Nondeformability of entire curves in projective hypersurfaces of high degree

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October 17, 2005

Abstract. In this article, we prove that there does not exist a family of entire curves in the universal family of hypersurfaces of degree $d \geq 2n$ in the complex projective space $\mathbb{P}^n$. This can be seen as a weak version of the Kobayashi conjecture asserting that a general projective hypersurface of high degree is hyperbolic in the sense of Kobayashi.

Let X be a hypersurface in the projective space $\mathbb{P}^n$. The Kobayashi conjecture claims that X is *hyperbolic*, provided that X is general and $d = \deg(X) \geq 2n - 1$. By the Brody criterion ([Bro]), the hyperbolicity of X is equivalent to the fact that every holomorphic map $\mathbb{C} \to X$ is constant. The conjecture has been proved for $n = 3$, $d \geq 21$ and X very general ([DE], [MQ]; see also [Bru1] for an account). An important and recent progress in the direction of the conjecture for all $n \geq 3$ was made by Y.-T. Siu in [S]: he obtains a confirmation of the conjecture under the assumption $d \gg n$.

Consider the universal family $\mathcal{X} \subset \mathbb{P}^n \times \mathbb{P}^{N_d}$ of hypersurfaces in $\mathbb{P}^n$ with fixed degree d (the number N_d is equal to $\binom{n+d}{d} - 1$). We will denote by X_t the fiber of $\mathcal{X}$ over the parameter $t \in \mathbb{P}^{N_d}$.

Theorem. *Let $U \subset \mathbb{P}^{N_d}$ be an open set and let $\Phi : \mathbb{C} \times U \to \mathcal{X}$ be a holomorphic map such that $\Phi(\mathbb{C} \times \{t\}) \subset X_t$ for all $t \in U$. If $d \geq 2n$, the rank of Φ cannot be maximal anywhere.*

Of course the theorem is an immediate consequence of the Kobayashi conjecture (and hence, if the degree d is big enough, of the result of Siu). So, the theorem above may be rephrased as follows : the Kobayashi conjecture may possibly fail only if there is an entire curve on a general hypersurface X which is not preserved by a deformation of X.

The question above is motivated by the "picture" in the algebraic situation: the existence of an algebraic cycle on the general member of the family implies its deformation on the nearby fibers. However, dealing with transcendental objects (e.g., entire curves) seems to be much more complicated. For entire curves tangent to a holomorphic foliation of dimension one, a substitute for the Hilbert scheme was found by M. Brunella in [Bru2].

Very roughly, the proof goes as follows. First of all, we consider the (non-zero) section of the holomorphic bundle $\Lambda^{1+N_d}\Phi^*T_{\mathcal{X}}$ given by the Jacobian of Φ (in fact, for some technical reasons, we will work with a sequence of reparametrizations of Φ, but we skip this point here, to keep the discussion clear). In order to use the positivity of the canonical bundle of the hypersurfaces, we take the wedge product of the previous section with an appropriate family of meromorphic vector fields on $\mathcal{X}$, and thus get a section σ of (a twist of) $\Phi^*K_{\mathcal{X}}^{-1}$. Next, we show that the laplacian of the logarithm of the norm of this section dominates a positive multiple of the norm of σ, and use negative curvature arguments to derive a contradiction, as soon as the degree d satisfies the numerical hypothesis of our theorem.

Proof of the theorem. The proof uses two ingredients: the first is that the vector bundle $T_{\mathcal{X}} \otimes p^*\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by its global sections (where p is the projection $\mathbb{P}^n \times \mathbb{P}^{N_d} \to \mathbb{P}^n$). The second relies on some negative curvature arguments, very much in the spirit of the Kobayashi-Ochiai theorem ([KO]).

We recall the following proposition, due to Siu ([S]).

Proposition 1 (Siu). *The vector bundle $T_{\mathcal{X}} \otimes p^*\mathcal{O}_{\mathbb{P}^n}(1)$ is globally generated.*

The proof of this proposition is given in [S]; we reproduce it here, for the convenience of the reader. Observe that the global generation of the restriction $T\mathcal{X}_{|X_t} \otimes \mathcal{O}_{X_t}(1)$ of the same bundle to a fiber X_t has been previously proved by Voisin ([V1], Prop. 1.1) who deduced from it important results about the algebraic hyperbolicity of a (very) general hypersurface (for an account of the subsequent developments of Voisin's approach, see [C]).

