Contractibility and sphericity of strictly pseudoconvex hypersurfaces.

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Abstract

It is a well-known result of Chern-Moser's theory that spherical hypersurfaces are the only strictly pseudoconvex ones which carry contracting germs of CR-automorphisms. In this paper, we recover this result from Pinchuk's dilations techniques. Our approach presents the advantage of being much more elementary and to work in the $\mathcal{C}^3$-category.

1 Introduction

In this paper, we study the CR-geometry of strictly pseudoconvex hypersurfaces of $\mathbb{C}^k$ with the help of Pinchuk techniques of dilations ([19], 1978). We are interested in the following question : can a germ of strictly pseudoconvex hypersurface be contracted to one of its points by a sequence of CR-automorphisms ?

The work by Chern-Moser ([11], 1974) leads to a complete list of CR-invariants of Levi non-degenerate hypersurfaces. In a contracting situation, these invariants are constant, suggesting some homogeneity of the surface. That is how Webster ([21], 1974), Burns-Shnider ([10], 1977), Beloshapka ([5], 1977) and Loboda ([13], 1979) established the following result :

Theorem 1. *Let S and S' be strictly pseudoconvex hypersurfaces in $\mathbb{C}^k$ (possibly with non-empty boundary). If there exists a sequence $(f_n)_n$ of CR-diffeomorphisms from S with value in S' which converge locally uniformly to a point of S' then S is spherical, which means that S is locally CR-diffeomorphic to an open set of the euclidean sphere.*

Webster and Burns-Shnider's approach is the following. The resolution of the G-structure problem done in [11] produces a CR-invariant tensor which vanishes if and only if the hypersurface is spherical. In the non-spherical situation, a normalisation of this invariant leads to a reduction of the structural group of the CR-structure of the hypersurface to $U(k-1)$. More concretely, it shows the existence of a CR-invariant distance on non-spherical hypersurfaces, and therefore prohibits any contracting behaviour for sequences of CR-diffeomorphisms. As for Beloshapka and Loboda, this theorem results from their study of the action of CR-diffeomorphisms on the chains, and more precisely from a local linearization of the hypersurface's CR-automorphism group. This last approach is only valid in the real-analytic framework.

An analog of this theorem for biholomorphisms of domains was obtained by Wong ([22], 1977). He characterized the unit ball among all other strictly pseudoconvex domains as the only one to have a non-compact group of automorphisms. Simplifying Wong's argument, Rosay got the following semi-local result ([20], 1979) :

Theorem (Wong-Rosay). *Let Ω be a bounded domain of $\mathbb{C}^k$ and $(f_n)_n$ a sequence of automorphisms of Ω. If there exists a point $z_0 \in \Omega$ whose orbit $\{f_n(z_0)\}$ clusters a boundary point in the neighbourhood of which $b\Omega$ is smooth and strictly pseudoconvex then Ω is biholomorphically equivalent to the unit ball of $\mathbb{C}^k$.*

Shortly afterwards, Pinchuk gave a proof of this theorem based on dilation techniques. They allowed to generalize this result in some weakly pseudoconvex situations. An analogous theorem was obtained so by Berteloot [6] when the cluster point is pseudoconvex of finite type in $\mathbb{C}^2$ and by Gaussier [12] when the cluster point is convex of finite type in any dimension. Bedford-Pinchuk had previously characterized the globally finite type domains in $\mathbb{C}^2$ or convex of finite type in $\mathbb{C}^k$ whose automorphism group is non-compact ([2, 1, 3]).

The purpose of this paper is to explain how to "localize" these dilation techniques in view of getting an elementary proof of theorem 1, which does not rely on Chern-Moser's work. Besides being much simpler than the original approach, our proof works for $\mathcal{C}^3$-regular hypersurfaces, for which Chern-Moser invariants are not defined.

