

A Wong-Rosay type theorem for the iterations of a proper holomorphic self-map.

Emmanuel Opshtein

Abstract

In this short paper, we show that the only proper holomorphic self-maps of bounded domains in \mathbb{C}^k whose dynamics escape to a strictly pseudoconvex point of the boundary are automorphisms of the euclidean ball. This Wong-Rosay type theorem, in a dynamical situation is non-trivial since the degrees of the considered maps are a priori unbounded.

1 Introduction.

In 1977, Wong proved that the only strictly pseudoconvex domain with non-compact group of automorphisms is the ball [17]. This result was generalized by Rosay [13] (see also [12]).

Theorem (Wong-Rosay). *Let Ω be a bounded domain in \mathbb{C}^k and (f^n) a sequence of its automorphisms. Assume that the orbit of a point of Ω under (f_n) accumulates a smooth strictly pseudoconvex point of $b\Omega$. Then Ω is biholomorphic to the euclidean ball.*

Ourimi showed that this version of Wong's theorem extends to proper maps, or even correspondences provided that the degrees of the maps f_n remain bounded [10, 11]. In this paper, we prove that the theorem above holds true when replacing the sequence of automorphisms by the iterates of a proper holomorphic self-map.

Theorem 1. *Let Ω be a bounded domain in \mathbb{C}^k with a proper holomorphic self-map f which extends smoothly to $b\Omega$. If there is a point y of Ω whose orbit under the iterates of f accumulates a smooth strictly pseudoconvex point a of $b\Omega$ (that is $f^{n_k}(y) \rightarrow a$), then Ω is biholomorphic to the euclidean ball and f is an automorphism.*

The hypothesis of this theorem may seem weaker than it is actually. It may be useful to have in mind the following classical fact from the beginning.

Remark 1.1. *Under the hypothesis of theorem 1, the subsequence f^{n_k} converges locally uniformly to a on Ω . Also, a is a fixed point of f .*

The main result on which relies the proof of theorem 1 is a local version of Wong-Rosay's theorem concerning sequences of CR-maps. It was first obtained by Webster in the wake of Chern-Moser's theory of strictly pseudoconvex hypersurfaces [16].

Theorem 2 (Webster). *Let (Σ, a) be a germ of strictly pseudoconvex hypersurface. Assume there is a sequence of CR-automorphisms of Σ whose images converge to a . Then S is spherical, i.e. locally CR-diffeomorphic to the euclidean sphere.*

The idea to prove theorem 1 is to consider the CR-map induced by f^n on the boundary rather than f^n itself. Using techniques developed in [9, 8], we study the way these CR-maps degenerate and check that theorem 2 applies. The local information it gives then propagates to the whole of Ω thanks to the dynamical situation. Notice that the same techniques were used in [8] to get an elementary proof of Webster's theorem.

The paper is organised as follows. We first prove theorem 1 under the additional assumption that the domain is smooth and convex. This part serves as a model proof since it exhibits the two basic properties we need to prove the general case of theorem 1. One is a local information on the sphericity of $b\Omega$ around a . The other is a dynamical information : we need to explain that (f^n) and not only a subsequence converges to a on Ω . In sections three and four we prove these two properties respectively, thus fixing the proof of theorem 1.

2 The convex case : a model proof.

In this section, we prove theorem 1 under the extra-assumption that Ω is a convex domain with smooth boundary. Almost all the technicity disappears in this context, so this section should help the reader to get a clear picture of the structure of the proof. The simplification comes from the fact that we understand the dynamics of proper holomorphic self-maps of these special domains. In [9, 7], the following was indeed proved :

Proposition 2.1. *Let Ω be a smoothly bounded convex domain in \mathbb{C}^k and $f : \Omega \rightarrow \Omega$ be a proper holomorphic self-map (then f extends smoothly to $b\Omega$, see [2]). Assume the f -orbit of a point in Ω accumulates a point a of $SC(b\Omega)$ ($SC(b\Omega)$ denotes the strongly convex part of $b\Omega$). Then the whole sequence $(f^n)_{n \in \mathbb{N}}$ converges locally uniformly to a on $\Omega \cup SC(b\Omega)$.*

Using this proposition we now proceed to the proof of theorem 1 when Ω is smooth and convex. Let us first fix the notation. By a complex affine change of coordinate, we can take a to the origin in \mathbb{C}^k , and arrange that $T_a b\Omega = \{\operatorname{Re} z_1 = 0\}$ and $\Omega \subset \{\operatorname{Re} z_1 > 0\}$. We then denote $U_\alpha := b\Omega \cap \{\operatorname{Re} z_1 < \alpha\}$ and $\Omega_\alpha := \Omega \cap \{\operatorname{Re} z_1 < \alpha\}$.

