Extensions of sections from a subvariety (after Takayama)

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Theorem (Takayama) — Let X be a smooth projective complex variety, let V be an irreducible subvariety of X, let $\mu : X' \to X$ be a modification with $V \not\subset \mu(\mathrm{Exc}(\mu))$, and let V' be the strict transform of V in X'.

Let D be a divisor on X with $\mu^* D \sim A' + E'$, where

a) A' is an ample $\mathbf{Q}$-divisor;*

b) E' is an effective $\mathbf{Q}$-divisor such that V' is a maximal log-canonical center for the pair (X', E').†

Then,

$$\mathrm{vol}_{X|V}(K_X + D) \geqslant \mathrm{vol}(K_V)$$

Moreover, if the canonical class of a desingularization of V is pseudo-effective, elements of $|m(K_X + D)|$ separate two general points of V for all integers m sufficiently big and divisible.

*It is enough to assume that A' is nef and big with $V' \not\subset \mathbf{B}_+(A')$.

†i.e., an irreducible component of $\mathrm{Nklt}(X', E')$ at a general point of which the pair is log-canonical.

Proof — We saw in the last talk that we may assume $X' = X$, μ is the identity and V is smooth. We write E instead of E' and A instead of A'. We may also assume that A is effective and that its support does not contain V.

The case where V is a hypersurface has already been treated, so we also assume $\operatorname{codim} V \geqslant 2$.

Let $\mu : X' \to X$ be a log-resolution for E. Write

$$\begin{aligned} \mu^* E &= E' + \sum_F e_F F \quad , \quad e_F \in \mathbf{Q}_{\geqslant 0} \\ K_{X'/X} &= \sum_F k_F F \qquad\qquad , \quad k_F \in \mathbf{Z}_{>0} \end{aligned}$$

where E' is the strict transform of E and F runs over the set of all prime μ-exceptional divisors in X'.

Since V is a maximal log-canonical center for the pair (X, E), we have

• if $V \subsetneq \mu(F)$, then $k_F - e_F > -1$;

• if $V = \mu(F)$, then $k_F - e_F \geqslant -1$, with equality for at least one F.

Set

$$\begin{aligned} I &= \{F \mid \mu(F) = V,\ k_F - e_F = -1\} \neq \varnothing \\ J &= \{F \mid \mu(F) = V\} - I \\ &= \{F \mid \mu(F) = V,\ k_F - e_F > -1\} \end{aligned}$$

Step 1: we may assume that I has only one element.

Choose $\delta_F \in \mathbf{Q}_{>0}$ small enough so that

$$\begin{aligned} \frac{1}{2}A + \frac{\delta_F}{e_F}E & \quad \text{is ample for all } F \in I; \\ \delta_F < 1 - e_F + k_F & \quad \text{for all } F \in J; \\ \frac{1}{2}\mu^* A - \sum_F \delta_F F & \quad \text{is ample;} \end{aligned}$$

and set

$$\varepsilon = \max_{F \in I} \frac{\delta_F}{e_F} > 0$$

where, by wiggling the δ_F, we may assume that the maximum is obtained for only one F in I, denoted by V'. By definition, $\mu(V') = V$.

Write

$$D \sim \left(\tfrac{1}{2}A + \varepsilon E\right) + \left(\tfrac{1}{2}A + (1-\varepsilon)E\right)$$

The $\mathbf{Q}$-divisor $A_1 = \frac{1}{2}A + \varepsilon E$ is ample.

For $m \gg 0$,

• pick $H \in |m(\frac{1}{2}\mu^* A - \sum_F \delta_F F)|$ general very ample and

•represent $\frac{1}{2}A + (1-\varepsilon)E$ by the effective $\mathbf{Q}$-divisor E_1 such that

$$\begin{aligned}\mu^* E_1 &= \frac{1}{m}H + \sum_F \delta_F F + (1-\varepsilon)\mu^* E \\ &= \frac{1}{m}H + (1-\varepsilon)E' + \sum_F (\delta_F + (1-\varepsilon)e_F)F\end{aligned}$$

which has simple normal crossing.

