

ON VOLUMES ALONG SUBVARIETIES OF LINE BUNDLES WITH NON-NEGATIVE KODAIRA-IITAKA DIMENSION

GIANLUCA PACIENZA AND SHIGEHARU TAKAYAMA

1. INTRODUCTION

We study the restricted volume along subvarieties of line bundles with non-negative Kodaira-Iitaka dimension. Our main interest is to compare it with a similar notion defined in terms of the asymptotic multiplier ideal sheaf, with which it coincides in the big case. We shall prove that the former is non-zero if and only if the latter is. We then study inequalities between them and prove that if they coincide on every very general curve the line bundle must have zero Kodaira-Iitaka dimension or be big.

Let X be a smooth projective variety, L a divisor or a line bundle on X with non-negative Kodaira-Iitaka dimension: $\kappa(L) \geq 0$. Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \text{SBs}(L)$, where $\text{SBs}(L) := \bigcap_{m>0} \text{Bs } |mL|$ is the stable base locus. We denote by $H^0(X|V, mL) = \text{Image}[H^0(X, mL) \rightarrow H^0(V, mL)]$ the image of restriction maps. The *restricted volume* of L along V is defined to be

$$\text{vol}_{X|V}(L) = \limsup_{m \rightarrow \infty} \frac{h^0(X|V, mL)}{m^d/d!}.$$

Similarly, we define the *reduced volume* of L along V as follows

$$\mu(V, L) = \limsup_{m \rightarrow \infty} \frac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^d/d!}.$$

Here $\mathcal{J}(\|mL\|) = \mathcal{J}(X, \|mL\|)$ is the asymptotic multiplier ideal sheaf of mL for every positive integer m ([L, 11.1.2]). When L is big, $\mu(V, L) = \text{vol}_{X|V}(L) > 0$ for any $V \not\subset \text{NAmp}(L)$ ([ELMNP3, 2.13] [T3, 3.1]), where $\text{NAmp}(L) := \bigcap_{m>0} \text{SBs}(mL - A)$ for any given ample divisor A on X , and is called the non-ample locus of L (in [L, 10.3.2], this is denoted by $\mathbf{B}_+(L)$ and called the augmented base locus). In the big case, the restricted volume has played an important rôle in the proof of the boundedness of pluricanonical maps (cf. [HM], [T3], [Ts2]) and the topic has been systematically studied by Ein, Lazarsfeld, Mustaa, Nakamaye and Popa in [L], [ELMNP1], [ELMNP2] and [ELMNP3]. On the other hand very little is known in the general case $\kappa(L) \geq 0$ and the present paper is an attempt to make the first basic steps in this direction, also with the hope that a better understanding of the restricted volume in the case $L = K_X$ could possibly lead to further progress in the study of pluricanonical maps for varieties with positive Kodaira dimension (for an attempt to adapt the arguments of [HM], [T3], [Ts2] to the case $\kappa(X) \geq 0$ see [P]; for results when $\kappa(X) \leq 2$, obtained with different techniques, see [VZ] and [Tod]).

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Our main concern is about the relationship between $\text{vol}_{X|V}(L)$ and $\mu(V, L)$, and the geometric meaning of their discrepancy for a line bundle L with $\kappa(L) \geq 0$. The basic relation $\mathfrak{b}(|mL|) \subset \mathcal{J}(\|mL\|)$ ([L, 11.1.8]), where $\mathfrak{b}(|mL|)$ is the base ideal of the linear system, leads to $\text{vol}_{X|V}(L) \leq \mu(V, L)$. By definition, $\text{vol}_{X|V}(L) > 0$ implies $\kappa(L) \geq \dim V$. However it is not clear at all that $\mu(V, L) > 0$ for $\dim V > 0$ implies $\kappa(L) > 0$. We have a natural product map $H^0(X|V, kL) \times H^0(X|V, mL) \longrightarrow H^0(X|V, (k+m)L)$, and also $\mathfrak{b}(|kL|) \cdot \mathfrak{b}(|mL|) \subset \mathfrak{b}(|(k+m)L|)$. However we do not know there exists a natural product map for $H^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)$ except when $V = X$, because we only have $\mathcal{J}(\|(k+m)L\|) \subset \mathcal{J}(\|kL\|) \cdot \mathcal{J}(\|mL\|)$ ([L, 11.2.4]). So we do not know about a natural ring structure on $\bigoplus_{m \geq 0} H^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)$. In spite of these difficulties, we think it is worth studying $H^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)$, as well as $H^0(X|V, mL)$, because $\mu(V, L)$ is a direct generalization of the usual intersection number. In fact, in case when L is semi-ample, $\mu(V, L) = L^d \cdot V$ for any $V \subset X$ (see Proposition 3.3, for a generalization to the case of a nef and abundant line bundle), whereas $\text{vol}_{X|V}(L) \neq L^d \cdot V$ in general (see [ELMNP3, 5.10]). We first describe their asymptotic behaviors.

Theorem 1.1. *Let X be a smooth projective variety, L a line bundle on X with $\kappa(L) \geq 0$, and $f : X \longrightarrow Y$ the Iitaka fibration associated to L . Let $V \subset X$ be a subvariety such that $V \not\subset \text{SBs}(L)$. Let $q = \dim f(V) \geq 0$.*

(1) *Assume that V contains a general point of X . Then*

$$0 < \limsup_{m \rightarrow \infty} \frac{h^0(X|V, mL)}{m^q} < +\infty.$$

(2) *Assume that V contains a very general point of X . Then*

$$0 < \limsup_{m \rightarrow \infty} \frac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^q} < +\infty.$$

The following is the main consequence in this paper.

Corollary 1.2. *Let X, L and $f : X \longrightarrow Y$ be as above. Let $V \subset X$ be a subvariety which contains a very general point of X .*

(1) *The following three conditions are equivalent: (o) the map $f|_V : V \dashrightarrow f(V)$ is generically finite, (i) $\text{vol}_{X|V}(L) > 0$, (ii) $\mu(V, L) > 0$.*

(2) *The condition $\limsup_{m \rightarrow \infty} h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)/m = 0$ implies the boundedness of $h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)$ as $m \rightarrow \infty$.*

Thus, the positivity of $\text{vol}_{X|V}(L)$ and $\mu(V, L)$ are equivalent to each other, and hence the weaker condition $\mu(V, L) > 0$ also implies $\kappa(L) \geq \dim V$. As for Corollary 1.2 (2) (which looks rather technical), it is the type of estimate appearing in the work of Nakayama [N, V.1.12], where it is used to prove the abundance conjecture in the case $\kappa = 0$.

We then try to describe their differences or the ratio $\mu(V, L)/\text{vol}_{X|V}(L)$ more precisely. In two extreme cases, it is known that they coincide. Let us recall the following.