The first step of the proof shows that $b\Omega$ is spherical around a . It should be noticed that it is the only point where we use the fact that the maps f^n converge to a on $SC(b\Omega)$.

Lemma 2.2. *If (f^n) converges to a on $SC(b\Omega)$ then a neighbourhood of a in $b\Omega$ is spherical.*

Proof : Theorem 2 explains that it is enough to find a contracting sequence of CR-automorphisms on a neighbourhood of a . Proposition 2.1 asserts that (f^n) is a sequence of contracting CR-maps on $SC(b\Omega)$, and f is in fact a local diffeomorphism (see [4]). We thus only need to prove that there is a fixed neighbourhood of a on which all f^n are injective. To see this, fix a neighbourhood U_{ε_0} of a on which f is injective. Since f^n converges locally uniformly to a on $SC(b\Omega)$, $f^n(U_{\varepsilon_0}) \subset U_{\varepsilon_0}$ for all large enough integers $n \geq n_0$. Consider now a neighbourhood U_ε of a in U_{ε_0} whose images $U_\varepsilon, f(U_\varepsilon), \dots, f^{n_0}(U_\varepsilon)$ are all contained in U_{ε_0} . Such a set exists because f is continuous and a is a fixed point of f . By construction $f^n(U_\varepsilon) \subset U_{\varepsilon_0}$ for all $n \in \mathbb{N}$, and the restriction of f^n to U_ε is injective as a composition of injective maps. \square

The second step of the proof consists in using the dynamics of f to propagate the local sphericity around a to produce a biholomorphism from Ω to B . We have just established the existence of a CR-diffeomorphism $\Phi : U_\varepsilon \rightarrow V \subset bB$. A classical extension theorem shows that Φ extends to a biholomorphism $\Phi : \Omega_\varepsilon \rightarrow D$ where D is an open set of B whose boundary contains V (see [3]). Transporting f by this biholomorphism gives a local automorphism of B , defined by

$$\begin{aligned} g : \Phi(\Omega_\varepsilon \cap f^{-1}(\Omega_\varepsilon)) &\longrightarrow \Phi(\Omega_\varepsilon) \\ x &\longmapsto \Phi \circ f \circ \Phi^{-1}(x). \end{aligned}$$

The key point of the whole proof is an extension phenomena discovered by Alexander [1] (see also [12, 15] for the form of the result we use here). The local biholomorphism g uniquely extends to a global automorphism of the ball. This extension of g allows to extend Φ to the union of the sets $O_i := f^{-i}(\Omega_\varepsilon)$:

$$\begin{aligned} \Phi &: \cup_{i \in \mathbb{N}} O_i \longrightarrow B \\ z \in O_i &\longmapsto g^{-i} \circ \Phi \circ f^i(z). \end{aligned}$$

Since f^n tends to a on Ω this formula defines in fact Φ on Ω , and we have the obvious functional equality : $\Phi \circ f = g \circ \Phi$.

Lemma 2.3. *The map Φ is a proper map from Ω to B .*

Proof : Recall that in the proof of lemma 2.2 we saw that a certain neighbourhood of a remains in U_ε under iteration and eventually ends to a . By conjugacy, the map g inherits of this local behaviour around $S := \Phi(a)$. The dynamics of ball's automorphisms being well-known (see proposition 4.1), this property is enough to guarantee that g is hyperbolic. It has exactly two fixed points S and N which attract all orbits of $\overline{B} \setminus \{N, S\}$ respectively positively and negatively. The basic consequence of this fact is that $\Phi(O_n) \setminus O_{n-1}$ goes to bB with n . Indeed, the f -orbit of a point z in this set reaches O_0 only at time n , so the g -orbit of $\Phi(z)$ only reaches D after the same time. If n is large, $\Phi(z)$ has to be very close to N which is on the boundary of B .

