

ON THE LOGARITHMIC KOBAYASHI CONJECTURE

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Abstract. We study the hyperbolicity of the log variety (\mathbb{P}^n, X) , where X is a very general hypersurface of degree $d \geq 2n + 1$ (which is the bound predicted by the Kobayashi conjecture). Using a positivity result for the sheaf of (twisted) logarithmic vector fields, which may be of independent interest, we show that any log-subvariety of (\mathbb{P}^n, X) is of log-general type, give a new proof of the algebraic hyperbolicity of (\mathbb{P}^n, X) , and exclude the existence of maximal rank families of entire curves in the complement of the universal degree d hypersurface. Moreover, we prove that, as in the compact case, the algebraic hyperbolicity of a log-variety is a necessary condition for the metric one.

1. INTRODUCTION

A complex manifold V is *hyperbolic* in the sense of S. Kobayashi if the hyperbolic pseudodistance defined on V (see section 2 for precise definitions) is a distance. A necessary condition for the hyperbolicity of a complex manifold is the constancy of holomorphic maps from \mathbb{C} to V . Hyperbolic complex manifolds have been studied in the two following contexts. One is the hyperbolicity of a compact complex manifold, in which case, thanks to a criterion due to R. Brody [1], the above necessary condition is also sufficient to guarantee the hyperbolicity of V . The other is the hyperbolicity of a compact complex manifold with an ample divisor removed. In the case of complements of projective hypersurfaces we have the Kobayashi conjecture [14]:

Conjecture 1. *The complement $\mathbb{P}^n \setminus X$ of a general hypersurface $X \subset \mathbb{P}^n$ of degree $\deg X \geq 2n + 1$, $n \geq 2$, is hyperbolic.*

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Notice that the lower bound in Conjecture 1 is sharp, since, as noticed first by M. Zaidenberg [26], there exists a line intersecting a general degree $2n$ hyperpersurface in two points. (For the Kobayashi conjecture on the hyperbolicity of a general hypersurface of high degree, and the related results, we refer the reader to §2.2).

In the present paper we study questions related to Conjecture 1 (which is proved for $n = 2$ and $d \geq 15$ in [10]), by extending to the logarithmic setting (part of) the techniques and ideas successfully used in the compact case.

Let \bar{V} be a smooth variety with a normal crossing divisor D . The pair (\bar{V}, D) is called a log-variety. Let $V = \bar{V} \setminus D$. We denote by $\bar{T}_V^* = T_V^*(\log D)$ its log-cotangent bundle and by $\bar{K}_V = \wedge^{\dim(V)} \bar{T}_V^* = K_{\bar{V}}(D)$ its log-canonical bundle. In the third section, in order to study the algebraic hyperbolicity properties of the log-variety (\mathbb{P}^n, X) , where X is a very general hypersurface, we prove the following non-vanishing result.

Theorem 2. *Let X be a very general hypersurface of arbitrary degree d in \mathbb{P}^n . Let Y be a k -dimensional subvariety in \mathbb{P}^n meeting X properly, $D := Y \cap X$ the induced divisor and $\nu : \tilde{Y} \rightarrow Y$ a log-resolution of (Y, D) i.e \tilde{Y} is smooth, ν is a projective birational morphism and $\nu^{-1}(D) + \text{Exc}(\nu)$ is a normal crossing divisor. Then*

$$h^0(\tilde{Y}, \bar{K}_{\tilde{Y}} \otimes \nu^* \mathcal{O}_{\mathbb{P}^n}(2n + 1 - k - d)) \neq 0,$$

where $\bar{K}_{\tilde{Y}}$ denotes the log-canonical bundle of the log-variety $(\tilde{Y}, \nu^{-1}(D))$.

In particular we deduce :

Corollary 3. *Let $X \subset \mathbb{P}^n$ be a very general hypersurface of degree $d \geq 2n + 2 - k$, $k \geq 1$. Then any k -dimensional log-subvariety (Y, D) of (\mathbb{P}^n, X) , for Y not contained in X , is of log-general type, that is, any log-resolution $\nu : \tilde{Y} \rightarrow Y$ of (Y, D) has big log-canonical bundle $K_{\tilde{Y}}(\nu^{-1}(D))$.*

Corollary 4. *Let $X \subset \mathbb{P}^n$ be a very general hypersurface of degree $d \geq 2n + 1$, and $C \subset \mathbb{P}^n$ a curve not contained in X . Then*

$$2g(\tilde{C}) - 2 + i(C, X) \geq (d - 2n) \deg C$$

where $\nu : \tilde{C} \rightarrow C$ is the normalization of C , $g(\tilde{C})$ its genus, and $i(C, X)$ is the number of distinct points in $\nu^{-1}(X)$.

The inequality in Corollary 4 has been previously proved by Xi Chen in [3] by means of a delicate degeneration argument (see also [24], Theorem 3.10, where it is proved that any C intersects a general hyper-surface of degree $d = 2n - 2 + r$, $r \geq 3$, in at least r points, as well

as [2] and [25] for the case $n = 2$). Note that, as a consequence, one gets that there is no entire curve $f : \mathbb{C} \rightarrow \mathbb{P}^n \setminus X$ in the complement of a very general hypersurface of degree $d \geq 2n + 1$, $n \geq 2$, if the Zariski closure $\overline{f(\mathbb{C})}$ is an algebraic curve. Notice moreover that both Corollaries 3 and 4 are sharp: the latter by the result of Zaïdenberg we have quoted above, and the former by a natural generalization of Zaïdenberg's result, which we present at the end of §3.

In the fourth section we prove that, as one expects, the hyperbolicity of a log-variety implies its algebraic hyperbolicity, thus answering a question raised by Xi Chen in [3].

