

In order to do so, recall the following consequence of Nadel's vanishing.

Corollary. *Let X be a smooth complex projective variety, D an effective $\mathbf{Q}$-divisor and L a divisor on X such that $L-D$ is big and nef. Then for any very ample divisor A on X, the sheaf*

$$\mathscr{I}_D(K_X + mA + L)$$

is globally generated for all $m \geqslant \dim(X)$.

Proof. By the classical Castelnuovo–Mumford argument, a coherent sheaf $\mathscr{F}$ on X verifying $H^i(X, \mathscr{F}(-iA)) = 0$ for all $i > 0$ is g.g. Then use Nadel to get $H^i(X, \mathscr{I}_D(K_X + (m-i)A + L)) = 0$ for $0 < i \leqslant m$, and conclude. QED.

As

$$[(r-1)(K_H+D|_H)+D|_H]-(\frac{r-1}{m}S+E|_H) \sim A|_H$$

is ample and

$$\mathscr{I}_r(r(K_H + D|_H) + nB|_H) =$$
$$\mathscr{I}_r(K_H + [(r-1)(K_H + D|_H) + D|_H] + nB|_H),$$

the Corollary above implies that

$$\mathscr{I}_r(r(K_H + D|_H) + nB|_H) \text{ is g.g.}$$

In particular

$$\mathscr{I}_r = \mathfrak{b}(|V||_H)$$

But $E|_H$ is klt, so, as remarked before,

$$\mathfrak{b}(|V||_H) \subset \mathscr{I}_{E|_H;|V||_H}.$$

So we have

$$\mathscr{I}_{r+1} \subset \mathscr{I}_r \subset \mathscr{I}_{E|_H;|V||_H}$$

hence the proof of (♣) ($\Rightarrow$ the inductive step $\Rightarrow$ the Lemma $\Rightarrow$ the Extension Theorem !) is completed. QED.

We now begin the

Proof of the Key Theorem :

First reduction. We claim that we may assume $X = X'$, $\mu = id$ and V smooth. Indeed take a modification $\nu : X'' \to X$ such that the strict transform V'' of V is smooth. Write

$$\begin{aligned} \nu^* A' &\sim A'' + E'' \\ &= \text{ample} + \nu\text{-exc. effective} \end{aligned}$$

Then $(\mu \circ \nu)^* D \sim A'' + (E'' + \nu^* E')$.

By the birational transformation rule

$$\mathscr{I}_{E'} = \nu_*(\mathscr{I}_{\nu^* E'}(K_{X''/X'}))$$

implies that V'' is a maximal lc center for the pair $(X'', \nu^* E')$, hence also for the pair $(X'', E'' + \nu^* E')$, as $\operatorname{Supp} E'' \not\supset V''$.

On the other hand, $\mathrm{vol}(K_V) = \mathrm{vol}(K_{V''})$ and, by the isomorphism

$$H^0(X'', m(K_{X''}+(\mu\circ\nu)^*D)) \simeq H^0(X, m(K_X+D)),$$

we also have

$$\mathrm{vol}_{X''|V''}(K_{X''}+(\mu\circ\nu)^*D) = \mathrm{vol}_{X|V}(K_X+D).$$

So that the inequality

$$\mathrm{vol}_{X''|V''}(K_{X''} + (\mu \circ \nu)^*D) \geqslant \mathrm{vol}(K_{V''})$$

and the one we want to prove

$$\mathrm{vol}_{X|V}(K_X + D) \geqslant \mathrm{vol}(K_V)$$

are equivalent.

The case $\mathrm{codim}\, V = 1$**.** By hypothesis V is a maximal lc center for (X, E). This simply means that V appears with multiplicity 1 in E. Let $\mu : X' \to X$ a modification such that $\mu^*E = V' + F$ has simple normal crossings (V' is the strict transform of V). Take an integer $m_0 > 0$ such that $m_0(\mu^*A + F - [F])$ has integer coefficients. The support of this divisor does not contain V', so we have an inclusion

$$H^0(V', mK_{V'}) \hookrightarrow H^0(V', m(K_{V'} + (\mu^*A + F - [F])|_{V'})) \quad (1)$$

for any integer $m > 0$ divisible by m_0. Since the pair $(X', F - [F])$ is klt, applying the Extension Theorem to the divisor $\mu^*A + F - [F]$ we get a surjection

$$H^0(X', m(K_{X'} + \mu^*D - [F])) \twoheadrightarrow H^0(V', m(K_{V'} + (\mu^*A + F - [F])|_{V'})) \quad (2)$$

In conclusion we have

$$\begin{array}{rcl} h^0(V, mK_V) & = & h^0(V', mK_{V'}) \\ (\ (1) + (2)\ ) & \leqslant & h^0(X'|V', m(K_{X'} + \mu^*D - [F])) \\ & \leqslant & h^0(X'|V', m(K_{X'} + \mu^*D)) \\ & = & h^0(X|V, m(K_X + D)) \end{array}$$

so the Key Theorem is proved in this case.