For an arbitrary sequence $(z_i)_{i \in \mathbb{N}} \in \Omega$ converging to $b\Omega$, we must show that $\Phi(z_i)$ tends to the boundary of B . For this, fix a positive real number ε and an integer n_0 such that $d(\Phi(O_n \setminus O_{n-1}), bB) \leq \varepsilon$ for all $n \geq n_0$. Split then (z_i) in two subsequences :

$$\begin{aligned} (z_i^1) &:= \{z_{n_i} \in \{z_n\} \mid z_{n_i} \notin O_{n_0}\} \\ \text{and } (z_i^2) &:= \{z_{n_i} \in \{z_n\} \mid z_{n_i} \in O_{n_0}\}. \end{aligned}$$

By construction, $d(\Phi(z_i^1), bB) \leq \varepsilon$. Since $f^{n_0}(z_i^2) \subset O_0$ and since f^{n_0} , $\Phi|_{O_0}$ and g are proper maps, $\Phi(z_i^2) = g^{-n_0} \circ \Phi|_{O_0} \circ f^{n_0}(z_i^2)$ is also ε -close to bB for i large enough. \square

Finally, we need to show that Φ is a biholomorphism. It is not yet clear since there obviously exist holomorphic coverings of the ball. Anyway we know that any proper map to a bounded domain has a finite degree (see [14], chap. 15). In particular, there is an integer d which bounds the numbers of preimages by Φ :

$$\#\Phi^{-1}(z) \leq d, \quad \forall z \in B.$$

Notice now that the degree of Φ bounds this of f^n for all n because $\Phi = g^{-n} \circ \Phi \circ f^n$. The degree of f^n is thus bounded on one hand and equal to $(\deg f)^n$ on the other. So f is an automorphism of Ω . The injectivity of Φ is now immediate because $\Phi|_{O_i} = g^{-n} \circ \Phi|_{O_0} \circ f^n$ is a composition of injective maps for all i . \square

Let us notice that the convexity of Ω was only used through proposition 2.1. Moreover, the convergence of f^n to a on $\mathcal{SC}(b\Omega)$ only served to get the sphericity near a .

Remark 2.4. *If Ω is a bounded domain in \mathbb{C}^n whose boundary is spherical in a neighbourhood of a point a , and f is a proper holomorphic self-map of $\overline{\Omega}$ whose dynamics converge to a inside Ω then f is an automorphism and Ω is the ball.*

The strategy for the general result is thus to prove the sphericity near a and the convergence of f^n to a on Ω .

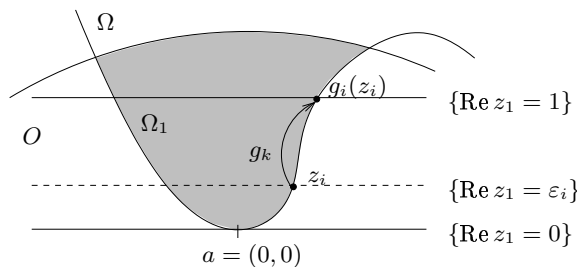
3 Local sphericity near the attractive point.

In this section we come back to the general situation of theorem 1 and we prove that a neighbourhood of a in $b\Omega$ is spherical. The idea behind this technical part of the proof is based on previous results concerning behaviours of sequences of CR-maps (see [9, 7]). Unformally speaking, they explain that non-equicontinuous sequences of CR-maps on strictly pseudoconvex hypersurface dilate a certain (anisotropic) distance. The proof of the sphericity then goes as follows. Either f^{n_k} converges to a on $S\mathcal{P}\mathcal{C}(b\Omega)$ and the situation is very similar to that of lemma 2.2 (see proposition 3.1). Or f^{n_k} is not equicontinuous on $S\mathcal{P}\mathcal{C}(b\Omega)$ and it is dilating. Then the inverse branches of f^{n_k} are contracting CR-diffeomorphisms and theorem 2 gives the sphericity. Let us first fix the easy situation where f^{n_k} converges to a on $S\mathcal{P}\mathcal{C}(b\Omega)$.