Proposition 1.3. (1) [ELMNP3, 2.13] [T3, 3.1]. *Assume $\kappa(L) = \dim X$. Then $\mu(V, L) = \text{vol}_{X|V}(L) > 0$ for any $V \not\subset \text{NAmp}(L)$.*

(2) [T1, 1.2]. $\kappa(L) = 0$ if and only if $\mu(C, L) = 0$ (then in particular $\text{vol}_{X|C}(L) = 0$) for any curve $C \not\subset \text{SBs}(L)$.

We can read [T1, 1.2] as (2) above, because of $\mu(C, L) = \|L; C\|$, where $\|L; C\|$ is the one in [T1, 2.7] (see Proposition 2.5 in this paper). Moreover, by using the arguments in [T1, 3.1], we can show that $\kappa(L) = 0$ if and only if $\mu(V, L) = \text{vol}_{X|V}(L) = 0$ for any subvarieties $V \not\subset \text{SBs}(L)$. We can show that these are the only cases when the two invariants are equal as follows.

Theorem 1.4. *Let X be a smooth projective variety and L a line bundle on X with $\kappa(L) \geq 0$. Let $x \in X$ be a very general point. Assume $\mu(C, L) = \text{vol}_{X|C}(L)$ for any curve C passing through x . Then either $\kappa(L) = 0$ or $\kappa(L) = \dim X$.*

In case L is semi-ample, this is quite easy. Our proof consists in fact in trying to generalize the argument in this case. We show that an inequality $\mu(C, L) \geq \delta \text{vol}_{X|C}(L)$ holds for every curve $C \not\subset \text{SBs}(L)$ with the map $f|_C : C \dashrightarrow f(C)$ is finite of degree δ .

Our methods of study in this paper depend on a careful study of various multiplier ideal sheaves, and dimension counting arguments. As it is mentioned in [L, 11.1.10], we do not know whether the definition of the asymptotic multiplier ideal $\mathcal{J}(\|L\|)$ is in the final form or not. This paper does not give a definitive answer on this. However we hope some results in this paper will help to understand it.

Notations and conventions. Throughout this paper, we let X be a smooth projective variety, L a divisor or a line bundle on X with $\kappa(L) \geq 0$, and $f : X \rightarrow Y$ the Iitaka fibration associated to L ([I] or [L, 2.1.33]). The Iitaka fibration is only defined up to birational equivalence. If a subvariety $V \subset X$ with $V \not\subset \text{SBs}(L)$ is given, we take a birational morphism $\pi : X' \rightarrow X$ from a smooth projective variety X' with a projective morphism $f' : X' \rightarrow Y'$ to a smooth projective variety Y' , so that π is isomorphic over the generic point of V and that f' is birational to the Iitaka fibration $f : X \rightarrow Y$ associated to L . Then we understand as $\dim f(V)$ to be $\dim f'(V')$, and also $f|_V : V \dashrightarrow f(V)$ to be $f'|_{V'} : V' \rightarrow f'(V')$. A curve in a general fiber of $f|_V : V \dashrightarrow f(V)$ means a curve C whose strict transform $C' \subset V'$ is contained in a general fiber of $f'|_{V'}$. By a *general* (resp. *very general*) *point* on X , we mean a point which belongs to the complement of a (resp. a countable union of) proper Zariski closed subset, which is determined by the divisor L .

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2. VOLUMES ALONG SUBVARIETIES

We shall study volumes along subvarieties and prove Theorem 1.1.

2.1. Intermediate restricted volumes. We shall prove Theorem 1.1 (1). Let $\dim f(V) = q$. We note that the space $H^0(X|V, mL)$ is unchanged under a birational morphism $\pi : X' \rightarrow X$ from a smooth projective variety X' , which is isomorphic over the generic

point of V (cf. the proof of [ELMNP3, 2.4]). We may assume, by taking an embedded resolution of V , that V is smooth and that there exists a projective morphism $f : X \rightarrow Y$ to a smooth projective variety Y , so that f is the Iitaka fibration associated to L .

Positivity: $\limsup h^0(X|V, mL)/m^q > 0$. Let A_Y be a very ample divisor on Y . We see $0 < \limsup_\ell h^0(f^*A_Y, \ell L)/\ell^{\kappa(L)-1} < +\infty$. By the same argument of Kodaira's lemma, we have $H^0(X, \ell L - f^*A_Y) \neq 0$ for some large ℓ . We take one such ℓ . Then $\ell L = f^*A_Y + E$ for some effective divisor E on X , and $H^0(Y, mA_Y) \cong H^0(X, mf^*A_Y) \subset H^0(X, mL)$ for any $m > 0$. If V contains a general point, we can assume $V \not\subset E$. (If L is big, this is equivalent to saying that $V \not\subset \text{NAmp}(L)$.) Since A_Y is ample, the restriction map $H^0(Y, mA_Y) \rightarrow H^0(f(V), mA_Y)$ is surjective for every large m . Then we have an inclusion $H^0(X|V, mL) \supset (f|_V)^*H^0(f(V), mA_Y)$ for every large m . Hence there exists a constant $c > 0$ such that $h^0(X|V, mL) \geq cm^q$ for every large m .

Finiteness: $\limsup h^0(X|V, mL)/m^q < +\infty$. In case $q = d = \dim V$, it is well-known. We may assume $q < d$. Assume on the contrary that $\limsup h^0(X|V, mL)/m^q = +\infty$. We take a sufficiently general complete intersection $W \subset V$ of $\dim W = q$ and $f(W) = f(V)$. By the same argument of Kodaira's lemma, the restriction map $H^0(X|V, mL) \subset H^0(V, mL) \rightarrow H^0(W, mL)$ has a non-trivial kernel for large m . This means that there exists a non-zero $s \in H^0(X, mL)$ such that $s|_V$ is not zero and vanishes along W . We may take m so large that the map $\Phi_{|mL|} : X \dashrightarrow \Phi_{|mL|}(X)$ is birational to the Iitaka fibration $f : X \rightarrow Y$. Since $f|_V$ is not generically finite, $(\text{div } s)|_V$ where $s \in H^0(X, mL)$, has to be in the direction of the ruling $f|_V : V \rightarrow f(V)$ plus some another fixed divisor $F_m|_V$, which independent of $s \in H^0(X, mL)$. On the other hand $W \subset V$ can be in arbitrary direction and $f(W) = f(V)$. The vanishing of $s|_V$ along W imposes the vanishing of $s|_V$ on V . This is a contradiction. \square

Remark 2.1. Theorem 1.1 (1) can be read as follows. Let $X, L, f : X \rightarrow Y$ and $V \subset X$ be as in Theorem 1.1 (1). Let p be an integer with $0 \leq p \leq d$. Then the following two conditions are equivalent: (0) $\dim f(V) = p$, (1) $0 < \limsup_m h^0(X|V, mL)/m^p < +\infty$.