Theorem 5. *Let X be a projective manifold and D an effective divisor on X such that $X \setminus D$ is hyperbolic and hyperbolically imbedded. Let ω be a hermitian metric on X . Then there exists $\varepsilon > 0$ such that*

$$2g(\tilde{C}) - 2 + i(C, D) \geq \varepsilon \deg_{\omega}(C)$$

for every compact irreducible curve $C \subset X$ with $C \not\subset D$, where \tilde{C} is the normalization of C , $g(\tilde{C})$ its genus and $\deg_{\omega}(C) = \int_C \omega$.

In the last section we strenghten the conclusion of Corollary 4, and prove that there is no entire curve, varying in a family of maximal rank, in the complement of a general hypersurface of degree at least $2n + 1$, without assumptions on the Zariski closure of the entire curve.

We denote by $\mathcal{X} \subset \mathbb{P}^n \times \mathbb{P}^{N_d}$ the universal family of hypersurfaces in \mathbb{P}^n with degree d . We will denote by X_t the fiber of \mathcal{X} over the parameter $t \in \mathbb{P}^{N_d}$.

Theorem 6. *Let $U \rightarrow \mathbb{P}^{N_d} := \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ be an étale cover of an open subset of \mathbb{P}^{N_d} , and let $\Phi : \mathbb{C} \times U \rightarrow \mathbb{P}^n \times U$ be a holomorphic map such that $\Phi(\mathbb{C} \times \{t\}) \subset \mathbb{P}^n \setminus X_t$ for all $t \in U$. If $d \geq 2n + 1$, the rank of Φ cannot be maximal anywhere.*

The above theorem is of course a consequence of Conjecture 1, and represents the logarithmic analog of the main result of [6]. Again, by Zaïdenberg's example, our result is sharp, as for $d = 2n$ one can consider the family of the exponential maps associated to the family of lines ℓ_t intersecting the hypersurface X_t in two points :

$$\exp_t : \mathbb{C} \times \{t\} \rightarrow \mathbb{C}^* = \ell_t \setminus (\ell_t \cap X_t) \subset \mathbb{P}^n \setminus X_t.$$

The proofs of Theorems 2 and 6 use the global generation of the sheaf of (twisted) vector fields on the log variety $(\mathbb{P}^n \times \mathbb{P}^{N_d}, \mathcal{X})$ (see §3, Proposition 11 for the precise statement). Once the logarithmic framework is set, our approach allows natural proofs, which are formally equal to those of the corresponding hyperbolicity properties of

$X \subset \mathbb{P}^n$. In that sense, it unifies the compact and the logarithmic cases. Notice moreover that the analogous global generation result in the compact case is the first step in Y.-T. Siu's proof of the hyperbolicity of a very general hypersurface $X \subset \mathbb{P}^n$, for $d \gg n$ (see [21], Lemma 4, and [22], Proposition 1.1). It seems then plausible that using a generalization of Proposition 11 to logarithmic jet bundles and following the strategy outlined in [21], one could prove Conjecture 1 for very high degree : this has been done for $n = 3$ in [20].

2. PRELIMINARIES

2.1. Log-manifolds. Let \bar{V} be a complex manifold with a normal crossing divisor D . The pair (\bar{V}, D) is called a *log-manifold*. Let $V = \bar{V} \setminus D$ be the complement of D .

Following [13], the logarithmic cotangent sheaf $\bar{T}_V^* = T_{\bar{V}}^*(\log D)$ is defined as the locally free subsheaf of the sheaf of meromorphic 1-forms on \bar{V} , whose restriction to V is T_V^* and whose localization at any point $x \in D$ is given by

$$\bar{T}_{V,x}^* = \sum_{i=1}^l \mathcal{O}_{\bar{V},x} \frac{dz_i}{z_i} + \sum_{j=l+1}^n \mathcal{O}_{\bar{V},x} dz_j$$

where the local coordinates z_1, \dots, z_n around x are chosen such that $D = \{z_1 \dots z_l = 0\}$.

Its dual, the logarithmic tangent sheaf $\bar{T}_V = T_{\bar{V}}(-\log D)$ is a locally free subsheaf of the holomorphic tangent bundle $T_{\bar{V}}$, whose restriction to V is T_V and whose localization at any point $x \in D$ is given by

$$\bar{T}_{V,x} = \sum_{i=1}^l \mathcal{O}_{\bar{V},x} z_i \frac{\partial}{\partial z_i} + \sum_{j=l+1}^n \mathcal{O}_{\bar{V},x} \frac{\partial}{\partial z_j}.$$

Recall that starting with an arbitrary divisor, Hironaka's theorem on resolution of singularities [12] guarantees that we can replace it by a normal crossing one after performing some blowing-ups.

Theorem 7. (*Hironaka, [12]*) *Let V be an irreducible complex algebraic variety (possibly singular), and let $D \subset V$ be an effective divisor on V . There is a projective birational morphism*

$$\mu : V' \rightarrow V,$$

where V' is non singular and μ has divisorial exceptional locus $\text{except}(\mu)$, such that

$$\mu^{-1}(D) + \text{except}(\mu)$$

is a normal crossing divisor.

One calls V' a *log-resolution* of (V, D) .