Proposition 3.1. *Assume f^{n_k} converges locally uniformly to a on a neighbourhood of a in $b\Omega$. Then $b\Omega$ is spherical near a .*

Proof : As a matter of fact, the convergence of the subsequence f^{n_k} to a implies the convergence of the whole dynamics of an iterate $g = f^p$ to a . To see this, pick a small neighbourhood U of a in $S\mathcal{P}\mathcal{C}(b\Omega)$ and an integer $p = n_{k_0}$ such that $f^p(U) \subset U$. The map $g := f^p$ restricts to U to a local diffeomorphism from U to itself, whose sequence of images $g^n(U)$ is obviously decreasing (*i.e.* $g^i(U) \supset g^{i+1}(U)$). Observe then that the subsequence $(g^{n'_k})$ defined by $n'_k := E(n_k/p) + 1$ converges uniformly to a on U . Indeed, $g^{n'_k} = f^{pn'_k} = f^{n_k+i}$ with $i < p$, so $g^{n'_k}(U) \subset \cup_{i \leq p} f^i(f^{n_k}(U))$. Since $f^{n_k}(U)$ is close to a by hypothesis (for k large enough) and a is a fixed point of f , the continuity of f implies that $g^{n'_k}(U)$ is also close to a . Since the sequence $g^n(U)$ decreases, it thus converges to a . Replacing f by g , we are therefore in the condition of application of lemma 2.2 and a neighbourhood of a is spherical. \square

Consider now the situation when f^{n_k} does not converge to a on a neighbourhood of a . Let us first describe the figure and notation. Since a is a strictly pseudoconvex point, it has a neighbourhood which is strongly convex when considered through a polynomial change of coordinates. Henceforth, we assume that Ω is strongly convex in a neighbourhood O of a , that a is the origin and that $\Omega \cap O$ is contained in $\{\text{Re } z_1 \geq 0\}$. Also, since all the arguments to come are purely local, we focus on Ω in O . We put $\Omega_\varepsilon := \Omega \cap O \cap \{\text{Re } z_1 \leq \varepsilon\}$ and $U_\varepsilon := b\Omega \cap O \cap \{\text{Re } z_1 \leq \varepsilon\}$. Assume without loss of generality that $\Omega_1 \Subset O$. The non-convergence of f^{n_k} means the existence of a sequence of points $z_i \in b\Omega$ tending to a , and integers k_i such that the points $f^{n_{k_i}}(z_i)$ lay out of a fixed neighbourhood of a , say U_1 . Since a is fixed by $f^{n_{k_i}}$, we can even assume that $f^{n_{k_i}}(z_i) \in bU_1 = b\Omega \cap O \cap \{\text{Re } z_1 = 1\}$ by moving z_i closer to a . Finally, put $g_i := f^{n_{k_i}}$ and define ε_i by $z_i \in \{\text{Re } z_1 = \varepsilon_i\}$.



The main point of this section is that f^{n_k} has a strong expanding behaviour.

Proposition 3.2. (see also [8]) For all ε there exists an integer $k = k(\varepsilon)$ such that $g_k(U_\varepsilon) \supset U_1 \setminus U_\varepsilon$.

The sphericity near a is a direct consequence of this proposition :

Corollary 3.3. If (f^{n_k}) does not converge to a in a neighbourhood of a then $b\Omega$ is spherical near a .

Proof : Fix an open contractible set V compact in $U_1 \setminus \{a\}$. For ε small enough, $V \subset U_1 \setminus U_\varepsilon$ and there is an integer k_ε such that $g_{k_\varepsilon}(U_\varepsilon) \supset V$. Moreover, there are no critical value of $g_{k_\varepsilon}|_{U_\varepsilon}$ inside V because both U_ε and V are strictly pseudoconvex (see [4]). Since V is simply connected, there exists an inverse branch of $g_{k_\varepsilon}|_{U_\varepsilon}$ on V , which means a CR-diffeomorphism $h_\varepsilon : V \rightarrow U_\varepsilon$ with $g_{k_\varepsilon} \circ h_\varepsilon = \text{Id}$. The sequence h_ε is therefore contracting on V , and theorem 2 implies that V is spherical. We have thus proved the local sphericity of $U_1 \setminus \{a\}$, which even proves the sphericity of U_1 because a is a strictly pseudoconvex point. Indeed, Chern-Moser's theory expresses the sphericity of an open strictly pseudoconvex hypersurface by the vanishing of a continuous invariant tensor. Since this tensor vanishes on $U_1 \setminus \{a\}$, it also vanishes on the whole of U_1 so U_1 itself is spherical. In the spirit of [8], It would be pleasant to get a more down-to-earth argument for this last point. \square

The proof of proposition 3.2 relies on the following lemma.