We have now

$$
\begin{aligned}
K_{X'/X} - \mu^* E_1 & \\
\sim\ & -\frac{1}{m}H - (1-\varepsilon)E' \\
& \qquad + \sum_F (k_F - \delta_F - (1-\varepsilon)e_F)F \\
\sim\ & -\frac{1}{m}H - (1-\varepsilon)E' - V' \\
& \qquad - \sum_{F\in I,\ F\neq V'} (1 - \varepsilon e_F + \delta_F)F \\
& \qquad - \sum_{F\notin I} (e_F - k_F + \delta_F - \varepsilon e_F)F \\
:=\ & -V' - \sum a_F F
\end{aligned}
$$

and by construction, $a_F < 1$ for any $F \in I \cup J$. We have

α) if $V \subset \mu(F)$, then $a_F < 1$, so that the "new" set I has only one element;

β) if $a_F < 0$, the divisor F is μ-exceptional;

γ) one $F = H$ is ample and $[a_H H] = 0$.

We write E instead of E_1 and A instead of A_1. The situation is the following:

$$\begin{array}{ccc} V' & \overset{\text{hypersurface}}{\hookrightarrow} & X' \\ f \downarrow & & \downarrow \mu \\ V & \hookrightarrow & X \end{array}$$

Step 2: apply the theorem about extensions of sections from a hypersurface to the smooth hypersurface $V' \subset X'$ and the (integral) divisor

$$\begin{aligned} D' &= -K_{X'/X} - V' - [F'] + \mu^* D \\ &\sim F' - [F'] + \mu^* A \\ &= (a_H H + \mu^* A) + (F' - [F'] - a_H H) \end{aligned}$$

where $F' = \sum a_F F$; since

- $A' = a_H H + \mu^* A$ is ample;

- $E' = F' - [F'] - a_H H$ is effective and its support does not contain V';

• the pair $(V', E'|_{V'})$ is klt;[‡]

the restriction

$$H^0(X', m(K_{X'}+V'+D')) \to H^0(V', m(K_{V'}+D'|_{V'}))$$

is surjective for all $m > 0$.

Write

$$[F'] = \sum [a_i] F_i = F_+ - F_-$$

with

• F_+ and F_- effective;

• F_- μ-exceptional (property β)).

The sheaf $\mu_* \mathscr{O}_{X'}(-[F']) \subset \mathscr{O}_X$ has cosupport contained in $\mu(\mathrm{Supp}(F_+))$, hence not containing V (property α)). We have

$$\begin{aligned} H^0(X', m(K_{X'} + V' + D')) \\ &= H^0(X', m(\mu^*(K_X + D) - [F'])) \\ &\hookrightarrow H^0(X, m(K_X + D)) \end{aligned}$$

[‡]Because $E' + V'$ has simple normal crossings and the coefficients of E' are in $(0, 1)$.

hence, by restriction, an injection

$$H^0(V', m(K_{V'}+D'|_{V'})) \hookrightarrow H^0(X|V, m(K_X+D))$$

where $H^0(X|V, \star)$ is the image of the restriction $H^0(X, \star) \to H^0(V, \star|_V)$.

Step 3: $f : V' \to V$ has connected fibers. Moreover, if $M' = F'|_{V'}$, the sheaf $f_* \mathscr{O}_{V'}(-[M'])$ has rank 1 on V.

Property α) implies that $-[M']$ is effective outside of the support of the F_i that do not dominate V, hence $f_* \mathscr{O}_{V'}(-[M'])$ has positive rank.

