Sketch of Takayama's proof

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We want to prove the following result.

Theorem 1 (Hacon-McKernan, Takayama) *For all n, there exists a positive constant m_n such that for any smooth complex projective variety X of general type of dimension n, the pluricanonical map $\varphi_{|mK_X|}$ is birational onto its image, for all $m \geq m_n$.*

This result is related to the existence of a unifom lower bound on the volume of the canonical divisor. Indeed, recalling that

$$\mathrm{vol}_X(D) := \limsup_{m \to +\infty} \frac{h^0(X, mD)}{m^n/n!}$$

we easily deduce the following.

Corollary. *For all n, there exists a positive constant v_n such that for any smooth complex projective variety X of general type of dimension n, we have :*

$$\mathrm{vol}(K_X) \geq v_n.$$

Theorem 1 $\Rightarrow$ Corollary :

Let m_n be as in Theorem 1. Let $\mu : X' \to X$ be the blow-up along the base locus of $|mK_X|$. Then

$$\mu^* mK_X = |M| + F = \text{bpf part} + \text{fixed part}.$$

In particular M is nef, so vol$(M) = M^n$. In conclusion we have :

$$\begin{aligned} \text{vol}(K_X) &= \frac{\text{vol}(\mu^* m_n K_X)}{m_n^n} \\ &\geq \frac{1}{m_n^n}\text{vol}(M) = \frac{1}{m_n^n}M^n \\ &= \frac{1}{m_n^n}\deg \varphi_{|M|}(X') \geq \frac{1}{m_n^n} \quad QED. \end{aligned}$$

On the other hand, Theorem 1 is a consequence of the following.

Theorem 2. *Let n be a positive integer. Suppose there exists a positive constant v such that $\mathrm{vol}(V) \geq v$, for any smooth complex projective variety V of general type of dimension $< n$. Then, there exists two positive constants $a := a_n$ and $b := b_n$ such that for any smooth complex projective variety X of general type of dimension n, the rational pluricanonical map $\varphi_{|mK_X|}$ is birational onto its image, for all $m \geq a + b/\mathrm{vol}(K_X)^{1/n}$.*

Theorem 2 $\Rightarrow$ Theorem 1 :

By induction on the dimension. Theorem 1 holds for $n = 1$. Suppose it holds for $n-1$. Then use the Corollary to deduce the existence of a positive lower bound :

$$\mathrm{vol}(V) \geq v_{n-1}$$

for all varieties of general type V of dimension $\leq n-1$, *i.e.* the hypotheses of Theorem 2 are fullfilled.

For the $n-$dimensional varieties of general type X such that the volume is bounded from below, say we have $\operatorname{vol}(K_X) \geq 1$, then

$$1 + a + b > a + b/\operatorname{vol}(K_X).$$

For those such that $\operatorname{vol}(K_X) < 1$, then *a priori* the quantity $a + b/\operatorname{vol}(K_X)$ may still be arbitrary big. This does not occur : using Theorem 2 and projecting down, we have that the variety X is birational to a subvariety of $\mathbf{P}^{2n+1}$ of degree :

$$\begin{aligned} &\leq \ (1 + a + \frac{b}{\operatorname{vol}(K_X)^{1/n}})^n \operatorname{vol}(K_X) \\ &= \ [(1 + a)\operatorname{vol}(K_X)^{1/n} + b]^n \leq (1 + a + b)^n. \end{aligned}$$

Such varieties are parametrized by an algebraic variety (the Chow scheme), so thanks to the upper semicontinuity of the plurigenera, their volumes are also bounded from below by a positive constant c_n (which is not effective!). Hence we may take

$$m_n := 1 + a + \frac{b}{c_n^{1/n}}$$

and the proof is finished QED.

So we are reduced to proving Theorem 2.

Idea of the proof :

Consider a modification $\mu : X' \to X$ such that

$$\mu^* K_X = A' + E' = \text{ample} + \text{effective}$$

is the approximate Zariski decomposition wrt a certain ε. Let x'_1, x'_2 two general points on X'. Then construct an effective $\mathbf{Q}-$divisor

$$D'_0 \sim t_0 A', \quad \text{with} \quad t_0 < a + b / \operatorname{vol}(K_X)^{1/n}$$

such that

(i) $x'_1, x'_2 \in \operatorname{Nklt}(X, D'_0)$ (ii) x'_1 is an isolated point in $\operatorname{Nklt}(X, D'_0)$