2.2. Hyperbolicity and algebraic hyperbolicity. Let X be a complex manifold. We denote by $f : \Delta \rightarrow X$ an arbitrary holomorphic map from the unit disk $\Delta \subset \mathbb{C}$ to X . The Kobayashi-Royden infinitesimal pseudo-metric [14] on X is the Finsler pseudometric on the tangent bundle T_X defined as follows. Fix a point $x \in X$ and a tangent vector $\xi \in T_{X,x}$. Then we define the infinitesimal Kobayashi length of ξ at x to be

$$k_X(\xi) = \inf\{\lambda > 0 \quad , \quad \text{such that there exists a holomorphic map} \\ f : \Delta \rightarrow X \text{ with } f(0) = x \text{ and } \lambda f'(0) = \xi\}.$$

The Kobayashi pseudo-distance d_X , is the geodesic pseudodistance obtained by integrating the Kobayashi-Royden infinitesimal pseudometric. The manifold X is *hyperbolic* in the sense of S. Kobayashi if the hyperbolic pseudodistance defined on X is a distance.

Directly from the definition of the Kobayashi pseudo-distance one can see that if $f : X \rightarrow Y$ is a holomorphic map of complex manifolds then it is distance decreasing i.e for $x, x' \in X$ we have

$$d_Y(f(x), f(x')) \leq d_X(x, x').$$

As mentioned in the introduction, in the case of a general projective hypersurface X , both X and its complement are conjectured to be hyperbolic, as soon as the degree of X is high enough.

Conjecture 8. *A general hypersurface $X \subset \mathbb{P}^n$, $n \geq 3$, of degree $\deg X \geq 2n - 1$ is hyperbolic.*

(For $n \geq 4$, the natural lower bound should be $2n - 2$).

The most important confirmation of Conjecture 8 has been obtained by Y.-T. Siu [21], who proved it for $d_n \gg n$. As for the known lower bounds on the degree, Conjecture 8 has been studied for $n = 3$ in [8] and [16], where the bound $d \geq 21$ (respectively $d \geq 36$) has been obtained (in the recent preprint [18], this bound has been improved to $d \geq 18$). In [19], the second author proved a weak form of Conjecture 8 for $n = 4$ and $d \geq 593$.

It is widely believed that when dealing with a projective variety V , there should exist a property of algebraic nature equivalent to the hyperbolicity of V . In the compact case, Demailly (see [7]) introduced the notion of *algebraic hyperbolicity*, and proved it is a necessary condition for the hyperbolicity. S. Lang proposed another property, namely the fact that any subvariety of V is of general type. Both properties have been checked for very general hypersurfaces of degree $d \geq 2n - 2$, in

[5] and [17], building on ideas and techniques introduced by C. Voisin [23] (see also [4] and [9]).

In analogy to the compact case, Xi Chen [3] studied the notion of algebraic hyperbolicity, in the sense of Demailly, for log-manifolds.

Definition 9. Let (X, D) be a log-manifold. For each reduced curve $C \subset X$ that meets D properly, let $\nu : \tilde{C} \rightarrow C$ be the normalization of C . Then $i(C, D)$ is the number of distinct points in the set $\nu^{-1}(D) \subset \tilde{C}$.

Definition 10. A logarithmic variety (X, D) is algebraically hyperbolic if there exists a positive number ε such that

$$2g(\tilde{C}) - 2 + i(C, D) \geq \varepsilon \deg_{\omega}(C)$$

for all reduced and irreducible curves $C \subset X$ meeting D properly where \tilde{C} is the normalization of C , $g(\tilde{C})$ its genus and $\deg_{\omega}(C) = \int_C \omega$ with ω a hermitian metric on X .

In the next section, we prove that the algebraic hyperbolicity in the sense of Demailly, as well as the algebraic property analogous to that proposed by Lang, hold for the complement of a very general projective hypersurface of degree at least equal to $2n + 1$.

3. ALGEBRAIC HYPERBOLICITY OF THE LOG VARIETY (\mathbb{P}^n, X)

In this section we give the proof of theorem 2, using logarithmic techniques, and the global generation of the sheaf of (twisted) logarithmic vector fields, which we now introduce.

Fix the following notations:

$\mathbb{P}^{N_d} := \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ denotes the parameter space for degree d hypersurfaces in \mathbb{P}^n .

$\mathcal{X} \subset \mathbb{P}^n \times \mathbb{P}^{N_d}$ denotes the universal hypersurface of degree d , and p and q the projections of $\mathbb{P}^n \times \mathbb{P}^{N_d}$ onto the two factors.

$X_F \subset \mathbb{P}^n_F$ is the hypersurface defined by the homogeneous polynomial $F \in \mathbb{P}^{N_d}$.

For a smooth hypersurface X_F we have the corresponding logarithmic manifold (\mathbb{P}^n_F, X_F) , with logarithmic tangent sheaf $\overline{T}_{\mathbb{P}^n_F} = T_{\mathbb{P}^n_F}(-\log X_F)$, logarithmic cotangent sheaf $\overline{\Omega}_{\mathbb{P}^n_F} = \Omega_{\mathbb{P}^n_F}(\log X_F)$ and logarithmic canonical sheaf $\overline{K}_{\mathbb{P}^n_F} = K_{\mathbb{P}^n_F} \otimes \mathcal{O}(d) = \mathcal{O}(d - n - 1)$.

3.1. The global generation result. We shall extend to the logarithmic setting an approach initiated by Clemens, Ein, Voisin and Siu (see [4], [9], [23], [21]):

Proposition 11. *The twisted logarithmic tangent bundle*

$$T_{\mathbb{P}^n \times \mathbb{P}^{N_d}}(-\log \mathcal{X})(1, 0) := T_{\mathbb{P}^n \times \mathbb{P}^{N_d}}(-\log \mathcal{X}) \otimes p^* \mathcal{O}_{\mathbb{P}^n}(1)$$

is generated by its global sections.