The approach we adopt consists in following at best the great lines of Pinchuk's proof of Wong-Rosay theorem (see [17]). We first extend the CR-diffeomorphisms $f_n : S \longrightarrow S'$ to holomorphic maps (still denoted by f_n) between domains Ω, Ω' obtained by filling S and S' with holomorphic discs. The fundamental difference with the global situation is the non-property, and consequently the lack of surjectivity, of these extensions f_n : $\Omega \longrightarrow \Omega'$: only the sequences of points approaching S are sent to points approaching $b\Omega'$. We then renormalize the sequence of maps f_n by post-componing them with anisotropic dilations centered at the cluster point a. We explain how to choose these dilations in section 2. Through these "lenses", the target domain Ω' looks like the euclidean ball of $\mathbb{C}^k$. This process provides us with a sequence of applications $F_n : \Omega \longrightarrow \widetilde{\Omega}_n$ ($\simeq B$) which is normal. After possible extraction, F_n converges to a holomorphic map $F : \Omega \longrightarrow \overline{B}$. The point is then to understand that F defines a biholomorphism between the source domain and an open set of the ball supported by an open portion of the sphere. The main difficulty, inexistant in the global setting of Wong-Rosay theorem, is precisely due to the non-surjectivity of the F_n. The non-degeneracy of F is no longer a simple matter of applying Hurwitz lemma but requires an analysis. For this, we establish in section 3 some metric properties for families of CR-maps. They are quantified versions of a more general principle : any family of CR maps between fixed hypersurfaces is either equicontinuous or dilating for a well-chosen metric. The final step (section 4) then consists in disqualifying all *a priori* possible reason for the limit map to degenerate. Thanks to the analysis of section 3, all these reasons are ruled out because they lead to a dilating behaviour of the F_n, which shows incompatible with the chosen renormalizations. We would like to notice that these techniques of "CR-explosions" proved also usefull for the injectivity problem considered in [16, 15].

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Notations and conventions :

- For $z \in \mathbb{C}^k$ ($k \geq 2$), we note $z = (z_1, z')$, where $z_1 \in \mathbb{C}$ and $z' \in \mathbb{C}^{k-1}$.
- For a smooth real hypersurface M in $\mathbb{C}^k$, we denote $\mathcal{SPC}(M)$ the set of strictly pseudoconvex points of M.
- When M is a (strictly) convex hypersurface in $\mathbb{C}^k$ and q is a point of M, we denote :

– $\vec{N}(q)$ the unit normal vector to M at q which points to the convex part of $\mathbb{C}^k \backslash M$,
– Λ_q the linear form $\Lambda_q(\cdot) = \langle \vec{N}(q), \cdot \rangle$,
– q_ε the point $q + \varepsilon \vec{N}(q)$,
– $B_\varepsilon(q)$ the ball centered at q_ε of radius ε; this ball is tangent to M at q,
– $B_\varepsilon^+(q) := B_\varepsilon(q) \cap \{\Lambda_q \geq \varepsilon\}$.

- When Ω and Ω' are domains in $\mathbb{C}^k$ and U is a part of $b\Omega$, we shall say that a map $F : \Omega \longrightarrow \Omega'$ is U-proper if $F \in \mathcal{C}^\infty(\Omega \cup U)$ and $F(U) \subset b\Omega'$.

- Finally, when D is a domain in $\mathbb{C}^k$ and $\lambda \in \mathbb{R}$, we denote

$$D_\lambda := D \cap \{\operatorname{Re} z_1 < \lambda\}$$
$$[bD]_\lambda := bD \cap \{\operatorname{Re} z_1 < \lambda\}$$

2 Renormalization of the sequence f_n.

Let us first describe and simplify the picture in theorem 1. Call a the accumulation point of (f_n) in S'. Let p be a point of S. The aim is to find a neighbourhood of p in S which is CR-diffeomorphic to an open set of the sphere. This problem does not depend on a holomorphic coordinates at p and a. Henceforth, the strictly pseudoconvex hypersurfaces S and S' can be supposed to be *convex* in neighborhoods U_1 and U' of p and a respectively. In the same way, we can suppose that $\Lambda_p = \Lambda_a = \operatorname{Re} z_1$, $U_1 = S \cap \{\operatorname{Re} z_1 < 1\}$ and $U' = S' \cap \{\operatorname{Re} z_1 < 1\}$. Then, there exist two bounded strictly convex domains Ω and Ω' such that $U_1 = [b\Omega]_1$ and $U' = [b\Omega']_1$ (see the list of notations above).