Lemma 3.4. There exists a diverging sequence $c_i \rightarrow +\infty$ such that for all $p \in U_1$ with $g_i(p) \notin U_\varepsilon$ we have :

$$\|g'_i(p)u\| \geq c_i \|u\| \quad \forall u \in T_p^{\mathbb{C}}b\Omega.$$

Proof : The idea is that Hopf's lemma gives estimates on the normal derivative of g_i , which transfer automatically to complex tangential estimates in strictly pseudoconvex geometry. For $p \in U_1$, let $\vec{N}(p)$ be the unit vector normal to $b\Omega$ pointing inside Ω and

$$B_\delta^+(p) := B(p + \delta\vec{N}(p), \delta) \cap \{\langle \vec{N}(p), \cdot \rangle \geq \delta\}.$$

When δ is small enough but fixed, $B_\delta^+(p)$ is in Ω and its image by g_i for i large is in Ω_ε . Thus if $g_i(p) \notin U_\varepsilon$, the non-positive *p.s.h* function $\varphi := -\langle \vec{N}(g_i(p)), g_i(y) - g_i(p) \rangle$ vanishes at p while it is less than $-c\varepsilon^2$ on $B_\varepsilon^+(p)$ (c is a constant depending only on the curvature of $b\Omega$ at a). Hopf's lemma then asserts that

$$n_i(p) := \langle g'_i(p)\vec{N}(p), \vec{N}(g_i(p)) \rangle = \|\vec{\nabla}\varphi(p)\| \geq \frac{c'\varepsilon^2}{\delta}.$$

Since δ was arbitrary, we could take it much smaller than ε^2 , so that the radial escape rate $n_i(p)$ is large. To transfer this radial estimate on the derivatives of g_i to complex tangential ones, consider the Levi form \mathcal{L} of $b\Omega$ defined by

$$\mathcal{L}(p, u) := \langle [u, iu], i\vec{N}(p) \rangle, \quad u \in T_p^{\mathbb{C}}b\Omega,$$

where u stands for the vector in $T_p^{\mathbb{C}}b\Omega$ as well as any extension of it to a vector field of $T^{\mathbb{C}}b\Omega$. The smoothness and strict pseudoconvexity of U_1 implies the existence of geometric constants c_1, c_2 such that

$$c_1 \|u\|^2 \leq \mathcal{L}(p, u) \leq c_2 \|u\|^2 \quad \forall p \in U_1, \forall u \in T_p^{\mathbb{C}}b\Omega.$$

Easy computations show that :

$$c_2 \|g'_i(p)u\|^2 \geq \mathcal{L}(g_i(p), g'_i(p)u) = n_i(p)\mathcal{L}(p, u) \geq c_1 n_i(p) \|u\|^2.$$

Since $n_i(p)$ is large when i is, this serie of inequalities imply lemma 3.4. \square

The previous lemma asserts that g_i dilates the complex tangential directions of $b\Omega$ if $g_i(p)$ is not close to a . The last observation we need to make in order to prove proposition 3.2 is that this “complex tangential dilation” property implies a genuine dilation.

A path γ in $b\Omega$ will be called a complex path if $\dot{\gamma}(t) \in T_{\gamma(t)}^{\mathbb{C}} b\Omega$ for all t . Its euclidean length will be denoted by $\ell(\gamma)$. For $x, y \in U_1$, define the CR-distance $d^{\text{CR}}(x, y)$ between x and y as the infimum of the lengths of complex paths joining x to y . The point is that the strict pseudoconvexity condition means that the complex tangential distribution is contact so complex paths can join any two points. Even more, the open set $U_1 \setminus U_\varepsilon$ is d^{CR} -bounded (see theorem 4 of [6], or [9]).

Proof of proposition 3.2 : Fix $\tau > 0$ such that $B^{\text{CR}}(z_i, \tau) \subset U_\varepsilon$ for all i large enough. Since $U_1 \setminus U_\varepsilon$ is d^{CR} -bounded, it is enough to prove that

$$bg_i(B^{\text{CR}}(z_i, \tau)) \cap B^{\text{CR}}(g_i(z_i), c_i\tau) \cap (U_1 \setminus U_\varepsilon) = \emptyset$$

because $c_i\tau$ can be made greater than the CR-diameter of $U_1 \setminus U_\varepsilon$. Take a point $x \in bg_i(B^{\text{CR}}(z_i, \tau)) \cap U_1 \setminus U_\varepsilon$ and let us prove that