2.2. Reduced volumes. Here we collect basic properties of reduced volumes.

Lemma 2.2. *Homogeneity* ([ELMNP3, 3.3]). *Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \text{SBs}(L)$. Then $\mu(V, pL) = p^d \mu(V, L)$ for every positive integer p .*

The previous lemma allows to define the quantity $\mu(V, L)$ for \mathbb{Q} -divisors.

Lemma 2.3. *Projection formula.* *Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \text{SBs}(L)$. Let $\pi : X' \rightarrow X$ be a birational morphism from a smooth projective variety X' . Let $V' \subset X'$ be a subvariety of $\dim V' = d$ with $f(V') = V$ and $V' \not\subset \text{Exc}(\pi)$ the exceptional locus of π . Then $\mu(V, L) = \mu(V', \pi^*L)$.*

Proof. Let $e = e(L) \geq 1$ be the exponent of L , which is the smallest positive integer so that $h^0(X, meL) \neq 0$ for all integer $m > 0$ ([L, 2.1.1]). We denote by $L' = \pi^*L$. We see $e(L) = e(L')$. We take a sufficiently large p such that $\mathcal{J}(\|mL\|) = \mathcal{J}(\frac{m}{ep} \cdot |epL|)$, and

$\mathcal{J}(\|mL'\|) = \mathcal{J}(\frac{m}{ep} \cdot |epL'|)$ ([L, 11.1.5]). We note basic relations [L, 9.5.8] and [L, 9.2.33]:

$$\begin{aligned} \mathcal{J}(\frac{m}{ep} \cdot |epL'|) &\subset \pi^{-1} \mathcal{J}(\frac{m}{ep} \cdot |epL|) \cdot \mathcal{O}_{X'}, \\ \mathcal{J}(\frac{m}{ep} \cdot |epL|) &= \pi_* (\mathcal{J}(\frac{m}{ep} \cdot |epL'|) \otimes K_{X'/X}). \end{aligned}$$

Here $K_{X'/X}$ is the relative canonical bundle of π . Since $\mathcal{J}(\frac{m}{ep} \cdot |epL'|) \otimes K_{X'/X}$ is torsion free, the natural homomorphism

$$\pi^* (\pi_* (\mathcal{J}(\frac{m}{ep} \cdot |epL'|) \otimes K_{X'/X})) \longrightarrow \mathcal{J}(\frac{m}{ep} \cdot |epL'|) \otimes K_{X'/X}$$

induces a homomorphism

$$\pi^{-1} (\pi_* (\mathcal{J}(\frac{m}{ep} \cdot |epL'|) \otimes K_{X'/X})) \cdot \mathcal{O}_{X'} \longrightarrow \mathcal{J}(\frac{m}{ep} \cdot |epL'|) \otimes K_{X'/X},$$

which is generically an isomorphism, since π is birational. Moreover, since $\pi^{-1} (\pi_* (\mathcal{J}(\frac{m}{ep} \cdot |epL'|) \otimes K_{X'/X})) \cdot \mathcal{O}_{X'}$ is torsion free, the last homomorphism is injective. Putting everything together, we have

$$\mathcal{J}(\|mL'\|) \subset \pi^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{O}_{X'} \longrightarrow \mathcal{J}(\|mL'\|) \otimes K_{X'/X},$$

with injective last homomorphism. Since $K_{X'/X}$ is independent of m , it is not difficult to see that

$$\mu(V', L') = \limsup \frac{h^0(V', \mathcal{O}_{V'}(mL') \otimes \mathcal{J}(\|mL'\|)|_{V'} \otimes K_{X'/X}|_{V'})}{m^d/d!}.$$

Hence we obtain

$$\mu(V', L') = \limsup \frac{h^0(V', \mathcal{O}_{V'}(mL') \otimes \pi^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'})}{m^d/d!}.$$

The right hand side is in fact $\mu(V, L)$ by Lemma 2.4 below and we are done. \square

Lemma 2.4. *Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \text{SBs}(L)$, and let $\nu : V' \longrightarrow V (\subset X)$ be a birational morphism from a proper variety V' . Then, as $m \rightarrow \infty$, one has*

$$\begin{aligned} (1) \quad \overline{\lim} \frac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^d/d!} &= \overline{\lim} \frac{h^0(V', \mathcal{O}_{V'}(m\nu^*L) \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'})}{m^d/d!}, \\ (2) \quad \underline{\lim} \frac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^d/d!} &= \underline{\lim} \frac{h^0(V', \mathcal{O}_{V'}(m\nu^*L) \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'})}{m^d/d!}. \end{aligned}$$

Proof. (1) We denote by $L' = \nu^*L$. Let $\mathcal{I} \subset \mathcal{O}_V$ be the annihilator of $\nu_* \mathcal{O}_{V'}/\mathcal{O}_V$, and let $\mathcal{I}' = \nu^{-1} \mathcal{I} \cdot \mathcal{O}_{V'} \subset \mathcal{O}_{V'}$. Then we have

$$H^0(V', \mathcal{O}_{V'}(mL') \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{I}') \subset \nu^* H^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)$$

as subspaces of $H^0(V', \mathcal{O}_{V'}(mL'))$, in particular

$$h^0(V', \mathcal{O}_{V'}(mL') \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{I}') \leq h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V).$$

Since $\dim(\text{Supp } \mathcal{O}_{V'}/\mathcal{I}') < d$, by an exact sequence argument, we have

$$\overline{\lim} \frac{h^0(V', \mathcal{O}_{V'}(mL') \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'})}{m^d} = \overline{\lim} \frac{h^0(V', \mathcal{O}_{V'}(mL') \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{I}')}{m^d}.$$

Hence we obtain

$$\overline{\lim} \frac{h^0(V', \mathcal{O}_{V'}(mL') \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'})}{m^d} \leq \overline{\lim} \frac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^d}.$$

The converse of this inequality follows from an elementary fact:

$$h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V) \leq h^0(V', \mathcal{O}_{V'}(mL') \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{O}_{V'}).$$

Thus we obtain our equality.