After eliminating finitely many terms in the sequence $(f_n)_n$, we can suppose that $f_n([b\Omega]_1) \subset U'$ for all $n \in \mathbb{N}$. The classical extension theorems for CR-maps show that f_n extends to a holomorphic map from Ω_1 to $\mathbb{C}^k$ (see [9], chap. 15). We still call this extension f_n. Since Ω'_1 has a *p.s.h* defining function ρ, the maximum principle apply to the restrictions of $\rho \circ f_n$ on the complex hyperplanes $\{z_1 = c\}$ ensures that $f_n(\Omega_1) \subset \Omega'_1$. A similar argument prove that (f_n) converges uniformly to a on Ω_1.

Finally, put $y_0 := (1/2, 0)$, $y_n := f_n(y_0)$, $p_n := f_n(p)$, so that y_n and p_n tend to a. Figure 1 illustrates the situation.

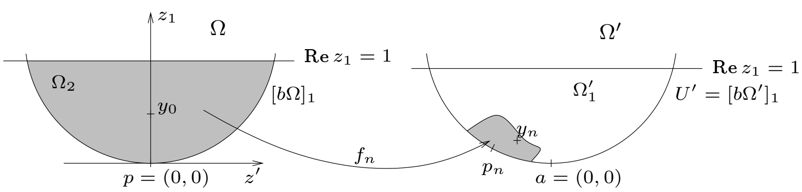

Figure 1: Geometric setting of theorem 1.

2.1 Dilations of coordinates.

We describe a technique of dilation of coordinates naturally associated to the pair (y_n, p_n), which is a slightly modified version of the techniques introduced by Pinchuk (see [17, 8]). The piece of surface U' can be supposed to possess an equation $\chi = 0$ with

$$\chi(z) := -\operatorname{Re} z_1 + \|z\|^2 + a(z)\|z\|^3 \text{ where } a(z) \in O(1).$$

The scaling process $\mathcal{S}_{q,\varepsilon}$ at point $q \in U'$ with parameter ε consists of a composition of two biholomorphisms of $\mathbb{C}^k$, Ψ_q and D_ε, and of a birational map Φ of $\mathbb{C}^k$:

$$\mathcal{S}_{q,\varepsilon} = \Phi \circ D_\varepsilon \circ \Psi_q,$$

• Ψ_q is composed of a translation which sends q to the origin and a unitary transformation which makes the tangent plane to $b\Omega$ at q horizontal (*i.e.* defined by $\operatorname{Re} z_1 = 0$). In these coordinates, $\chi_q := \chi \circ \Psi_q^{-1}$ is of the form :

$$\chi_q(z) = -\operatorname{Re} z_1 + Q_q(z) + a_q(z)\|z\|^3 \tag{1}$$

where Q_q is the real Hessian of χ_q at 0. As U' is convex, it is a positive-definite real-quadratic form on $\mathbb{C}^k = \mathbb{R}^{2k}$. The function $a_q(z)$ is bounded from above on Ω'_1, independantly of $q \in U'$ because $b\Omega'$ is $\mathcal{C}^3$. Notice that $\Psi_a = Id$, $Q_a = \|\cdot\|^2$ and that the maps $q \mapsto \Psi_q$ and $q \mapsto Q_q$ are continuous because of the regularity of U'.