Proof. Consider global coordinates $(Z_j)_{0\leq j\leq n}$ (resp. $(a_\alpha)_{|\alpha|=d}$) on $\mathbb{C}^{n+1}$ (resp. on $\mathbb{C}^{N_d+1}$). The equation of the manifold $\mathcal{X}$ in $\mathbb{P}^n \times \mathbb{P}^{N_d}$ can be written as

$$\sum a_\alpha Z^\alpha = 0$$

where we use here the multi-index notation $Z^\alpha = \prod Z_j^{\alpha_j}$. Consider the open set $U_0 = \{Z_0 \neq 0\} \times \{a_{d0\ldots 0} \neq 0\}$ in $\mathbb{P}^n \times \mathbb{P}^{N_d}$. For the rest of the proof, we will work on U_0, with the induced nonhomogeneous coordinates.

Consider a multi-index $\alpha \in \mathbb{N}^d$ and an integer j such that $\alpha_j \geq 1$. On the set U_0, consider the vector field

$$V_{\alpha,j} = \frac{\partial}{\partial a_\alpha} - z_j \frac{\partial}{\partial \hat{a}_\alpha}$$

where $z_j = Z_j/Z_0$, $\hat{a}_k = a_k$ if $k \neq j$, and $\hat{a}_j = \alpha_j - 1$. The vector field $V_{\alpha,j}$ is tangent to $\mathcal{X}_0 = \mathcal{X} \cap U_0$, as a quick verification shows. On the other hand, we can extend it to the whole manifold $\mathcal{X}$ as a meromorphic vector field and its pole order is equal to 1. Remark that $V_{\alpha,j}$ is a meromorphic section of the kernel of the differential of the first projection $p_{|\mathcal{X}} : \mathcal{X} \to \mathbb{P}^n$.

We also have a "lifting" property for the vector fields, as follows. Consider a vector field

$$V_0 = \sum_{j=1}^{n} v_j \frac{\partial}{\partial z_j}$$

on $\mathbb{C}^n$, where $v_j = \sum_k v_k^{(j)} z_k + v_0^{(j)}$ is a polynomial of degree at most one in the z_j-variables. There exists a vector field

$$V = \sum_{|\alpha| \le d} v_\alpha \frac{\partial}{\partial a_\alpha} + V_0$$

that is tangent to $\mathcal{X}_0$ and that extends to the whole manifold $\mathcal{X}$ as a holomorphic section of the tangent bundle. Indeed, if we want V to be tangent to $\mathcal{X}_0$, the condition to be satisfied is

$$\sum_\alpha v_\alpha z^\alpha + \sum_{\alpha,j} a_\alpha v_j \frac{\partial z^\alpha}{\partial z_j} = 0$$

and the complex numbers v_α are simply chosen such that the coefficient of the monomial z^α in the above equation is equal to zero. The extension property is also quickly verified, as well as the global generation of the bundle $T_{\mathcal{X}} \otimes p^*\mathcal{O}_{\mathbb{P}^n}(1)$ by the vector fields already constructed. The proposition is thus proved. □

Consider a holomorphic map $\Phi : \mathbb{C} \times U \to \mathcal{X}$ over the base $U \subset \mathbb{P}^{N_d}$ as in the theorem. We suppose that Φ has maximal rank. If $d \ge 2n$, we are going to derive a contradiction.

As U is an open set, we can shrink it and suppose that it is equal to a polydisc $\mathbb{B}(\delta_0)^{N_d}$. We will consider the following sequence of maps

$$\Phi_k : \mathbb{B}(\delta_0 k)^{N_d+1} \to \mathcal{X}$$

given by $\Phi_k(z, \xi_1, \ldots, \xi_{N_d}) = \Phi(zk^{N_d}, \frac{1}{k}\xi_1, \ldots, \frac{1}{k}\xi_{N_d})$. The technical reason for which we need to change the radius of the disc will be clear in a moment. Notice that the initial map $\Phi_1 = \Phi$ is of maximal rank, thus the section

$$J_\Phi(z,\xi) = \frac{\partial \Phi}{\partial z} \wedge \frac{\partial \Phi}{\partial \xi_1} \wedge \cdots \wedge \frac{\partial \Phi}{\partial \xi_{N_d}}(z,\xi) \in \Lambda^{1+N_d} T_{\mathcal{X},\Phi(z,\xi)} \tag{1}$$

of $\Phi^*\Lambda^{1+N_d}T_{\mathcal{X}}$ is not identically zero. Let us assume that $J_\Phi(\underline{0})$ is nonzero in the corresponding vector space. Remark that $J_{\Phi_k}(\underline{0}) = J_\Phi(\underline{0})$, for any