(2) is obtained by replacing “ $\overline{\lim}$ ” by “ $\underline{\lim}$ ” in the proof of (1) above. \square

In the case the subvariety V is a curve we have a more explicit description of the reduced volume, which will be used in the proof of Theorem 1.1 (2). Let us first recall the definition of $\|L; C\|$ ([T1, 2.7]). Let $\mathcal{J} \subset \mathcal{O}_C$ be an ideal sheaf. For the normalization $\nu : C' \rightarrow C$, we define $\deg_C \mathcal{J}$ as the degree of the invertible sheaf $\nu^{-1} \mathcal{J} \cdot \mathcal{O}_{C'}$. Then, $mL \cdot C + \deg_C \mathcal{J}(\|mL\|)|_C \geq 0$ for any $m > 0$ ([T1, 2.6 (1)]), and we can define

$$\|L; C\| = L \cdot C + \limsup_{m \rightarrow \infty} \frac{1}{m} \deg_C \mathcal{J}(\|mL\|)|_C.$$

Proposition 2.5. *Let $C \subset X$ be a curve with $C \not\subset \text{SBs}(L)$. Then $\mu(C, L) = \|L; C\|$ holds.*

Proof. The proof will be done in the same way as in [T2, 3.1]. Let $\nu : C' \rightarrow C \subset X$ be the normalization. We consider a family of invertible sheaves $\{\mathcal{G}_m\}_{m \in \mathbb{N}}$ on C' , where $\mathcal{G}_m = \mathcal{O}_{C'}(m\nu^*L) \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{O}_{C'}$ with degree $d_m := \deg \mathcal{G}_m \geq 0$ ([T1, 2.6 (1)]). By the subadditivity [DEL, [L, 11.2.4]: $\mathcal{J}(\|(\ell + m)L\|) \subset \mathcal{J}(\|\ell L\|) \cdot \mathcal{J}(\|mL\|)$, it follows $d_{\ell+m} \leq d_\ell + d_m$. Then by [T2, 3.4], their limits $\lim_{m \rightarrow \infty} h^0(C', \mathcal{G}_m)/m$ and $\lim_{m \rightarrow \infty} d_m/m$ exist and they coincide: $\lim_{m \rightarrow \infty} h^0(C', \mathcal{G}_m)/m = \lim_{m \rightarrow \infty} d_m/m$. By definition $\deg_C (\mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C) = \deg_{C'} (\mathcal{O}_{C'}(m\nu^*L) \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{O}_{C'})$. Hence

$$\lim_{m \rightarrow \infty} \frac{h^0(C', \mathcal{O}_{C'}(m\nu^*L) \otimes \nu^{-1} \mathcal{J}(\|mL\|) \cdot \mathcal{O}_{C'})}{m} = \lim_{m \rightarrow \infty} \frac{\deg_C (\mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C)}{m}.$$

Then by Lemma 2.4, we obtain our assertion. \square

2.3. Intermediate reduced volumes. We shall prove Theorem 1.1 (2). We need a refinement of [T1, 3.1].

Lemma 2.6. *Assume $\kappa(L) = 0$. Let $C \subset X$ be a curve with $C \not\subset \text{SBs}(L)$. Then $h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C)$ and $\deg_C (\mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C)$ are bounded as $m \rightarrow \infty$.*

Proof. Let $e = e(L) \geq 1$ be the exponent of L . Since $\kappa(L) = 0$, there exists a non-zero effective divisor $D \in |eL|$ such that $|meL|$ is generated by mD for any m . In general we have $\mathcal{J}(\|mL\|) = \mathcal{J}(\frac{1}{pe} |pemL|)$ for sufficiently large p ([L, 11.1.5]), and $\mathcal{J}(\frac{1}{pe} |pemL|) = \mathcal{J}(\frac{1}{pe} |pmD|) = \mathcal{J}(\frac{m}{e} D)$ ([L, 9.2.26]).

We have at least $\mathcal{J}(\|mL\|) = \mathcal{J}(\frac{m}{e} D) \subset \mathcal{J}(\lfloor m/e \rfloor D) = \mathcal{O}_X(-\lfloor m/e \rfloor D)$. Here $\lfloor a \rfloor$ stands for the integral part of a non-negative number a . Then it is enough to bound $h^0(C, \mathcal{O}_C(mL - \lfloor m/e \rfloor D))$ and $\deg_C \mathcal{O}_C(mL - \lfloor m/e \rfloor D)$. Let $\nu : C' \rightarrow C$ be the normalization. We have $h^0(C, \mathcal{O}_C(mL - \lfloor m/e \rfloor D)) \leq h^0(C', \mathcal{O}_{C'}(m\nu^*L - \lfloor m/e \rfloor \nu^*D))$, and

$\deg_C \mathcal{O}_{C'}(m\nu^*L - \lfloor m/e \rfloor \nu^*D) = (m - e\lfloor m/e \rfloor)L \cdot C$. Since $0 \leq m - e\lfloor m/e \rfloor < e$, the invertible sheaves $\mathcal{O}_{C'}(m\nu^*L - \lfloor m/e \rfloor \nu^*D)$ have non-negative bounded degrees as $m \rightarrow \infty$. The following sublemma implies our assertion. \square

Sublemma 2.7. *Let C be a smooth projective curve, and $\{\mathcal{G}_m\}_m$ be a family of invertible sheaves on C with $\deg \mathcal{G}_m \geq 0$. Then, as $m \rightarrow \infty$, $\deg \mathcal{G}_m$ is bounded if and only if $h^0(C, \mathcal{G}_m)$ is bounded.*

Proof. We denote by g the genus of C , $\chi(\mathcal{O}_C) = 1 - g$, and by $d_m = \deg \mathcal{G}_m \geq 0$.

If d_m is unbounded, for every $k > 0$, we have m_k such that $d_{m_k} > 2g - 2 + k$. Then, by Riemann-Roch and vanishing, $h^0(C, \mathcal{G}_{m_k}) = \deg \mathcal{G}_{m_k} + \chi(\mathcal{O}_C)$, and which is unbounded.

Assume $d_m < b$ for any m . We claim that $h^0(C, \mathcal{G}_m) \leq \max\{b, 2g - 1\} + \chi(\mathcal{O}_C)$. Take m . If $d_m > 2g - 2$, Riemann-Roch and vanishing imply $h^0(C, \mathcal{G}_m) = \deg \mathcal{G}_m + \chi(\mathcal{O}_C) \leq b + \chi(\mathcal{O}_C)$. If $d_m \leq 2g - 2$, we take an effective divisor D_m on C with $\deg D_m + d_m = 2g - 1$. Then $h^0(C, \mathcal{G}_m) \leq h^0(C, \mathcal{G}_m \otimes \mathcal{O}(D_m)) = 2g - 1 + \chi(\mathcal{O}_C)$. \square

For the finiteness we will need the following.