Proof. Let $\mathcal{X} \subset \mathbb{P}^n \times \mathbb{P}^{N_d}$ be the universal hypersurface of degree d given by the equation

$$\sum_{|\alpha|=d} a_\alpha Z^\alpha = 0$$

where $[a] \in \mathbb{P}^{N_d}$ and $[Z] \in \mathbb{P}^n$, for $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, $|\alpha| = \sum_i \alpha_i$ and if $Z = (Z_0, \dots, Z_n)$ are homogeneous coordinates on \mathbb{P}^n , then $Z^\alpha = \prod Z_j^{\alpha_j}$. Notice that \mathcal{X} is a smooth hypersurface of bidegree $(d, 1)$ in $\mathbb{P}^n \times \mathbb{P}^{N_d}$. We consider the log-manifold $(\mathbb{P}^n \times \mathbb{P}^{N_d}, \mathcal{X})$. Let us consider

$$\mathcal{Z} = (a_{0\dots 0d} Z_{n+1}^d + \sum_{|\alpha|=d, \alpha_{n+1}=0} a_\alpha Z^\alpha = 0) \subset \mathbb{P}^{n+1} \times U$$

where $\alpha \in \mathbb{N}^{n+2}$, and

$$U := (a_{0\dots 0d} \neq 0) \cap \left(\bigcup_{|\alpha|=d, \alpha_{n+1}=0} (a_\alpha \neq 0) \right) \subset \mathbb{P}^{N_d+1}.$$

Consider the natural projection $\pi : \mathcal{Z} \rightarrow \mathbb{P}^n \times \mathbb{P}^{N_d}$ and set

$$\mathcal{H} := \pi^{-1}(\mathcal{X}) = \{Z_{n+1} = 0\}.$$

Therefore we obtain a dominant log-morphism $\pi : (\mathcal{Z}, \mathcal{H}) \rightarrow (\mathbb{P}^n \times \mathbb{P}^{N_d}, \mathcal{X})$ which induces a map

$$\pi_* : \bar{T}_{\mathcal{Z}}(1, 0) := T_{\mathcal{Z}}(-\log \mathcal{H})(1, 0) \rightarrow \bar{T}_{\mathbb{P}^n \times \mathbb{P}^{N_d}}(1, 0) := T_{\mathbb{P}^n \times \mathbb{P}^{N_d}}(-\log \mathcal{X})(1, 0).$$

Therefore we are reduced to prove that $\bar{T}_{\mathcal{Z}}(1, 0)$ is generated by its global sections. Consider the open set $U_0 = \{Z_0 \neq 0\} \times U$ in $\mathbb{P}^{n+1} \times U$ with the induced inhomogeneous coordinates. The equation of \mathcal{Z} on U_0 becomes

$$\mathcal{Z}_0 := \{z_{n+1}^d + \sum_{|\alpha| \leq d, \alpha_{n+1}=0} a_\alpha z^\alpha = 0\}.$$

Consider the vector field

$$V_{\alpha, j} = \frac{\partial}{\partial a_\alpha} - z_j \frac{\partial}{\partial a_{\tilde{\alpha}}}$$

where $\alpha \in \mathbb{N}^{n+1}$, $\alpha_{n+1} = 0$, $\exists j \alpha_j \geq 1$, $\tilde{\alpha}_k = \alpha_k$ if $k \neq j$ and $\tilde{\alpha}_j = \alpha_j - 1$. Notice that $V_{\alpha, j}$ is a logarithmic vector field of $(\mathcal{Z}_0, \mathcal{H}_0 := \mathcal{H} \cap U_0)$ which extends to $(\mathcal{Z}, \mathcal{H})$ with a pole order equal to 1.

Consider a vector field

$$V_0 = \sum_{j=1}^n v_j \frac{\partial}{\partial z_j} + v_{n+1} z_{n+1} \frac{\partial}{\partial z_{n+1}}$$

where

$$v_j = \sum_{k=1}^n v_k^{(j)} z_k + v_0^j, 1 \leq j \leq n$$

is linear in the variables z_k , and $v_{n+1} \in \mathbb{C}$. We claim that there exists a logarithmic vector field

$$V = \sum_{|\alpha| \leq d, \alpha_{n+1} = 0} v_\alpha \frac{\partial}{\partial a_\alpha} + V_0$$

tangent to \mathcal{L}_0 . Indeed, the condition to be satisfied is

$$\sum_{\alpha} v_\alpha z^\alpha + \sum_{\alpha, j} a_\alpha v_j \frac{\partial z^\alpha}{\partial z_j} = 0$$

and the complex numbers v_α are chosen such that the coefficient of z^α in the above equation is equal to zero. This logarithmic vector field of $(\mathcal{L}_0, \mathcal{H}_0)$ extends to $(\mathcal{L}, \mathcal{H})$.

The previous vector fields give the global generation of $\overline{T}_{\mathcal{X}}(1, 0)$. \square

3.2. Sharp algebraic hyperbolicity properties for (\mathbb{P}^n, X) . Having recorded in the previous subsection the needed positivity result, we can now prove Theorem 2, together with its corollaries. We also show, in Example 13, that our results are sharp in the degree.

Proof of Theorem 2. Let $U \subset \mathbb{P}^N$ be the open subset parametrizing smooth hypersurface. We want to study families of k -dimensional irreducible subvarieties inside $\mathbb{P}^n \times U$, intersecting properly the family of hypersurfaces. So, eventually passing to an étale cover of U , we consider an irreducible subvariety $\mathcal{Y} \subset \mathbb{P}^n \times \mathbb{P}^N$ such that the projection map $\mathcal{Y} \rightarrow U$ is dominant of relative dimension k , and such that \mathcal{Y} intersects properly \mathcal{X} (and so does its generic fiber Y_F with X_F). Let $\mathcal{D} \subset \mathcal{Y}$ the family of divisors induced by the intersections $D_F := Y_F \cap X_F$. Let $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ be a log resolution of $(\mathcal{Y}, \mathcal{D})$ i.e $\widetilde{\mathcal{Y}}$ is smooth and $\widetilde{\mathcal{D}} = \nu^{-1}(\mathcal{D})$ is a normal crossing divisor, and so is its general fiber $\widetilde{D}_F \subset \widetilde{Y}_F$.