On the other hand, the cosupport of the ideal sheaf $\mu_* \mathscr{O}_{X'}(-[F'])$ does not contain V and

$$-[F'] - V' \sim K_{X'} + \lceil -\mu^*(K_X + E) \rceil$$

Since $-\mu^*(K_X + E)$ is μ-nef and μ-big,

$$R^1 \mu_* \mathscr{O}_{X'}(-[F'] - V') = 0$$

by the relative Kawamata–Viehweg vanishing theorem, hence

$$\mu_* \mathscr{O}_{X'}(-[F']) \twoheadrightarrow f_* \mathscr{O}_{V'}(-[M'])$$

and $f_* \mathscr{O}_{V'}(-[M'])$ has rank exactly 1. It follows that $f_* \mathscr{O}_{V'}$ has rank 1 and f has connected fibers by Zariski's Main Theorem.

Next, we construct a commutative diagram

$$\begin{array}{ccc} W' & \xrightarrow{\tau'} & V' \\ g \downarrow & & \downarrow f \\ W & \xrightarrow{\tau} & V \end{array}$$

where τ and τ' are modifications such that

a) any hypersurface Z of W' with $\operatorname{codim}_W g(Z) \geqslant 2$ is τ'-exceptional;

b) τ' is an isomorphism above a dense open subset of V.

Indeed, by a theorem of Raynaud, there exists a modification $W \to V$ such that the component W'_0 of $W \times_V V'$ that dominates V' is flat, hence equidimensional, above W. Take for W' a desingularization of W'_0. Since singularities of W'_0 are above Exc(τ), we may assume that τ' is an isomorphism off g^{-1}(Exc(τ)), hence above $V - \tau$(Exc(τ)).

Decompose

$$N' = \tau'^* M' - K_{W'/V'}$$

as the sum of

• a g-vertical part N'_v and

• a g-horizontal part N'_h, which is the strict transform of M'_h. In particular, $[N'_h] \leqslant 0$.

Summary:

$$\begin{array}{ccccc} W' & \xrightarrow{\tau'} & V' & \hookrightarrow & X' \\ g \downarrow & & f \downarrow & & \downarrow \mu \\ W & \xrightarrow{\tau} & V & \hookrightarrow & X \end{array}$$

Step 4: Kawamata's theorem.

Let $g : W' \to W$ be a morphism with connected fibers between smooth irreducible projective varieties, let $Q' \subset W'$ be $Q \subset W$ hypersurfaces with normal crossings, and let N' be a $\mathbf{Q}$-divisor on W'. Assume:

i) $g^{-1}(Q) \subset Q'$, so that g induces a morphism $W' - Q' \to W - Q$;

ii) $\mathrm{Supp}(N') \subset Q'$;

iii) g is smooth above $W - Q$;

iv) $g|_{\mathrm{Supp}(N'_h)}$ has relative normal crossings above $W - Q$;

v) $g(N'_v) \subset Q$;

vi) $[N'_h] \leqslant 0$;

vii) $g_* \mathscr{O}_{W'}(-[N'])$ has rank 1;

viii) there exists a $\mathbf{Q}$-divisor N on W such that $K_{W'} + N' \sim g^*(K_W + N)$.

If

$$\begin{aligned} Q &= \sum_i Q_i \quad , \quad & Q' &= \sum_j Q'_j \\ g^* Q_i &= \sum_j q_{i,j} Q'_j \quad , \quad & N' &= \sum_j n'_j Q'_j \end{aligned}$$

$$\delta_i = \max_{g(Q'_j) = Q_i} \frac{n'_j + q_{i,j} - 1}{q_{i,j}}$$

and $\Delta = \sum_i \delta_i Q_i$, the $\mathbf{Q}$-divisor $N - \Delta$ is nef.

Upon composing τ with a modification $W_1 \to W$, and replacing W' by a desingularization of the component of $W_1 \times_W W'$ that dominates W', we may assume that there exist $Q' \subset W'$ and $Q \subset W$ as above satisfying i) thourh v).