Proposition 2.8. *Let $x \in X$ be a very general point, and let $C \subset X$ be a curve passing through x . If $\mu(C, L) = 0$, then $h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C)$ and $\deg_C(\mathcal{O}_C(mL) \otimes \mathcal{J}(\|mL\|)|_C)$ are bounded as $m \rightarrow \infty$.*

Proof. By virtue of Lemma 2.3 and 2.4, possibly after taking a modification of X , we may assume that C is smooth. We may moreover assume that there exists a projective morphism $f : X \rightarrow Y$ to a smooth projective variety Y , which is birational to the Iitaka fibration associated to L . Let Y_0 be a countable union of subvarieties of Y such that X_y is smooth and $\kappa(X_y, L_y) = 0$ for any $y \in Y \setminus Y_0$, where $X_y = f^{-1}(y)$ and $L_y = L|_{X_y}$. Let $e_y = e(L_y)$ be the exponent of L_y for $y \in Y \setminus Y_0$. As in the proof of Lemma 2.6, for every $y \in Y \setminus Y_0$, we have $B_y \in |e_y L_y|$ such that $\mathcal{J}(X_y, \|mL_y\|) = \mathcal{J}(X_y, \frac{m}{e_y} B_y)$ for any m .

We fix m . Let $e = e(L)$ be the exponent of L . We take a sufficiently large integer $p = p_m$ such that $\mathcal{J}(X, \|mL\|) = \mathcal{J}(X, \frac{1}{pe} |pemL|)$, and then take a general member $D = D_m \in |pemL|$ such that $\mathcal{J}(X, \frac{1}{pe} |pemL|) = \mathcal{J}(X, \frac{1}{pe} D)$. By the generic restriction theorem [L, 9.5.35], there exists a subvariety $Y_m \subset Y$ such that $\mathcal{J}(X, \|mL\|)|_{X_y} = \mathcal{J}(X_y, \frac{1}{pe} D|_{X_y})$ for any $y \in Y \setminus Y_m$.

We then take a very general point $y \in Y \setminus \bigcup_{m \geq 0} Y_m$. We can write $e = e_y q_y$ for a positive integer q_y . For every m , $|p_m e m L_y| = |p_m m q_y e_y L_y| = \langle p_m m q_y B_y \rangle$. Hence $D_m|_{X_y} = p_m m q_y B_y$. Thus $\mathcal{J}(X_y, \frac{1}{p_m e} D_m|_{X_y}) = \mathcal{J}(X_y, \frac{m}{e_y} B_y)$. Finally we have $\mathcal{J}(X, \|mL\|)|_{X_y} = \mathcal{J}(X_y, \|mL_y\|)$ for every $m > 0$.

By [T1, 1.3], $\mu(C, L) = 0$ imposes that C is contained in a fiber $X_y = f^{-1}(y)$, where $y \in Y$ is also very general. In particular $\mathcal{O}_C(mL) \otimes \mathcal{J}(X, \|mL\|)|_C = \mathcal{O}_C(mL_y) \otimes \mathcal{J}(X_y, \|mL_y\|)|_C$. Since we know the boundedness properties for L_y by Lemma 2.6, we have our assertion. \square

Proof of Theorem 1.1 (2). The positivity: $\limsup_m h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)/m^q > 0$ follows from $h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V) \geq h^0(X|V, mL)$ and Theorem 1.1 (1). We shall

prove the finiteness: $\limsup_m h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)/m^q < +\infty$. In case $q = d$, it is well-known ([I], or [L, 2.1.38]). Hence we assume $q < d$. The proof will be done by induction on $d - q \geq 0$. The first step $d - q = 0$ is already over.

We assume Theorem 1.1 (2) is true for any subvariety $W \subset X$ containing a very general point of X with $\dim W \leq d - 1$ and $\dim f(W) = q (\leq d - 1)$. Let $V \subset X$ be a subvariety containing a very general point of X with $\dim V = d$ and $\dim f(V) = q$. Let A be a very ample Cartier divisor on V . Let k be a positive integer. We take a general member $W_k \in |kA|$ so that $f(W_k) = f(V)$, W_k is smooth where V is, and $\dim \text{Sing } W_k = \dim \text{Sing } V - 1 \leq d - 2$. For every $m > 0$, we consider a restriction map $r_m : H^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V) \longrightarrow H^0(W_k, \mathcal{O}_{W_k}(mL) \otimes \mathcal{J}(\|mL\|)|_{W_k})$. By the induction hypothesis, we know $0 < \limsup_m h^0(W_k, \mathcal{O}_{W_k}(mL) \otimes \mathcal{J}(\|mL\|)|_{W_k})/m^q < +\infty$. If $\limsup_m h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)/m^q = +\infty$, there exists a positive integer m_k such that the map r_{m_k} has a non-trivial kernel. We take such m_k and a non-zero $s_k \in \ker r_{m_k}$. We do the same process for every k .

We can find a curve C in a very general fiber of $f|_V : V \dashrightarrow f(V)$ such that $C \not\subset W_k$, C intersects W_k where V (and hence W_k) is smooth, and $s_k|_C \neq 0$ for all $k > 0$. Since $\dim f(C) = 0$, we have $\|C; L\| = 0$ by [T1, 1.3]. From Proposition 2.5 we deduce that $\mu(C, L) = 0$. Let $\nu : C' \longrightarrow C (\subset V)$ be the normalization. Then $\nu^*(s_k|_C)$ defines a non-zero element of $H^0(C', \mathcal{O}_{C'}(m_k \nu^* L - k \nu^* A) \otimes \nu^{-1} \mathcal{J}(\|m_k L\|) \cdot \mathcal{O}_{C'})$. In particular, $\deg_C(\mathcal{O}_C(m_k L) \otimes \mathcal{J}(\|m_k L\|)|_C) \equiv \deg_{C'} \mathcal{O}_{C'}(m_k \nu^* L) \otimes \nu^{-1} \mathcal{J}(\|m_k L\|) \cdot \mathcal{O}_{C'} \geq kA \cdot C$, hence it is unbounded. It may be $\limsup_k m_k/k = 0$, however this contradicts Proposition 2.8. Thus $\limsup_m h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)/m^q < +\infty$. \square

Proof of Corollary 1.2. (1) Assume (o) (resp. (i), resp. (ii)). Then $q = \dim f(V)$ in Theorem 1.1 has to be $q = d = \dim V$. Then by Theorem 1.1, we have (i) and (ii) (resp. (o) and (ii), resp. (o) and (i)).