Then for all $m > t_0$:

$$m\mu^* K_X = (m - t_0) A' + (D'_0 + mE') = \text{ample} + \text{eff},$$

hence by Nadel vanishing :

$$H^1(X', \mathscr{I}_{D'_0+mE'}(K_{X'} + m\mu^* K_X)) = 0$$

so, setting $V := V(\mathscr{I}_{D'_0+mE'})$

$$H^0(X', K_{X'}+m\mu^* K_X) \twoheadrightarrow H^0(V, (K_{X'}+m\mu^* K_X)_{|V}).$$

In particular, by (i) and (ii), and the fact that $x'_1, x'_2 \not\in E'$, there exists a section $s \in H^0(X', K_{X'} + m\mu^* K_X)$ such that

$$s(x'_1) \neq 0 \text{ and } s(x'_2) = 0.$$

Then, using the isomorphisms

$$\begin{aligned} & H^0(X', K_{X'} + m\mu^* K_X) \\ \simeq\ & H^0(X, (\mu_* \mathscr{O}_{X'}(K_{X'/X}))((m+1)K_X)) \\ \simeq\ & H^0(X, (m+1)K_X) \end{aligned}$$

we get a global section of $\mathscr{O}_X((m+1)K_X)$ separating $\mu(x'_1)$ and $\mu(x'_2)$.

Details of the Proof of Theorem 2 :

We will proceed by descending induction on $d \in \{1, \ldots, n\}$ to produce an effective $\mathbf{Q}-$divisor $D'_d \sim t_d A'$ such that

Needed properties :

1. $x'_1, x'_2 \in \mathsf{Nklt}(X, D'_d)$;
2. (X', D'_d) is lc at a non-empty subset I of $\{x'_1, x'_2\}$;
3. $dim\, \mathsf{Nklt}(X', D'_d, I) \leq d$.

For $d = 0$ we will get a divisor D'_0 exactly as in the idea of the proof outlined before and we will be done.

We start using the approximate Zariski decomposition (with $\varepsilon = 1/2$), that is, we consider a modification $\mu : X' \to X$ such that

$$\mu^* K_X = A' + E' = \text{ample} + \text{effective}$$

with

$$A'^d . V' \geq \frac{1}{2^d} \mathrm{vol}_{X|V}(K_X)$$

for all $d-$dimensional irreducible subvariety $V \subset X$ not contained in $\mathbf{B}_+(K_X)$ ($V' \subset X'$ denotes the strict transform). In particular

$$A'^n \geq \frac{1}{2^n} \mathrm{vol}(K_X).$$

Take an integer

$$t'_{n-1} > 2n \cdot 2^{\frac{1}{n}} \cdot \mathrm{vol}(K_X)^{-\frac{1}{n}}.$$

Then

$$(t'_{n-1} A')^n \geq \frac{t'^n_{n-1}}{2^n} \mathrm{vol}(K_X) > 2n^n.$$

So, as Catriona explained, there exists an effective $\mathbf{Q}-$divisor

$$D'_{n-1} \sim t_{n-1} A' \text{ with } t_{n-1} \leq t'_{n-1}$$

such that (X', D'_{n-1}) is nklt at x'_1 and x'_2 and lc at one of these points, say at x'_1.

Suppose now that we have constructed an effective $\mathbf{Q}-$divisor $D'_d \sim t_d A'$ verifying the three properties above. Then, again thanks to what Catriona explained, up to slightely modifying t_d, we may assume that

$$V' := \mathsf{Nklt}(X', D'_d, x'_1) \text{ is irreducible.}$$

As x'_1 is general, V' is of general type (by the adjunction formula, families of subvarieties not of general type are contained in a countable union of proper subvarieties of X').

Notice that

$$\mu^*(t_d + 1)K_X \sim A' + [D'_d + (t_d + 1)E'].$$

Since x'_1, x'_2 are general, they are not contained in E', hence V' is an irreducible component of $\mathsf{Nklt}(X', D'_d + (t_d + 1)E')$. Moreover, again by the generality of the points, V' is still a maximal lc center for the pair $(X', D'_d + (t_d + 1)E')$.

Now we get to the key result in Takayama's proof.

Key Theorem. *Let X be a complex projective variety, let $V \subset X$ be an irreducible subvariety and $\mu : X' \to X$ a modification with $V \not\subset \mu(\mathrm{Exc}(\mu))$. Let $V' \subset X'$ be the strict transform of V. Let D be a divisor on X and $\mu^* D = A' + E'$ a decomposition such that :*

a) A' is an ample $\mathbf{Q}-$divisor ;

b) E' is an effective $\mathbf{Q}-$divisor such that V' is a maximal lc center for the pair (X', E').