For general $F \in U$, and arbitrary degree d we want to produce a non zero element in $H^0(\widetilde{Y}_F, \overline{K}_{\widetilde{Y}_F}(2n+1-k-d))$, where $\overline{K}_{\widetilde{Y}_F} = K_{\widetilde{Y}_F}(\widetilde{D}_F)$.

We have:

$$(1) \quad \overline{K}_{\tilde{Y}_F} \simeq \overline{\Omega}_{\tilde{\mathcal{Y}}}^{N+k} \Big|_{\tilde{Y}_F}$$

Indeed:

$$\Omega_{\tilde{\mathcal{Y}}}^{N+k} \otimes \mathcal{O}_{\tilde{\mathcal{Y}}}(\tilde{\mathcal{D}}) \Big|_{\tilde{Y}_F} = \Omega_{\tilde{\mathcal{Y}}}^{N+k} \Big|_{\tilde{Y}_F} \otimes \mathcal{O}_{\tilde{Y}_F}(\tilde{D}_F),$$

and by the adjunction formula

$$K_{\tilde{Y}_F} = \Omega_{\tilde{\mathcal{Y}}}^{N+k} \Big|_{\tilde{Y}_F},$$

since the normal bundle of a fiber in a family is trivial.

Using standard linear algebra, we have:

$$(2) \quad \left(\bigwedge^{n-k} T_{\mathbb{P}^n \times \mathbb{P}^N}(-\log \mathcal{X}) \right) \Big|_{\mathbb{P}_F^n} \otimes \overline{K}_{\mathbb{P}_F^n} \simeq \Omega_{\mathbb{P}^n \times \mathbb{P}^N}^{N+k}(\log \mathcal{X}) \Big|_{\mathbb{P}_F^n}$$

The generically surjective map $K_{\mathbb{P}^n \times \mathbb{P}^N}(\mathcal{X}) \rightarrow K_{\tilde{\mathcal{Y}}}(\tilde{\mathcal{D}})$ induces a map

$$\Omega_{\mathbb{P}^n \times \mathbb{P}^N}^{N+k}(\log \mathcal{X}) \Big|_{\mathbb{P}_F^n} (2n+1-k-d) \rightarrow \overline{\Omega}_{\tilde{\mathcal{Y}}}^{N+k} \Big|_{\tilde{Y}_F} (2n+1+k-d)$$

that is non zero for F general in U .

Recalling that

$$\overline{K}_{\mathbb{P}_F^n} = \mathcal{O}_{\mathbb{P}_F^n}(d-n-1)$$

and using (1) and (2), it is enough to show that

$$\left(\bigwedge^{n-k} T_{\mathbb{P}^n \times \mathbb{P}^N}(-\log \mathcal{X}) \right) \Big|_{\mathbb{P}_F^n} \otimes \mathcal{O}_{\mathbb{P}_F^n}(n-k)$$

is globally generated. To conclude, we notice that

$$\left(\bigwedge^{n-k} T_{\mathbb{P}^n \times \mathbb{P}^N}(-\log \mathcal{X}) \right) \Big|_{\mathbb{P}_F^n} \otimes \mathcal{O}_{\mathbb{P}_F^n}(n-k) = \bigwedge^{n-k} \left(T_{\mathbb{P}^n \times \mathbb{P}^N}(-\log \mathcal{X}) \Big|_{\mathbb{P}_F^n} (1) \right)$$

and invoke the global generation of $T_{\mathbb{P}^n \times \mathbb{P}^N}(-\log \mathcal{X}) \Big|_{\mathbb{P}_F^n} (1)$, that follows from Proposition 11.

Letting the family \mathcal{Y} vary, that is, varying the Hilbert polynomial, we obtain that for F outside a countable union of proper closed subvarieties of U , all the k -dimensional subvarieties Y intersecting properly X_F verify

$$(3) \quad h^0(\tilde{Y}, \overline{K}_{\tilde{Y}} \otimes \nu^* \mathcal{O}_{\mathbb{P}^n}(2n+1-k-d)) \neq 0.$$

□

Let us now show how to deduce Corollaries 3 and 4.

Proof of Corollary 3. By the above theorem, the logarithmic canonical bundle of (\tilde{Y}, \tilde{D}) may be written as the sum of the *effective* line bundle $\overline{K}_{\tilde{Y}} \otimes \nu^* \mathcal{O}_{\mathbb{P}^n}(2n+1-k-d)$ and the line bundle $\nu^* \mathcal{O}_{\mathbb{P}^n}(d-(2n+1-k))$. The latter is *big*, as soon as $d \geq 2n+2-k$, so the corollary is proved. □

Proof of Corollary 4. If $C \subset \mathbb{P}^n$ is a curve intersecting properly the general hypersurface X_F , $f: \tilde{C} \rightarrow C$ its desingularization, $D := C \cap X_F$ the divisor given by the intersection with the hypersurface, and $\tilde{D} = f^{-1}(D)$, then by (3) we have

$$0 \leq \deg(K_{\tilde{C}}(\tilde{D}) \otimes f^* \mathcal{O}_{\mathbb{P}^n}(2n-d)) = 2g(C) - 2 + i(C, X) - (d-2n) \deg C,$$

and we are done. □

A further consequence is the following.