(2) Assume $\limsup_m h^0(V, mL \otimes \mathcal{J}(\|mL\|))/m = 0$. Then by Theorem 1.1 (2), we have $q = \dim f(V) = 0$, and then Theorem 1.1 (2) with $q = 0$ implies the boundedness of $h^0(V, mL \otimes \mathcal{J}(\|mL\|))$. \square

3. RELATION AMONG VARIOUS VOLUMES ALONG SUBVARIETIES

3.1. Proof of Theorem 1.4. To prove Theorem 1.4, we introduce another more geometric notion.

Notation 3.1. Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \text{SBs}(L)$. Let m be a sufficiently large integer such that $\text{Bs } |mL| = \text{SBs}(L)$, and that the rational map $\Phi_{|mL|} : X \dashrightarrow \mathbb{P} = \mathbb{P}^{N_m}$ with $N_m = \dim |mL|$ is birational to the Iitaka fibration $f : X \longrightarrow Y$ associated to L . Let $\pi_m : X_m \longrightarrow X$ be a birational morphism from a smooth projective variety X_m such that $\pi_m^* |mL| = |M_m| + F_m$, where $|M_m|$ is base point free (the moving part) and F_m is the fixed part. Denote by $\psi_m = \Phi_{|M_m|} : X_m \longrightarrow \mathbb{P}$ the induced morphism, and by $\mathcal{O}(1)$ the hyperplane bundle on \mathbb{P} such that $\mathcal{O}_{X_m}(M_m) = \psi_m^* \mathcal{O}(1)$. We

can take π_m so that it is an isomorphism over the generic point of V . We then denote by $V_m \subset X_m$ the strict transform of V . \square

Definition 3.2. [ELMNP3, 2.6, 2.7]. Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \text{SBs}(L)$. In the notation above, we define

$$\|L^d \cdot V\| := \limsup_{m \rightarrow \infty} \frac{M_m^d \cdot V_m}{m^d} = \limsup_{m \rightarrow \infty} \frac{\#(V \cap D_{m,1} \cap \dots \cap D_{m,d} \setminus \text{SBs}(L))}{m^d}.$$

Here $D_{m,1}, \dots, D_{m,d} \in |mL|$ are general members. This number $\|L^d \cdot V\|$ is called the *asymptotic intersection number* of L and V ([ELMNP3, 2.6]), or the *asymptotic moving intersection number* for the right hand side ([ELMNP3, 2.7]). \square

This $\|L \cdot C\|$ for curves is different from $\|L; C\|$ in [T1, 2.7] in general. In case L is big, $\|L^d \cdot V\| = \mu(V, L) = \text{vol}_{X|V}(L)$ holds for any subvariety $V \subset X$ of $\dim V = d > 0$ with $V \not\subset \text{NAmP}(L)$ ([ELMNP3, 2.13] [T3, 3.1]). In another ideal case, these quantities relate to each other as follows. See [L, 2.3.17] for nef and abundant divisors.

Proposition 3.3. *Assume L is nef and abundant. Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that V contains a general point of X (in particular $V \not\subset \text{SBs}(L)$), and the map $f|_V : V \dashrightarrow f(V)$ is generically finite of degree δ . Then $\mu(V, L) = \delta \text{vol}_{X|V}(L) = \|L^d \cdot V\|$.*

Proof. By Kawamata [K, 2.1], there exists a birational morphism $\pi : X' \rightarrow X$ from a smooth projective variety X' , and a surjective morphism $f : X' \rightarrow Y$ with connected fibers to a smooth projective variety Y with a nef and big divisor L_Y on Y , such that $\pi^*L \sim_{\mathbb{Q}} f^*L_Y$, where $\sim_{\mathbb{Q}}$ stands for the \mathbb{Q} -linear equivalence. Since V contains a general point, we may assume that π is an isomorphism over the generic point of V . Hence, using this fact and the homogeneity of $\mu(V, L)$ we may assume from the beginning that $X' = X$ and also $L = f^*L_Y$ for a nef and big divisor L_Y .

(1) We claim that $\text{vol}_{X|V}(L) = \text{vol}_{Y|f(V)}(L_Y)$. This follows from the fact that there are natural isomorphisms $H^0(Y, mL_Y) \cong H^0(X, mL)$ by pull-back for all m , and hence $H^0(Y|f(V), mL_Y) \cong H^0(X|V, mL)$ for all m .

(2) We claim that $\|L^d \cdot V\| = \delta \|L_Y^d \cdot f(V)\|$. This follows from the alternative definition (Definition 3.2) of $\|L^d \cdot V\|$. By taking general members $D_{m,1}, \dots, D_{m,d} \in |mL_Y| (\cong |mL|)$ for every large m , we have

$$\begin{aligned} \delta \|L_Y^d \cdot f(V)\| &= \delta \limsup_m \frac{\#(f(V) \cap D_{m,1} \cap \dots \cap D_{m,d} \setminus \text{SBs}(L_Y))}{m^d} \\ &= \limsup_m \frac{\#(V \cap f^*D_{m,1} \cap \dots \cap f^*D_{m,d} \setminus \text{SBs}(L))}{m^d} = \|L^d \cdot V\|. \end{aligned}$$

(3) We claim that $\mu(V, L) = \|L^d \cdot V\|$. By Wilson [W, 2.2], since L_Y is nef and big, there exists an effective divisor D on Y such that the linear system $|mL_Y - D|$ is base point free for all sufficiently large m . Hence so is $|mL - f^*D|$ for all sufficiently large m . Then it is not difficult to see that $\mathcal{J}(X, |mL|) = \mathcal{O}_X$ for all m . Then $\mu(V, L) = \limsup_m h^0(V, \mathcal{O}_V(mL)) / (m^d/d!) = \|L^d \cdot V\| = \delta \|L_Y^d \cdot f(V)\|$. We can find this type of argument in [MR, §2].

(4) For the nef and big L_Y , we know that $L_Y^d \cdot f(V) = \text{vol}_{Y|f(V)}(L_Y) = \|L_Y^d \cdot f(V)\|$ (cf. [ELMNP3, 2.13]). Then $\delta L_Y^d \cdot f(V) = \delta \text{vol}_{Y|f(V)}(L_Y) = \delta \|L_Y^d \cdot f(V)\|$. By (1) and (2), we have $L^d \cdot V = \delta \text{vol}_{X|V}(L) = \|L^d \cdot V\|$. Then we have our assertion by (3). \square

Remark 3.4. In a similar situation as above, if $L = f^*L_Y$ for a big (not necessarily nef) divisor L_Y on Y , we have $\mu(V, L) \geq \delta \text{vol}_{X|V}(L) = \|L^d \cdot V\|$. In fact, for L_Y big, it is known that $\text{vol}_{Y|f(V)}(L_Y) = \|L_Y^d \cdot f(V)\|$ ([ELMNP3, 2.13]). The claims (1) and (2) in the proof above still hold, because we do not use L to be nef. Then we have $\delta \text{vol}_{X|V}(L) = \|L^d \cdot V\|$. Since $\mu(V, L) \geq \|L^d \cdot V\|$ in general by Lemma 3.5 below, we have our assertion. \square

We shall study relationships among three notions of volumes along subvarieties in case the divisor is neither big, nor nef and abundant (namely in very bad situations), and prove Theorem 1.4 as a consequence.