• D_ε is the $\mathbb{C}$-linear anisotropic dilation

$$D_\varepsilon : (z_1, z') \longmapsto \left(\frac{z_1}{\varepsilon}, \frac{z'}{\sqrt{\varepsilon}}\right).$$

A simple computation shows that the defining function $\chi_{q,\varepsilon}$ of $D_\varepsilon \circ \Psi_q(U')$ defined by $\chi_{q,\varepsilon} := \frac{1}{\varepsilon}\chi_q \circ D_\varepsilon^{-1}$ is :

$$\chi_{q,\varepsilon}(z_1, z') = -\operatorname{Re} z_1 + Q_q\big((0, z')\big) + O(\sqrt{\varepsilon}).$$

• Φ is the Cayley transform

$$\Phi : (z_1, z') \longmapsto \left(\frac{z_1 - 1}{z_1 + 1}, \frac{2z'}{z_1 + 1}\right)$$

which sends $\Sigma := \{\operatorname{Re} z_1 \geq \|z'\|^2\}$ to $B := \{|z_1|^2 + \|z'\|^2 \leq 1\}$. Notice that $D_\varepsilon \circ \Psi_q(\Omega')$ lies in the half-space $\{\operatorname{Re} z_1 > 0\}$ because of the convexity of Ω'. The mapping $\mathcal{S}_{q,\varepsilon} = \Phi \circ D_\varepsilon \circ \Psi_q$ is thus well defined on Ω'.

We now define a sequence of scalings $\mathcal{S}_n := \mathcal{S}_{p_n,\varepsilon_n}$ associated to the sequence (p_n, y_n) (see figure 2). For this, observe that $\{\mathcal{S}_{q,\varepsilon}^{-1}(\{\operatorname{Re} z_1 = 0\}),\ \varepsilon > 0\}$ defines a foliation of $\mathbb{C}^k \backslash T_q^{\mathbb{C}} b\Omega'$ whose leaves are euclidean cylinders centered at $T_q^{\mathbb{C}} b\Omega'$:

$$\mathcal{F}_{q,\varepsilon} := \mathcal{S}_{q,\varepsilon}^{-1}(\{\operatorname{Re} z_1 = 0\}) = \{z \in \mathbb{C}^k,\ d(z, T_q^{\mathbb{C}} b\Omega') = c(\varepsilon)\}.$$

Since Ω' is convex, each point $z \in \Omega'$ belongs to a unique leaf $\mathcal{F}_{q,\varepsilon_q(z)}$, $\varepsilon_q(z) > 0$, justifying the following definition :

Definition 2.1. *The scaling $\mathcal{S}_n$ associated to the sequence (p_n, y_n) is defined by $\mathcal{S}_n := \mathcal{S}_{p_n,\varepsilon_n}$ where $\varepsilon_n := \varepsilon_{p_n}(y_n)$ is the unique real number such that $y_n \in \mathcal{F}_{p_n,\varepsilon_n}$. We set $F_n := \mathcal{S}_n \circ f_n$, $\widetilde{\Omega}_n := \mathcal{S}_n(\Omega')$ and $\widetilde{y}_n := \mathcal{S}_n(y_n)$.*

These choices provide us with a sequence of maps F_n such that (see figure 2) :

$$\begin{array}{rcccl} F_n & : & \Omega_2 & \longrightarrow & \widetilde{\Omega}_n \\ & & p & \longmapsto & a = (-1, 0) \\ & & y_0 & \longmapsto & \widetilde{y}_n \in \{\operatorname{Re} z_1 = 0\}. \end{array}$$

Notice that $\varepsilon_{p_n}(y_n)$ goes to 0 when p_n and y_n converge to a.

We say that a sequence of smooth domains D_n converges in the $\mathcal{C}^2$-sense to a domain D on a compact set K if $K \cap \overline{D_n}$ converges to $K \cap \overline{D}$ in the sense of the Hausdorf topology and if the defining functions $\chi_{i,n}$ of D_n on the open sets U_i of a covering of D converge in the $\mathcal{C}^2$-norm to defining functions χ_i of $D \cap U_i$.

It is well known that the sequence of domains $\widetilde{\Omega}_n$ converges to the ball B in the $\mathcal{C}^2$-sense on the compacts of $\mathbb{C}^{k+1} \backslash \{z_1 = 1\}$. The purpose of the next proposition is to control better this convergence. It will show useful in the fourth section.