$k \geq 1$, where $J_{\Phi_k} \in \Phi_k^* \Lambda^{1+N_d} T_{\mathcal{X}}$ is the section associated to the map Φ_k as indicated in (1). is not identically zero, as a section. The positivity of the vector bundle $T_{\mathcal{X}}$ in the parameter space directions now comes into the picture: thanks to Proposition 1, we can choose $n-2$ vector fields

$$V_1, \ldots, V_{n-2} \in T_{\mathcal{X}} \otimes p^* \mathcal{O}_{\mathbb{P}^n}(1)$$

such that

$$J_{\Phi_k}(\underline{0}) \wedge \Phi_k^* \big(V_1 \wedge \cdots \wedge V_{n-2} \big) \neq 0$$

in $K_{\mathcal{X}}^{-1} \otimes p^* \mathcal{O}_{\mathbb{P}^n}(n-2)_{\Phi_k(0)}$.

With the vector fields previously chosen, we consider the following section

$$\sigma_k = J_{\Phi_k} \wedge \Phi_k^* \big(V_1 \wedge \cdots \wedge V_{n-2} \big)$$

of the bundle $\Phi_k^* (K_{\mathcal{X}}^{-1} \otimes p^* \mathcal{O}_{\mathbb{P}^n}(n-2)) \big)$ over the polydisc. Its value at the origin is independent of k, and of course nonzero. If q is the projection of $\mathcal{X}$ on the parameter space $\mathbb{P}^{N_d}$, then under the assumption $d \geq 2n$, the restriction of $K_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}^n}(2-n))$ to $\pi_2^{-1}(U)$ is ample (eventually after shrinking once again the open subset U), hence we can endow this bundle with a metric h of positive curvature.

We now define a sequence of functions $f_k : \mathbb{B}(\delta_0 k)^{N_d+1} \to \mathbb{R}_+$ as follows

$$\forall w \in \mathbb{B}(\delta_0 k)^{N_d+1} \quad f_k(w) = \| \sigma_k(w) \|_{\Phi_k^* h^{-1}}^{2/(N_d+1)}.$$

Remark 2. Notice that, by construction, there exists a positive number c such that for each $k \geq 1$, we have $f_k(0) = c$.

We have the following lemma.

Lemma 3. *For each $k \geq 1$, there exists a positive constant C such that we have $\Delta \log f_k \geq C f_k$ pointwise over the polydisc $\mathbb{B}(\delta_0 k)^{N_d+1}$.*

Proof. First, remark that by construction, the image of the map Φ_k lies inside $q^{-1}(U)$, for each $k \geq 1$, so that

$$i \partial \bar{\partial} \log \| \sigma_k \|_{\Phi_k^* h^{-1}}^2 \geq \Phi_k^* \Theta_h \big(K_{\mathcal{X}} \otimes p^* \mathcal{O}_{\mathbb{P}^n}(2-n) \big). \tag{2}$$

In the inequality (2), take the trace with respect to the flat metric on the polydisc. We get

$$\begin{aligned} \Delta \log \| \sigma_k \|_{\Phi_k^* h^{-1}}^2 &\geq C \Big(\Big\| \frac{\partial \Phi_k}{\partial z} \Big\|_\omega^2 + \sum_{j=1}^{N_d} \Big\| \frac{\partial \Phi_k}{\partial \xi_j} \Big\|_\omega^2 \Big) \\ &\geq C \| J_{\Phi_k} \|_{\Lambda^{1+N_d} \omega}^{2/(1+N_d)} \\ &\geq C \| \sigma_k \|_{\Phi_k^* h^{-1}}^{2/(1+N_d)} \end{aligned}$$

The constant C in the previous sequence of inequalities varies from one line to another, but we still denote it by C as it is independent of k. The above relations are obtained using the vector inequalities

$$\|W_1 \wedge \cdots \wedge W_s\| \leq \|W_1\| \ldots \|W_s\| \leq s^{-s}(\|W_1\| + \cdots + \|W_s\|)^s.$$

So the lemma is proved. □

Using the previous lemma we will prove a result whose proof is very close to that of the Ahlfors-Schwarz lemma.