$$d^{\text{CR}}(g_i(z_i), x) \geq c_i\tau. \quad (1)$$

Consider an arc-length parameterized complex path γ in $U_1 \setminus U_\varepsilon$ joining $g_i(z_i)$ to x . Since g_i is a local CR-diffeomorphism at each point of $B^{\text{CR}}(z_i, \tau)$ whose image lies in the strictly pseudoconvex part of $b\Omega$, the connected component of $g_i(z_i)$ in $\gamma \cap g_i(B^{\text{CR}}(z_i, \tau))$ can be lifted to a complex path $\tilde{\gamma}$ through g_i . Thus there exists $l \leq \ell(\gamma)$ and $\tilde{\gamma} : [0, l] \rightarrow B^{\text{CR}}(z_i, \tau)$ joining z_i to $bg_i(B^{\text{CR}}(z_i, \tau))$ such that $g_i \circ \tilde{\gamma}(t) = \gamma(t)$ for all $t \in [0, l]$. Since $\tilde{\gamma}(t) \in U_1$ and $g_i(\tilde{\gamma}(t)) \in U_1 \setminus U_\varepsilon$ for all t , the estimates obtained in lemma 3.4 yield :

$$\ell(\gamma) \geq l = \int_0^l \|\dot{\gamma}(t)\| dt = \int_0^l \|g'_i(\tilde{\gamma}(t))\dot{\tilde{\gamma}}(t)\| dt \geq c_i \int_0^l \|\dot{\tilde{\gamma}}(t)\| dt \geq c_i \ell(\tilde{\gamma}).$$

This proves (1) since $\tilde{\gamma}$ joins z_i to $bg_i(B^{\text{CR}}(z_i, \tau))$ (so $\ell(\tilde{\gamma}) \geq \tau$) and γ is any complex path joining $g_i(z_i)$ to x . \square

4 Dynamics of f inside Ω .

In this section we explain why the maps f^n - and not only f^{n_k} - converge to a inside Ω . By remark 2.4, and since we already know that $b\Omega$ is spherical near a , it is exactly what remains at this point to conclude the proof of theorem 1. We get this information by conjugating locally f to an automorphism of the ball, whose dynamics is easily understood thanks to the following classical result (see [14]).

Proposition 4.1. *The dynamics of a ball automorphism g is necessarily of one of the following types.*

1. *Hyperbolic (or North-South) : There exist exactly two fixed points $N, S \in bB$ of g and g^n converges locally uniformly to S on $\overline{B} \setminus \{N\}$.*

2. *Parabolic (or South-South) :* There exists a unique fixed point $S \in bB$ of g and g^n converges locally uniformly to S on $\overline{B} \setminus \{S\}$.
3. *Recurrent (or compact) :* The g -orbits remain at fixed distance from bB . If g has a fixed point on bB then it has a whole complex line through this point, pointwise fixed by g (see also [5]).

We keep the same notation as in the previous part. Consider a CR-diffeomorphism $\Phi : U_\varepsilon \longrightarrow V \subset bB$ and extend it to a biholomorphism $\Phi : \Omega_\varepsilon \longrightarrow D \subset B$ exactly as in the convex case. Define the Φ -conjugate of f

$$g : \begin{array}{ccc} \Phi(\Omega_\varepsilon \cap f^{-1}(\Omega_\varepsilon)) & \longrightarrow & \Phi(\Omega_\varepsilon) \\ y & \longmapsto & \Phi \circ f \circ \Phi^{-1}(y), \end{array}$$

and extend it to a global automorphism of the ball.

Lemma 4.2. *The dynamics of g is either parabolic or hyperbolic with south pole $\Phi(a)$.*

Proof : Since f has no fixed point in Ω , g cannot have any fixed point in $\Phi(\Omega_\varepsilon)$, so $\Phi(a)$ is an isolated fixed point and the dynamics of g cannot be recurrent. We therefore only need to check that if g is hyperbolic then $\Phi(a)$ is the south pole of the dynamics. Assume by contradiction it is not : $\Phi(a)$ is a repulsive fixed point of g , so f is also repulsive at a . After shrinking Ω_ε if necessary, we can therefore assume that f^{-1} is defined on Ω_ε , with values in Ω_ε and even that $d(f|_{\Omega_\varepsilon}^{-1}(y), a) < d(y, a)$ for any $y \in \Omega_\varepsilon$. By assumption, there is a point $y_0 \in \Omega$ such that $f^{n_k}(y_0) \in \Omega_\varepsilon$ as soon as k is large enough. Define then