Proposition 2.2. *The sequence of domains $(\widetilde{\Omega}_n)$ converge to B in the $\mathcal{C}^2$-sense on every compact sets of $\mathbb{C}^k \backslash \{(1,0)\}$.*

Proof : As ε_n goes to 0, the sequence of domains $D_{\varepsilon_n} \circ \Psi_{p_n}(\Omega')$ is known to converge to Σ in the $\mathcal{C}^2$-sense on any compact sets of $\mathbb{C}^k$ (see [17]). The domains $\widetilde{\Omega}_n$ being obtained from these by applying the Cayley transform, which sends the infinity to $\{z_1 = 1\}$, they obviously converge to B on any compact sets of $\mathbb{C}^k \backslash \{z_1 = 1\}$. It is thus sufficient to check that

$$\widetilde{\Omega}_n \cap \{|z_1 - 1| < \delta\} \subset B[(1,0), r(\delta)] \qquad \forall n \in \mathbb{N},$$

with $r(\delta)$ going to zero with δ. This last point results from the obvious observation that Ω' lies inside every parabola which is tangent to $b\Omega'$ of sufficiently small curvature. Precisely, there exists a constant $c > 0$ such that $\Psi_q(\Omega') \subset \{\operatorname{Re} z_1 > c\|z\|^2\}$ for all points $q \in b\Omega'$. Actually, one easily sees that $\Phi \circ D_{\varepsilon_n}(\{\operatorname{Re} z_1 > c\|z'\|^2\}) = E_c := \{|z_1|^2 + c\|z'\|^2 < 1\}$. The wanted inclusion thus immediately comes from

$$\widetilde{\Omega}_n \cap \{|z_1 - 1| < \delta\} \subset E_c \cap \{|z_1 - 1| < \delta\}.$$

□

The following picture describes the scaling process.

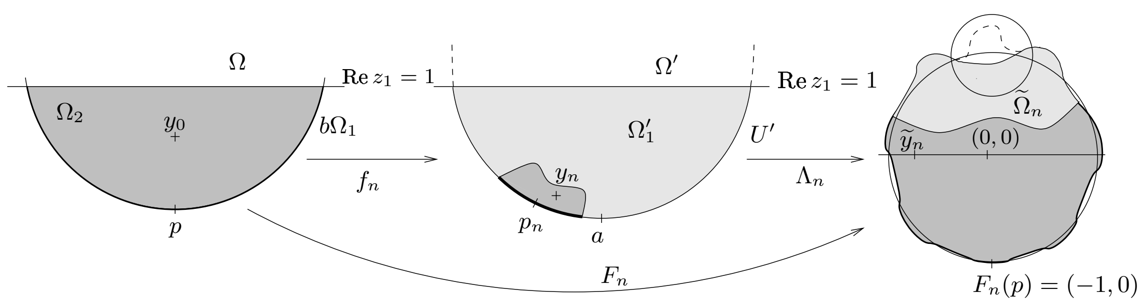

Figure 2: Situation after the n'th scaling.

Observe that the proposition 2.2 guarantees the normality of the family (F_n), as it implies that $\widetilde{\Omega}_n$ is a sequence of uniformly bounded domains.

Corollary 2.3. *The sequence (F_n) is a normal family. After possible extraction, we can assume that F_n converges uniformly to $F : \Omega_2 \longrightarrow \overline{B}$ on the compact sets of Ω_2. Moreover, the following alternative holds. Either F takes value inside B, or $F(\Omega_2)$ is a point of bB.*

Remark 2.4. *The image of U' by the scaling process at $q \simeq a$ of parameter $\varepsilon \ll 1$ is a hypersurface with boundary of $\mathbb{C}^k$. Its boundary is non-empty and belongs to a neighbourhood of $(1,0)$ whose diameter goes to zero with ε.*

2.2 Reduction of the problem.

The proof of theorem 1 consists in showing that F_n converges to a biholomorphism from a neighbourhood of p in $\overline{\Omega}$ to a neighbourhood of $(-1, 0)$ in $\overline{B}$. We explain here how to deduce it from some convergence properties of the images of F_n and of their inverses G_n. Let us first show how to define these inverses.