Corollary 12. *For a very general hypersurface X of degree $d \geq 2n+1$ in \mathbb{P}^n , $\mathbb{P}^n \setminus X$ does not contain any algebraic torus \mathbb{C}^* . Therefore a holomorphic map $f: \mathbb{C} \rightarrow \mathbb{P}^n \setminus X$ is constant if $f(\mathbb{C})$ is contained in an algebraic curve.*

We end the present section by discussing an example which generalizes [26], and shows that also Corollary 3 is sharp.

Example 13. Given a degree d hypersurface $X \subset \mathbb{P}^n$ and an integer $r \geq 1$, consider the bicontact locus $\Delta_{r,d-r,X} \subset X$ consisting of points $x \in X$ through which passes a line ℓ such that ℓ has contact at least r at x and, if it is not contained in X , ℓ intersects X at most another point x' . In other words, generically we have

$$\ell \cap X = r \cdot x + (d-r) \cdot x'.$$

If X is general of degree $d \leq 2n$, then $\Delta_{r,d-r,X}$ is non empty and of the expected dimension $2n-d$ (see [23], or [17], §4, for the proof of this fact, and for a description of (a desingularization of) $\Delta_{r,d-r,X}$ as the zero locus of a section of a vector bundle). Hence, taking $d = 2n$, we recover the existence of a line intersecting the general degree d hypersurface in two points, as first observed in [26]. Now take $d = 2n+1-k$. In this case the dimension of $\Delta_{r,d-r,X}$ equals $k-1$. Let Y be the $(k$ -dimensional) subvariety of \mathbb{P}^n spanned by the lines parametrized by $\Delta_{r,d-r,X}$. If its desingularization \tilde{Y} were of log-general type, then by restriction to the general line ℓ in the ruling of Y , we would get non constant sections of $H^0(\ell, \overline{K}_{\ell}^{\otimes m})$. So we would get to a contradiction since $\overline{K}_{\ell} = \mathcal{O}_{\ell}$, as ℓ intersects X at two points.

4. HYPERBOLICITY AND ALGEBRAIC HYPERBOLICTY OF LOG VARIETIES

In this section, we would like to answer a question raised by X. Chen in [3]:

Let X be a projective manifold and D an effective divisor on X . Is it true that if $X \setminus D$ is hyperbolic and hyperbolically imbedded then (X, D) is algebraically hyperbolic?

The answer is positive. Namely we have theorem 5.

To give a proof we need to recall a certain number of basic definitions and results taken from [11] and [15]. Specialists may safely skip them and go directly to the proof of the theorem.

First, we need a Gauss-Bonnet formula in the non-compact case. We follow the approach of [11] which we recall for the convenience of the reader. Let M be a Riemann surface. M is said to be of *finite type* if there exists a compact Riemann surface M' containing M such that $M' \setminus M$ consists of finitely many points. The genus of M is defined as the genus of M' .

A puncture of M is defined to be a domain $D_0 \subset M$ conformally equivalent to $\{z \in \mathbb{C}; 0 < |z| < 1\}$. The data of a puncture is thus equivalent to that of a point in $M' \setminus M$.

Recall that a Kleinian group G is a subgroup of PGL_2 whose action on \mathbb{P}^1 is discontinuous at some point and that a Kleinian group is called Fuchsian if there is a disc invariant under the action. Let G be a Fuchsian group acting on the unit disk Δ . Let $\{x_1, x_2, \dots, x_n\}$ be the set of points of Δ/G that are either punctures or ramified points of the projection $\pi : \Delta \rightarrow \Delta/G$. Let ν_j be the ramification index of $\pi^{-1}(x_j)$ and set $\nu_j = \infty$ for punctures. Let us assume that Δ/G is of finite type. If π is ramified over finitely many points, then we will say that G is of *finite type over Δ* . We let g be the genus of Δ/G . We can define the characteristic of G :

$$\chi = 2g - 2 + \sum_{j=1}^n \left(1 - \frac{1}{\nu_j}\right).$$

We can project the Poincaré metric $\frac{4|dz|^2}{(1-|z|^2)^2}$ on Δ/G which gives the hyperbolic metric of constant curvature -1 . We have the following theorems:

Theorem 14. ([11], p.233) *The area of Δ/G with respect to the hyperbolic metric is finite and*

$$Area(\Delta/G) = 2\pi\chi.$$

Theorem 15. ([11], p.234) *Let M be a Riemann surface and $\{x_1, x_2, \dots\}$ a discrete sequence on M . To each point x_k we assign an integer $\nu_k \geq 2$ or ∞ .*

If $M = \mathbb{P}^1$ we exclude two cases:

- (i) $\{x_1, x_2, \dots\}$ consists of one point and $\nu_1 \neq \infty$.
- (ii) $\{x_1, x_2, \dots\}$ consists of two points and $\nu_1 \neq \nu_2$.

Let $M' = M \setminus \bigcup_{\nu_k=\infty} \{x_k\}$. Then there exists a simply connected Riemann surface \widetilde{M} , a Kleinian group G of self mappings of \widetilde{M} such that

- (a) $\widetilde{M}/G \cong M'$
- (b) *the natural projection $\pi : \widetilde{M} \rightarrow M'$ is unramified except over the points x_k with $\nu_k < \infty$ where the branch numbers verify $b_\pi(\tilde{x}) = \nu_k - 1$ for all $\tilde{x} \in \pi^{-1}(\{x_k\})$.*

The third result we need is related to the notion of hyperbolic imbedding (see [15]). Let Z be a complex manifold and Y a complex submanifold with compact closure \overline{Y} . Y is hyperbolically imbedded in Z if for every pair of distinct points p, q in $\overline{Y} \subset Z$, there exist neighborhoods U_p and U_q of p and q in Z such that $d_Y(U_p \cap Y, U_q \cap Y) > 0$ which is equivalent to say that $d_Y(p_n, q_n)$ cannot converge to zero when two sequences $\{p_n\}$ and $\{q_n\}$ in Y approach two distinct points p and q of the boundary $\partial Y = \overline{Y} \setminus Y$.