Lemma 3.5. *Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \text{SBs}(L)$. Then $\mu(V, L) \geq \|L^d \cdot V\|$.*

Proof. We use Notation 3.1. We take a sufficiently large m . We denote by $\nu_m : V_m \rightarrow V \subset X$ the induced morphism, and by $L' = \pi_m^*L$. Let k be a positive integer. We have $\mathcal{O}_{X_m}(-kF_m) = \mathfrak{b}(|mL'|)^k \subset \mathfrak{b}(|kmL'|)$, where $\mathfrak{b}(|\bullet|)$ stands for the base ideal of a linear system, $\mathfrak{b}(|kmL'|) \subset \mathcal{J}(\|kmL'\|)$ ([L, 11.1.8(iv)]), and $\mathcal{J}(\|kmL'\|) \subset \pi_m^{-1}\mathcal{J}(\|kmL\|) \cdot \mathcal{O}_{X_m}$ ([L, 9.5.8]). Therefore $\mathcal{O}_{X_m}(kM_m) = \mathcal{O}_{X_m}(kmL' - kF_m) \subset \mathcal{O}_{X_m}(kmL') \otimes \pi_m^{-1}\mathcal{J}(\|kmL\|) \cdot \mathcal{O}_{X_m}$, and then $\mathcal{O}_{V_m}(kM_m) \subset \mathcal{O}_{V_m}(kmL') \otimes (\pi_m^{-1}\mathcal{J}(\|kmL\|) \cdot \mathcal{O}_{X_m})|_{V_m} = \mathcal{O}_{V_m}(kmL') \otimes \nu_m^{-1}\mathcal{J}(\|kmL\|) \cdot \mathcal{O}_{V_m}$.

Now we have

$$\begin{aligned} M_m^d \cdot V_m &= \limsup_{k \rightarrow \infty} \frac{h^0(V_m, \mathcal{O}_{V_m}(kM_m))}{k^d/d!} \\ &\leq \limsup_{k \rightarrow \infty} \frac{h^0(V_m, \mathcal{O}_{V_m}(kmL') \otimes \nu_m^{-1}\mathcal{J}(\|kmL\|) \cdot \mathcal{O}_{V_m})}{k^d/d!}. \end{aligned}$$

By Lemma 2.4, we know that the last term is $\mu(V, mL)$. Thus we have $M_m^d \cdot V_m/m^d \leq \mu(V, L)$, and letting $m \rightarrow \infty$ we have $\|L^d \cdot V\| \leq \mu(V, L)$. \square

Lemma 3.6. *Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \text{SBs}(L)$ and the map $f|_V : V \dashrightarrow f(V)$ is generically finite of degree δ . Then $\delta \text{vol}_{X|V}(L) \geq \|L^d \cdot V\|$.*

Proof. We use Notation 3.1. We may assume, by taking m to be large enough, that $\psi_m|_{V_m} : V_m \rightarrow \psi_m(V_m) \subset \mathbb{P}$ is generically finite of degree δ . We take one such m . We see

$$\text{vol}_{X|V}(L) = \text{vol}_{X_m|V_m}(m\pi_m^*L)/m^d \geq \text{vol}_{X_m|V_m}(M_m)/m^d.$$

We know by [ELMNP3, 2.5] that

$$\text{vol}(V_m, M_m|_{V_m}) = \delta \text{vol}(\psi_m(V_m), \mathcal{O}(1)|_{V_m}).$$

Since $\mathcal{O}(1)$ is ample, for every sufficiently large k , we have $H^0(\psi_m(X_m), \mathcal{O}(k)) = H^0(\mathbb{P}|\psi_m(X_m), \mathcal{O}(k))$ and $H^0(\psi_m(V_m), \mathcal{O}(k)) = H^0(\mathbb{P}|\psi_m(V_m), \mathcal{O}(k))$, and hence

$H^0(\psi_m(V_m), \mathcal{O}(k)) = H^0(\psi_m(X_m)|_{\psi_m(V_m)}, \mathcal{O}(k))$. By pull-back, $H^0(\psi_m(X_m), \mathcal{O}(k)) \subset H^0(X_m, \mathcal{O}(kM_m))$. Thus $h^0(\psi_m(V_m), \mathcal{O}(k)) \leq h^0(X_m|_{V_m}, \mathcal{O}(kM_m))$. Then

$$\begin{aligned} \delta \operatorname{vol}_{X_m|_{V_m}}(M_m) &= \delta \limsup_k h^0(X_m|_{V_m}, \mathcal{O}(kM_m))/(k^d/d!) \\ &\geq \delta \limsup_k h^0(\psi_m(V_m), \mathcal{O}(k)|_{\psi_m(V_m)})/(k^d/d!) \\ &= \delta \operatorname{vol}(\psi_m(V_m), \mathcal{O}(1)|_{V_m}) \\ &= \operatorname{vol}(V_m, M_m|_{V_m}) = M_m^d \cdot V_m. \end{aligned}$$

Hence $\delta \operatorname{vol}_{X|V}(L) \geq M_m^d \cdot V_m/m^d$. \square

Remark 3.7. We have an additional remark in the proof above. Denote by $\psi_m : X_m \rightarrow Y_m := \psi_m(X_m) \subset \mathbb{P}$. Let $\nu_m : Y'_m \rightarrow Y_m$ be the normalization, and $\psi'_m : X_m \rightarrow Y'_m$ be the induced morphism (with connected fibers!). Then $H^0(X_m, \mathcal{O}(kM_m)) = H^0(X_m, \psi'^*_m \nu'^*_m \mathcal{O}(k)) = H^0(Y'_m, \nu'^*_m \mathcal{O}(k)) \supset H^0(Y_m, \mathcal{O}(k))$. We see $\operatorname{vol}(Y'_m, \nu'^*_m \mathcal{O}(1)) = \operatorname{vol}(Y_m, \mathcal{O}(1))$, since $\mathcal{O}(1)$ is ample. Hence

$$\limsup_k \frac{h^0(X_m, \mathcal{O}(kM_m))}{(k^d/d!)} = \operatorname{vol}(Y_m, \mathcal{O}(1))$$

with $d = \kappa(L)$. \square

For curves, we can show the converse of Lemma 3.6.