Lemma 2.5. *Let D, D' be two pseudoconvex domains in $\mathbb{C}^k$ with smooth boundaries. With the notation of p.3, assume that D_1 and D'_1 are connected non-empty sets, and $[bD]_1$, $[bD']_1$ are strictly convex. Then any holomorphic $[bD]_1$-proper map $F : D_1 \longrightarrow D'$ is a local diffeomorphism on D_1. Moreover, if $F([bD]_1) \supset [bD']_1$ and $F_{|[bD]_1}$ is injective, then $F(D_1) \supset D'_1$ and F has a right inverse $G : D'_1 \longrightarrow D_1$ (i.e. $F \circ G = Id_{D'_1}$).*

Proof : Were F not to be a local diffeomorphism on D_1, the critial set $\{\mathrm{Jac}\, F = 0\}$ would be an analytic set whose intersection with $[bD]_1 \subset \mathcal{SPC}(bD)$ would be empty [18]. Restricted to this analytic set, the function $-\mathrm{Re}\, z_1$ would then violate the maximum principle.

From the strict convexity of $[bD]_1$ and the injectivity of F on it, it follows that $F_{|[bD]_1}$ is a CR-diffeomorphism on its image, which contains $[bD']_1$. Therefore, there exists a CR-diffeomorphism $\overline{G}$ from $[bD']_1$ to $[bD]_1$ such that $F \circ \overline{G} = \mathrm{Id}$. This map $\overline{G}$ extends as a $[bD']_1$-proper holomorphic mapping $G : D'_1 \longrightarrow D$. From the maximum principle, its image is moreover obviously contained in D_1. The mapping $F \circ G$ is therefore well defined on D'_1, and coincides with the identity on $[bD']_1$. The unicity principle thus shows that $F \circ G = \mathrm{Id}_{D'_1}$. We conclude from this both that $F(D_1) \supset D'_1$ and that G is the right inverse of F on D'_1. □

This lemma guarantees the existence of an inverse G_n to F_n naturally defined on a maximal set of the form $\widetilde{\Omega}_{n,t}$. The following definition brings light to the properties of those applications, or rater of their images, which we will have to check.

Definition 2.6. *We denote by $\mathcal{K}(F_n)$, $\mathcal{O}(F_n)$, $\mathcal{K}(G_n)$, $\mathcal{O}(G_n)$ the following properties :*

- $\mathcal{K}(F_n)$ *: F_n converge to a map F et $F(\Omega_2) \subset B$,*
- $\mathcal{O}(F_n)$ *: there exist a positive real number λ such that $F_n(\Omega_1) \supset \widetilde{\Omega}_{n,\lambda}$ for all $n \in \mathbb{N}$,*
- $\mathcal{K}(G_n)$ *: G_n converge to a map G and $G(B_\lambda) \subset \Omega_2$,*
- $\mathcal{O}(G_n)$ *: there exist a monotone continuous function $\lambda' :]0, \lambda] \to \mathbb{R}_*^+$ such that*

$$G_n(\widetilde{\Omega}_{n,t}) \supset \Omega_{\lambda'(t)} \text{ for all } n.$$

We are now able to formulate the promised reduction :

Proposition 2.7. *If the renormalized sequence $(F_n)_n$ and the sequence of inverses $(G_n)_n$ check the properties $\mathcal{K}(F_n)$, $O(F_n)$, $\mathcal{K}(G_n)$, $O(G_n)$ then the limit map $F : \Omega_2 \to B$ extends to a CR-diffeomorphism between a neighbourhood of p in $[b\Omega]_1$ and a neighbourhood of $(-1, 0)$ in bB.*