Let us prove the following proposition which is another version of Theorem 3.3.3 of [15]:

Proposition 16. *Let Y be a relatively compact complex submanifold (i.e \overline{Y} is compact) of a complex manifold Z . Then the following are equivalent:*

- (a) Y is hyperbolically imbedded in Z .
- (b) *Given a length function L on Z there is a positive constant ε such that $k_Y \geq \varepsilon L$ on Y .*

Proof. Let us prove that (a) implies (b). If ε does not exist then, from the definition of the Kobayashi infinitesimal pseudometric, there exists a sequence $\{f_n\}$ of holomorphic functions from Δ to Y such that

$$L(f'_n(0)) > n.$$

Since \overline{Y} is compact we may assume that $\{f_n(0)\}$ converges to a point $p \in \overline{Y}$.

Let U be a complete hyperbolic neighborhood of p in Z . Assume that there exists a positive number $r < 1$ such that $f_n(\Delta_r) \subset U$ for $n \geq n_0$. Then $\{f_n|_{\Delta_r} : \Delta_r \rightarrow U\}$ would be relatively compact and would have a

subsequence which converges to a holomorphic function from Δ_r to U , which contradicts $L(f'_n(0)) > n$.

This means that for each positive integer k , there exist a point $z_k \in \Delta$ and an integer n_k such that $|z_k| < \frac{1}{k}$ and $f_{n_k}(z_k) \notin U$. Let $p_k = f_{n_k}(0)$ and $q_k = f_{n_k}(z_k)$. By taking a subsequence we may assume that $\{q_k\}$ converges to a point q not in U . Therefore we have

$$d_Y(p_k, q_k) \leq d_\Delta(0, z_k) \rightarrow 0 \text{ for } k \rightarrow \infty,$$

and this contradicts the fact that Y is hyperbolically imbedded in Z .

Let us prove that (b) implies (a). Let δ be the distance function on Z induced by L . Then

$$\varepsilon\delta \leq d_Y \text{ on } Y$$

which implies obviously that Y is hyperbolically imbedded in Z . \square

Now, we can prove Theorem 5.

Proof of Theorem 5. Let $\nu : \tilde{C} \rightarrow C$ be the normalization and $\tilde{D} = \nu^{-1}(D)$. As $X \setminus D$ is hyperbolic $C' = \tilde{C} \setminus \tilde{D}$ is hyperbolic and admits the unit disk as its universal cover $\rho : \Delta \rightarrow C'$. Let $k_{C'}$ be the hyperbolic metric of constant curvature -1 with $\mu_{C'}$ its area element. From the distance decreasing property of Kobayashi metrics and proposition 16 we have

$$k_{C'}(t) \geq k_{X \setminus D}(\nu_*(t)) \geq \varepsilon \|\nu_*(t)\|_\omega, \quad \forall t \in T_{C'}.$$

Therefore from theorems 14 and 15 we have

$$2\pi(2g(\tilde{C}) - 2 + i(C, D)) = \int_{C'} \mu_{C'} \geq \varepsilon^2 \int_C \omega.$$

\square

5. FAMILIES OF ENTIRE CURVES IN THE COMPLEMENT OF THE UNIVERSAL HYPERSURFACE.

The goal of this section is to prove that a family of entire curves in the complement of the universal degree d hypersurface $\mathcal{X} \subset \mathbb{P}^n \times \mathbb{P}^{N_d}$ cannot have maximal rank, as soon as $d \geq 2n + 1$ (as predicted by the log Kobayashi conjecture). As in the compact case, which has been treated in [6], such a result points out to the lack of a good parameter space for entire curves. The proof goes exactly as in the compact case and relies on the global generation of the bundle $T_{\mathbb{P}^n \times \mathbb{P}^{N_d}}(-\log \mathcal{X})(1, 0)$, proved in Proposition 11.

Let $U \rightarrow \mathbb{P}^{N_d}$ be an étale cover of the open subset parametrizing smooth hypersurfaces. As before, to render the notation less heavy we will simply write $U \subset \mathbb{P}^{N_d}$

Consider a holomorphic map $\Phi : \mathbb{C} \times U \rightarrow (\mathbb{P}^n \times U) \setminus \mathcal{X}$ over the base $U \subset \mathbb{P}^{N_d}$.

As U is an open set, we can shrink it and suppose that it is equal to a polydisc $\mathbb{B}(\delta_0)^{N_d}$. Consider the following sequence of maps

$$\Phi_k : \mathbb{B}(\delta_0 k)^{N_d+1} \rightarrow (\mathbb{P}^n \times U) \setminus \mathcal{X}$$

given by $\Phi_k(z, \xi_1, \dots, \xi_{N_d}) = \Phi(zk^{N_d}, \frac{1}{k}\xi_1, \dots, \frac{1}{k}\xi_{N_d})$. We will point out where the change of the radius of the disc is used in the proof.