Then : $$\mathrm{vol}_{X|V}(K_X + D) \geq \mathrm{vol}(K_V).$$

Recall that

$$\mathrm{vol}_{X|V}(D) := \limsup_{m \to +\infty} \frac{\dim[\mathrm{Im}(H^0(X, mD) \to H^0(V, mD|_V))]}{m^d/d!}$$

Let us see how to use this result. We have

$$\begin{aligned} v &\le \text{vol}(K_{V'}) \quad \text{(hypothesis of Thm. 2)} \\ &\le \text{vol}_{X|V}(K_X + (t_d+1)K_X) \quad \text{(Key Thm.)} \\ &= (t_d+2)^d \text{vol}_{X|V}(K_X) \\ &\le 2^d(t_d+2)A'^d.V' \quad \text{(approx. Zariski dec.)} \end{aligned}$$

In conclusion, taking

$$t'_{d-1} > 2d \cdot 2^{\frac{1}{d}} \cdot v^{-1/d}(t_d+2)$$

we get

$$(\star) \qquad (t'_{d-1}A')^d.V' > 2d^d.$$

Using $(\star)$, the Angehrn-Siu arguments allow to construct an effective $\mathbf{Q}-$divisor $D'_{d-1} \sim t_{d-1}A'$, with $t_{d-1} < t_d + t'_{d-1}$, verifying the three needed properties, and the inductive step is proved.

Details of the Angehrn-Siu arguments :

We need the following result.

Proposition. *Let X be a complex projective variety, D an effective $\mathbf{Q}-$divisor on X and V a $d-$dimensional irreducible component of* Nklt(X,D). *There exists a open dense subset U in the smooth locus of V and a rational number $\varepsilon_0 \in]0,1[$ such that, for any $y \in U$, any rational $\varepsilon \in]0,\varepsilon_0[$, and any effective $\mathbf{Q}-$divisor $B \not\supset V$ verifying*

$$\text{mult}_y B_{|V} > d$$

the locus Nklt$(X,(1-\varepsilon)D+B)$ *contains y.*

For simplicity, we suppose (X',D'_d) is lc at both points x'_1 and x'_2 and that the irreducible component $V' = \text{Nklt}(X,D'_d,x'_1)$ also passes through x'_2.

Let U as in the previous Proposition w.r. t. the pair (X', D'_d). We choose :

- two 1$-$parameter families of points $(y^i_\lambda)_{\lambda\in\Lambda}$ such that $y^i_\lambda \in U$ for $\lambda \neq 0$, and $y^i_0 = x'_i$;

and using condition ($\star$) (again as Catriona explained !)

- a flat family of divisors $(G_{V',\lambda})_{\lambda\in\Lambda\backslash 0}$, with $G_{V',\lambda} \in |t'_{d-1}mA'_{|V'}|$ having multiplicity md at both y^1_λ and y^2_λ.

Denote by $G_{V',0}$ the limit divisor.

Moreover we may take $m >> 0$ so that

α) we may lift the $G_{V',\lambda}$'s to a flat family of divisors $G_\lambda \in |t'_{d-1}mA'|$;

β) The sheaf $\mathscr{I}_{V'}(mt'_{d-1}A')$ is g.g.

By Bertini-Kollár, condition β) implies that, for any ε and any $t \in]0,1[$, we have

$$\text{Nklt}(X', (1-\varepsilon)D'_d + tG_0) \subset \text{Nklt}(X, D'_d)$$

outside V'.

Set

$$B_\lambda := \frac{1}{m} G_\lambda \in |t'_{d-1} A'|$$

and, for $\varepsilon \in]0, \varepsilon_0[$

$$E_\lambda := (1-\varepsilon)D'_d + B_\lambda \quad (\leq (t_d + t'_{d-1})A')$$

Thanks to the Proposition, for all $\lambda \neq 0$, the loci $\text{Nklt}(X, E_\lambda)$ contain y^1_λ and y^2_λ. Then by a semicontinuity result :

$$\text{Nklt}(X', E_0) \ni x'_1, x'_2.$$

Hence, multiplying E_0 by the max of the lc thresholds at x'_1 and x'_2, we get the desired divisor

$$D'_{d-1} \sim t_{d-1} A', \quad \text{with} \quad t_{d-1} < t_d + t'_{d-1}.$$

verifying the three needed properties, and the inductive step is completed QED.