Properties vi) and vii) hold, and if $N = \tau^*(K_X + E)|_V - K_W$, we have

$$\begin{aligned} K_{W'} + N' &= \tau'^*(K_{V'} + M') \\ &= \tau'^*(K_{X'} + V' + F')|_{V'} \\ &= g^*\tau^*(K_X + E)|_{V'} \\ &= g^*(K_W + N) \end{aligned}$$

(property viii)). By Kawamata, $N - \Delta$ is nef (on W).

End of the proof.

Let j such that $g(Q'_j)$ has codimension 1 in W; it is a $Q_{i(j)}$ and

$$\begin{aligned} q_{i,j} &= 0 \quad \text{for } i \neq i(j) \\ q_{i(j),j} &\geqslant 1 \\ \delta_{i(j)} q_{i(j),j} &= n'_j + q_{i(j),j} - 1 \end{aligned}$$

By property a), the Q'_j for which $g(Q'_j)$ has codimension > 1 in W are τ'-exceptional, hence

$$\begin{aligned} N' - g^*\Delta &= N'_h + \sum_j (1 - q_{i(j),j}) Q'_j + \tau'\text{-exc} \\ &\leqslant N'_h + \tau'\text{-exc} \end{aligned}$$

and

$$\begin{aligned} &K_{W'} + N' - g^*\Delta \\ &\quad \leqslant K_{W'} + N'_h + \tau'\text{-exc} \\ &\quad = \tau'^* K_{V'} + \tau'^* M'_h + \tau'\text{-exc} \\ &\quad \leqslant \tau'^*(K_{V'} + M'_h - [M'_h]) + \tau'\text{-exc} \\ &\quad \leqslant \tau'^*(K_{V'} + M'_h - [M'_h] + M'_v - [M'_v]) \\ &\qquad\qquad + \tau'\text{-exc} \\ &\quad = \tau'^*(K_{V'} + M' - [M']) + \tau'\text{-exc} \end{aligned}$$

For all $m \in \mathbf{N}$ such that $m(N - \Delta + \tau^* A|_V)$ are $m(M' + f^* A|_V)$ integral, we have, using $g_* \mathscr{O}_{W'} = \mathscr{O}_W$,

$$\begin{aligned} & H^0(W, m(K_W + N - \Delta + \tau^* A|_V)) \\ & \quad \simeq \quad H^0(W', m(K_{W'} + N' - g^* \Delta + g^* \tau^* A|_V)) \\ & \quad \hookrightarrow \quad H^0(W', m(\tau'^*(K_{V'} + M' - [M'] + f^* A|_V) \\ & \qquad\qquad\qquad\qquad\qquad\qquad + \tau'\text{-exc})) \\ & \quad \simeq \quad H^0(V', m(K_{V'} + M' - [M'] + f^* A|_V)) \\ & \quad \simeq \quad H^0(V', m(K_{V'} + D'|_{V'})) \\ & \quad \hookrightarrow \quad H^0(X|V, m(K_X + D)) \end{aligned}$$

The $\mathbf{Q}$-divisor $N - \Delta + \tau^* A|_V$, sum of the nef $N - \Delta$ and the nef and big $\tau^* A|_V$, is big. For all m sufficiently big and divisible, we have

$$H^0(W, mK_W) \hookrightarrow H^0(W, m(K_W + N - \Delta + \tau^* A|_V))$$

hence

$$\begin{aligned} h^0(V, mK_V) &= h^0(W, mK_W) \\ &\leqslant h^0(X|V, m(K_X + D)) \end{aligned}$$

Let $d = \dim V$, divide this inequality by $m^d/d!$, and let m go to infinity.

The left-hand-side goes to $\mathrm{vol}(K_V)$ and the upper limit of the right-hand-side is $\leqslant \mathrm{vol}_{X|V}(K_X + D)$.

Finally, if K_V (hence also K_W) is pseudo-effective, $K_W + N - \Delta + \tau^* A|_V$ is big, hence sections of sufficiently big and divisible multiples separate two general points of W. The same holds for sections of $K_X + D$ and general points of V.