Proposition 4. *For each $k \geq 1$ we have $f_k(0) \leq Ck^{-2}$. In particular, as $k \to \infty$, we have $f_k(0) \to 0$.*

Proof. Consider the volume form of the Poincaré metric on the polydisc

$$\psi_k = \frac{1}{\left(1 - \frac{|z|^2}{\delta_0^2 k^2}\right)^2} \prod_{j=1}^{N_d} \frac{1}{\left(1 - \frac{|\xi_j|^2}{\delta_0^2 k^2}\right)^2}$$

A quick computation shows that

$$i\partial\bar{\partial} \log \psi_k = (\delta_0 k)^{-2} \left(\frac{idz \wedge d\overline{z}}{\left(1 - \frac{|z|^2}{\delta_0^2 k^2}\right)^2} + \sum_{j=1}^{N_d} \frac{id\xi_j \wedge d\overline{\xi_j}}{\left(1 - \frac{|\xi_j|^2}{\delta_0^2 k^2}\right)^2} \right)$$

so if we take the trace of this equality with respect to the flat metric, we get

$$\Delta \log \psi_k \leq Ck^{-2}\psi_k. \tag{3}$$

Remark that the previous inequality can be obtained precisely because we have the same radius $\delta_0 k$ for the components of the polydisc which is the domain of ψ_k. This is why we had to reparametrize our map Φ from the very beginning.

Consider the function $(z, \xi) \mapsto \dfrac{f_k(z,\xi)}{\psi_k(z,\xi)}$. Its maximum cannot be achieved at a boundary point of the domain, since ψ_k goes to infinity as (z, ξ) goes to the boundary. So at the maximum point (z_0, ξ_0), we have

$$\Delta \log f_k/\psi_k \leq 0. \tag{4}$$

This inequality, combined with Lemma 3 and (3), gives

$$f_k(z_0, \xi_0) \leq Ck^{-2}\psi_k(z_0, \xi_0). \tag{5}$$

Since the relation (5) is verified at the maximum point of the quotient, it follows that the same is true at an arbitrary point, so we get

$$f_k(z, \xi) \leq Ck^{-2}\psi_k(z, \xi). \tag{6}$$

To finish the proof, it is sufficient to write the inequality (6) at the origin. □

Since Proposition 4 and Remark 2 contradict each other, the theorem is proved. □

References

[Bro] R. Brody, *Compact manifolds in hyperbolicity*, Trans. Amer. Math. Soc. **235** (1978), 213–219.

[Bru1] M. Brunella, *Courbes entières dans les surfaces algébriques complexes (d'après McQuillan, Demailly-El Goul,...)*, Séminaire Bourbaki, Exp. No. 881, 2000/2001, Astérisque **282** (2002), 39–61.

[Bru2] M. Brunella, *On entire curves tangent to a foliation*, preprint 2004.

[C] H. Clemens, *Lower bounds on genera of subvarieties of generic hypersurfaces*. Special issue in honor of Steven L. Kleiman. Comm. Algebra **31** (2003), 3673–3711.

[D] J.-P. Demailly, *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*, Algebraic geometry—Santa Cruz 1995, 285–360, Proc. Sympos. Pure Math. **62**, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[DE] J.-P. Demailly, J. El Goul, *Hyperbolicity of generic surfaces of high degree in projective 3-space*, Amer. J. Math. **122** (2000), 515–546.

[K] S. Kobayashi, *Hyperbolic complex spaces*. Grundlehren der Mathematischen Wissenschaften **318**, Springer-Verlag, Berlin, 1998.

[KO] S. Kobayashi, T. Ochiai, *Meromorphic mappings onto compact complex spaces of general type*, Invent. Math. **31** (1975), 7–16.

[MQ] M. McQuillan, *Holomorphic curves on hyperplane sections of* 3*-folds*, Geom. Funct. Anal. **9** (1999), 370–392.

[S] Y.T. Siu, *Hyperbolicity in complex geometry*, The legacy of Niels Henrik Abel, 543–566, Springer, Berlin, 2004.

[V1] C. Voisin, *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Diff. Geom. **44** (1996), 200–214.

[V2] C. Voisin, *A correction: "On a conjecture of Clemens on rational curves on hypersurfaces"*, J. Diff. Geom. **49** (1998), 601–611.

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