$$n'_k := \min\{n \mid f^i(y_0) \in \Omega_\varepsilon, \quad \forall i \in [n, n_k]\},$$

so that $f^{n'_k-1}(y_0) \notin \Omega_\varepsilon$. Since $f|_{\Omega_\varepsilon}^{-1}$ is contracting, the point $f^{n'_k}(y_0)$ is closer to a than $f^{n_k}(y_0)$, so it tends to a (in particular (n'_k) is an extraction). Equivalently $f^{n'_k-1}(f(y_0))$ tends to a , so $f^{n'_k-1}$ converges locally uniformly to a by remark 1.1. This is in contradiction with $f^{n'_k-1}(y_0) \notin \Omega_\varepsilon$. \square

This lemma implies the existence of a point $x \in \Phi(\Omega_\varepsilon)$ whose orbit remains in $\Phi(\Omega_\varepsilon)$ and tends to $\Phi(a)$. The orbit of $y = \Phi^{-1}(x)$ is thus the preimage of the g -orbit of x by Φ so $f^n(y)$ tends to a . By remark 1.1, we thus get the

Corollary 4.3. *Under the assumption of theorem 1, the sequence (f^n) converges to a on Ω .*

Proof of theorem 1. Theorem 1 is now a formal consequence of remark 2.4, proposition 3.1 and corollaries 3.3 and 4.3.

References

- [1] H. Alexander. Holomorphic mappings from the ball and polydisc. *Math. Ann.*, 209:249–256, 1974.
- [2] H. Boas and E. Straube. Sobolev estimates for the $\bar{\partial}$ -Neumann operator on domains in \mathbf{C}^n admitting a defining function that is plurisubharmonic on the boundary. *Math. Z.*, 206(1):81–88, 1991.
- [3] A. Boggess. *CR manifolds and the tangential Cauchy-Riemann complex*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1991.
- [4] J. E. Fornaess. Biholomorphic mappings between weakly pseudoconvex domains. *Pacific J. Math.*, 74(1):63–65, 1978.
- [5] B. D. MacCluer. Iterates of holomorphic self-maps of the unit ball in \mathbf{C}^N . *Michigan Math. J.*, 30(1):97–106, 1983.

- [6] A. Nagel, E. M. Stein, and S. Wainger. Balls and metrics defined by vector fields. I. Basic properties. *Acta Math.*, 155(1-2):103–147, 1985.
- [7] E. Opshtein. Approche dynamique du problème de l’injectivité des applications holomorphes propres. *Thèse*, 2005.
- [8] E. Opshtein. Sphericity and contractibility of strictly pseudoconvex hypersurfaces. *Prepublication, arXiv math.CV/0504054*, 2005.
- [9] E. Opshtein. Dynamique des applications holomorphes propres des domaines réguliers et problème de l’injectivité. *Math. Ann.*, 133(1):1–30, 2006.
- [10] N. Ourimi. A local version of Wong-Rosay’s theorem for proper holomorphic mappings. *Proc. Amer. Math. Soc.*, 128(3):831–836, 2000.
- [11] N. Ourimi. Some compactness theorems of families of proper holomorphic correspondences. *Publ. Mat.*, 47(1):31–43, 2003.
- [12] S. I. Pinčuk. The analytic continuation of holomorphic mappings. *Mat. Sb. (N.S.)*, 98(140)(3(11)):416–435, 495–496, 1975.
- [13] J.-P. Rosay. Sur une caractérisation de la boule parmi les domaines de \mathbf{C}^n par son groupe d’automorphismes. *Ann. Inst. Fourier (Grenoble)*, 29(4):ix, 91–97, 1979.
- [14] W. Rudin. *Function theory in the unit ball of \mathbf{C}^n* , volume 241 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York, 1980.
- [15] A. E. Tumanov and G. M. Khenkin. Local characterization of holomorphic automorphisms of Siegel domains. *Funktsional. Anal. i Prilozhen.*, 17(4):49–61, 1983.
- [16] S. M. Webster. On the transformation group of a real hypersurface. *Trans. Amer. Math. Soc.*, 231(1):179–190, 1977.
- [17] B. Wong. Characterization of the unit ball in \mathbf{C}^n by its automorphism group. *Invent. Math.*, 41(3):253–257, 1977.