Lemma 3.8. *Let $C \subset X$ be a curve such that $C \not\subset \text{SBs}(L)$ and the map $f|_C : C \dashrightarrow f(C)$ is finite of degree δ . Then $\delta \operatorname{vol}_{X|C}(L) \leq \|L \cdot C\|$.*

Proof. We may assume C is smooth, by taking an embedded resolution of C and by using Lemma 2.3. Since $\dim f(C) > 0$, we know $\operatorname{vol}_{X|C}(L) > 0$, in particular we have $\limsup_m h^0(X|C, mL) = +\infty$. We use Notation 3.1 with $V = C$. We denote by $\nu_m : C'_m \rightarrow \psi_m(C_m) = \Phi_{|mL}(C) (\subset \mathbb{P})$ the normalization, and by $\alpha_m : C_m \rightarrow C'_m$ the induced morphism. We note $C_m \cong C$. We may assume, by taking m to be large enough, that the map α_m has degree δ . We have $M_m|_{C_m} = \alpha_m^* \nu_m^* \mathcal{O}(1)$ and $\deg M_m|_{C_m} = \delta \deg \nu_m^* \mathcal{O}(1)$. Then

$$H^0(X|C, mL) \cong H^0(X_m|C_m, M_m) \subset \alpha_m^* H^0(C'_m, \nu_m^* \mathcal{O}(1)).$$

(We note that there is an isomorphism $H^0(X_m|C_m, M_m) \xrightarrow{\sim} H^0(X|C, mL)$.)

We note that $\limsup_m h^0(X|C, mL) = +\infty$ implies $\limsup_m (\deg M_m|_{C_m}) = +\infty$. In fact, since $h^1(C_m, M_m|_{C_m}) = h^0(C_m, K_{C_m} - M_m|_{C_m}) \leq h^0(C, K_C)$ is bounded, if we had $\deg M_m|_{C_m} < d_0$ for all m , Riemann-Roch would imply that $h^0(C_m, M_m|_{C_m}) \geq h^0(X|C, mL)$ is also bounded. Thus we can take m so large that $\deg M_m|_{C_m} > \delta \deg K_C$. In particular, $\deg \nu_m^* \mathcal{O}(1) > \deg K_{C'_m}$. Then by Riemann-Roch and vanishing, we have $h^0(C_m, M_m|_{C_m}) = \deg M_m|_{C_m} + \chi(\mathcal{O}_{C_m})$ and $h^0(C'_m, \nu_m^* \mathcal{O}(1)) = \deg \nu_m^* \mathcal{O}(1) + \chi(\mathcal{O}_{C'_m})$. Therefore

$$\delta h^0(C'_m, \nu_m^* \mathcal{O}(1)) = \delta (\deg \nu_m^* \mathcal{O}(1) + \chi(\mathcal{O}_{C'_m})) = \deg M_m|_{C_m} + \delta \chi(\mathcal{O}_{C'_m}).$$

Hence $\delta \frac{1}{m} h^0(X|C, mL) \leq \frac{1}{m} \deg M_m|_{C_m} + \frac{\delta}{m} \chi(\mathcal{O}_{C'_m}) \leq \frac{1}{m} \deg M_m|_{C_m} + \delta/m$ holds. Since this holds for infinitely many m , by letting $m \rightarrow \infty$, we have $\delta \operatorname{vol}_{X|C}(L) \leq \|L \cdot C\|$. \square

The previous three lemmas imply immediately the following.

Corollary 3.9. *Let $C \subset X$ be a curve such that $C \not\subset \text{SBs}(L)$ and the map $f|_C : C \dashrightarrow f(C)$ is finite of degree δ . Then $\mu(C, L) \geq \delta \text{vol}_{X|C}(L) = \|L \cdot C\|$ holds.*

Corollary 3.10. *(=Theorem 1.4) Let $x \in X$ be a very general point. Assume $\mu(C, L) = \text{vol}_{X|C}(L)$ for any curve C passing through x . Then either $\kappa(L) = 0$ or $\kappa(L) = \dim X$.*

Proof. Assume $0 < \kappa(L) < \dim X$. Then we can find a curve $C \subset X$, for example as a general complete intersection over a general curve C' in Y , with $\deg(f|_C : C \dashrightarrow C') > 1$. By Corollary 3.9, we have $\mu(C, L) > \text{vol}_{X|C}(L)$ and get a contradiction. \square

3.2. Concluding remarks. Here are some remarks to pursue the arguments above.

Remark 3.11. There is a missing piece for a better understanding the asymptotic property of linear series $\{|mL|\}_{m>0}$ for general L with $\kappa(L) \geq 0$. In case L is big, there exists an effective (very ample) divisor G such that $\mathfrak{b}_m(-G) \subset \mathcal{J}(\|mL\|)(-G) \subset \mathfrak{b}_m$ for all $m > 0$ [L, 11.2.21] ([ELMNP3, 3.1]), where $\mathfrak{b}_m = \mathfrak{b}(|mL|)$ is the base ideal. This “uniformity” was crucial in the asymptotic study of big divisors. We would like to see whether or not, this “uniformity” still holds in case L is not big. A counter-example will also be interesting.

In any case let us point out that arguing as in the proof of [ELMNP3, 2.13], one obtains the following.

Proposition 3.12. *Suppose there exists an effective divisor G on X such that $\mathfrak{b}_m(-G) \subset \mathcal{J}(\|mL\|)(-G) \subset \mathfrak{b}_m$ for all $m > 0$. Then $\|L^d \cdot V\| \geq \mu(V, L)$ (which implies $\|L^d \cdot V\| = \mu(V, L)$), for any subvariety V of $\dim V = d > 0$ such that $V \not\subset (\text{SBs}(L) \cup \text{Supp } G)$.*

It would also be interesting to understand whether the converse of Lemma 3.6 holds in general as follows.

Question 3.13. Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \text{SBs}(L)$ and the map $f|_V : V \dashrightarrow f(V)$ is generically finite of degree δ . Then, does the inequality $\delta \text{vol}_{X|V}(L) \leq \|L^d \cdot V\|$ hold?

Remark 3.14. Notice that if Question 3.13 is affirmative, we would get a relation $\mu(V, L) \geq \delta \text{vol}_{X|V}(L) = \|L^d \cdot V\|$ for V as in the question (cf. Corollary 3.9), and get a natural generalization of Theorem 1.4. Precisely the same arguments given in the proof of Corollary 3.10 would yield the following: Let $x \in X$ be a very general point. Assume $\mu(V, L) = \text{vol}_{X|V}(L)$ for any d -dimensional subvariety V passing through x . Then either $\kappa(L) < d$ or $\kappa(L) = \dim X$.

Remark 3.15. A parallel analytic approach, in the spirit of [B], to the questions studied in the present paper would be possible.

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Gianluca PACIENZA

Institut de Recherche Mathématique Avancée

Université L. Pasteur et CNRS

Rue R. Descartes - 67084 Strasbourg Cedex, France

e-mail: pacienza@math.u-strasbg.fr

Shigeharu TAKAYAMA

Graduate School of Mathematical Sciences

University of Tokyo

3-8-1 Komaba, Tokyo

153-8914, Japan

e-mail: taka@ms.u-tokyo.ac.jp