If $\Phi_1 = \Phi$ is of maximal rank, its Jacobian gives a nonzero section

$$J_\Phi(z, \xi) = \frac{\partial \Phi}{\partial z} \wedge \frac{\partial \Phi}{\partial \xi_1} \wedge \dots \wedge \frac{\partial \Phi}{\partial \xi_{N_d}}(z, \xi) \in \bigwedge^{1+N_d} \Phi^* T_{\mathbb{P}^n \times \mathbb{P}^{N_d}, \Phi(z, \xi)}$$

Let us assume that $J_\Phi(\mathbf{0})$ is nonzero in the corresponding vector space. Remark that, by construction, $J_{\Phi_k}(\mathbf{0}) = J_\Phi(\mathbf{0})$, for any $k \geq 1$, hence $J_{\Phi_k} \in \Phi_k^* \Lambda^{1+N_d} T_{\mathbb{P}^n \times \mathbb{P}^{N_d}}$ is not identically zero. Thanks to Proposition 11, we can choose $(n-1)$ logarithmic vector fields

$$V_1, \dots, V_{n-1} \in H^0(T_{\mathbb{P}^n \times \mathbb{P}^N}(-\log \mathcal{X})(1, 0))$$

such that the sections

$$\sigma_k := J_{\Phi_k} \wedge \Phi_k^*(V_1 \wedge \dots \wedge V_{n-1}) \in \Phi_k^*(\overline{K}_{\mathbb{P}^n \times \mathbb{P}^{N_d}}^{-1} \otimes p^* \mathcal{O}_{\mathbb{P}^n}(n-1))$$

are nonzero at the origin.

If q is the projection of $\mathbb{P}^n \times \mathbb{P}^{N_d}$ on the parameter space \mathbb{P}^{N_d} , then under the assumption $d \geq 2n+1$, the restriction of $\overline{K}_{\mathbb{P}^n \times \mathbb{P}^{N_d}} \otimes p^* \mathcal{O}_{\mathbb{P}^n}(1-n)$ to $q^{-1}(U)$ is ample (eventually after shrinking once again the open subset U), hence we can endow this bundle with a metric h of positive curvature.

For any $w \in \mathbb{B}(\delta_0 k)^{N_d+1}$ set

$$(4) \quad f_k(w) = \|\sigma_k(w)\|_{\Phi_k^* h^{-1}}^{2/(N_d+1)}.$$

Notice that, by construction, there exists a positive number c such that for each $k \geq 1$, we have

$$(5) \quad f_k(\mathbf{0}) = c > 0.$$

On the other hand we have

Proposition 17. *For each $k \geq 1$ we have $f_k(\mathbf{0}) \leq C \cdot k^{-2}$. In particular, as $k \rightarrow \infty$, we have $f_k(\mathbf{0}) \rightarrow 0$.*

Theorem 6 follows from the fact that (5) and Proposition 17 contradict each other.

We now give the proof of Proposition 17, which is very close to that of the classical Ahlfors-Schwarz lemma.

Proof of Proposition 17. First, notice that for each $k \geq 1$, there exists a positive constant C such that we have

$$(6) \quad \Delta \log f_k \geq C \cdot f_k$$

pointwise over the polydisc $\mathbb{B}(\delta_0 k)^{N_d+1}$. Indeed, by construction, the image of the map Φ_k lies inside $q^{-1}(U)$, for each $k \geq 1$, so that

$$i\partial\bar{\partial} \log \|\sigma_k\|_{\Phi_k^* h^{-1}}^2 \geq \Phi_k^* \Theta_h (\overline{K}_{\mathbb{P}^n \times \mathbb{P}^{N_d}} \otimes p^* \mathcal{O}_{\mathbb{P}^n}(1-n)).$$

Hence we get

$$\begin{aligned} \Delta \log \|\sigma_k\|_{\Phi_k^* h^{-1}}^2 &\geq C'' \cdot \left(\left\| \frac{\partial \Phi_k}{\partial z} \right\|_{\omega}^2 + \sum_{j=1}^{N_d} \left\| \frac{\partial \Phi_k}{\partial \xi_j} \right\|_{\omega}^2 \right) \\ &\geq C' \cdot \|J_{\Phi_k}\|_{\Lambda^{1+N_d} \omega}^{2/(1+N_d)} \\ &\geq C \cdot \|\sigma_k\|_{\Phi_k^* h^{-1}}^{2/(1+N_d)} \end{aligned}$$

and (6) is proved (the above relations are obtained using the vector inequalities

$$\|W_1 \wedge \cdots \wedge W_s\| \leq \|W_1\| \cdots \|W_s\| \leq s^{-s} (\|W_1\| + \cdots + \|W_s\|)^s).$$

Then, consider the volume form of the Poincaré metric on the polydisc

$$\psi_k = \frac{1}{\left(1 - \frac{|z|^2}{\delta_0^2 k^2}\right)^2} \prod_{j=1}^{N_d} \frac{1}{\left(1 - \frac{|\xi_j|^2}{\delta_0^2 k^2}\right)^2}$$

A computation shows that

$$(7) \quad \Delta \log \psi_k \leq C \cdot k^{-2} \psi_k.$$

(Remark that the previous inequality can be obtained precisely because, thanks to the reparameterization, we have the same radius $\delta_0 k$ for the components of the polydisc which is the domain of ψ_k .)

Consider the function $(z, \xi) \mapsto \frac{f_k(z, \xi)}{\psi_k(z, \xi)}$. Its maximum cannot be achieved at a boundary point of the domain, since ψ_k goes to infinity as (z, ξ) goes to the boundary. So at the maximum point (z_0, ξ_0) , we have

$$\Delta \log f_k / \psi_k \leq 0.$$

This inequality, combined with (6) and (7), gives

$$f_k(z_0, \xi_0) \leq C \cdot k^{-2} \psi_k(z_0, \xi_0)$$

Since the relation (5) is verified at the maximum point of the quotient, the same is true at an arbitrary point, thus, in particular, at the origin:

$$f_k(\underline{0}) \leq C \cdot k^{-2}.$$

□

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