Proof : This proposition is a consequence of classical theorems on extensions of biholomorphisms to strictly pseudoconvex points. It results from the following three facts :

$$F \text{ induces a biholomorphism } F : G(B_\lambda) \xrightarrow{\sim} B_\lambda, \tag{2}$$

$$\Omega_{\lambda'(\lambda)} \subset G(B_\lambda) \subset \overline{\Omega}, \tag{3}$$

$$F\big(\Omega_{\lambda'(t)}\big) \subset B_t \quad \text{for } t \le \lambda. \tag{4}$$

Actually, (4) implies that F has a continuous extension at p sending p to $(-1,0)$. The points (2) and (3) show that F is a biholomorphism between domains for which p and $(-1,0)$ are strictly pseudoconvex points. From [7], it follows that F has a Hölder continuous extension to a neighbourhood of p in $b\Omega$ into bB. Now a theorem of Bell [4] asserts that the smooth extension to strictly pseudoconvex points is a consequence of the continuous extension, so F extends actually to a CR-diffeomorphism from a neighbourhood of p in $b\Omega$ to a neighbourhood of $(-1,0)$ in bB. We thus only have to prove (2), (3) and (4).

From the hypothesis $O(F_n)$ and lemma 2.5, we know that G_n is defined on $\widetilde{\Omega}_{n,\lambda}$, where it checks :

$$F_n \circ G_n = \mathrm{Id}. \tag{5}$$

We deduce immediately from (5) and $O(G_n)$ that $F_n(\Omega_{\lambda'(t')}) \subset \widetilde{\Omega}_{n,t'}$ for $t' < t \leq \lambda$. In view of the hypothesis $\mathcal{K}(F_n)$, we get $F(\Omega_{\lambda'(t')}) \subset \overline{B}_{t'} \cap B \subset B_t$ when passing to the limit. The point (4) is then an obvious consequence of the continuity of λ' :

$$F\big(\Omega_{\lambda'(t)}\big) \subset B_t \quad \text{for } t \leq \lambda. \qquad (4)$$

We also get from (5) that $G_n \circ F_n = \mathrm{Id}$ on $G_n\big(\widetilde{\Omega}_{n,\lambda}\big)$ and therefore on $\Omega_{\lambda'(\lambda)}$ thanks to the hypothesis $O(G_n)$:

$$G_n \circ F_n = Id \quad \text{on } \Omega_{\lambda'(t)} \text{ for } t \leq \lambda. \tag{6}$$

Now the hypothesis $\mathcal{K}(G_n)$ allows to go to the limit in (5) because F_n converges uniformly to F on compact sets of Ω_2. We thus get :

$$F \circ G = \mathrm{Id} \quad \text{on } B_t \text{ for } t \leq \lambda. \tag{7}$$

Finally, the inclusion (4) enables to go to the limit in (6) :

$$G \circ F = \mathrm{Id} \ \text{ on } \ \Omega_{\lambda'(t)} \ \text{ for } \ t \leq \lambda. \tag{8}$$

The identity (7) then shows (2) while (4) and (8) show (3). ☐

3 CR-maps *vs* CR-dilations.

Our objective, which we defer to next section, is to check the compactness properties "$\mathcal{K}$" and "$\mathcal{O}$"for the sequences (F_n) and (G_n). We now explain estimates on derivatives and images of CR-maps which will enable us to control these sequences. The main result of this section is a precise realization of the following principle :

A sequence of CR maps between fixed hypersurfaces is either equicontinuous or dilating with respect to the CR-distance.

The CR-distance, which we introduce now, has the advantage of fitting much more to the CR-geometry than the euclidean distance, but still defining the same topology on strictly pseudoconvex hypersurface. Let X be a smooth submanifold of $\mathbb{C}^k$, x and y two points on X. We call complex path between x and y any piecewise $\mathcal{C}^1$-smooth path $\gamma : [0,l] \longrightarrow X$ joining x to y, and such that $\dot\gamma(t) \in T^{\mathbb{C}}_{\gamma(t)}S$ whenever it makes sense. The euclidean length of this path will be denoted $\ell(\gamma)$. We then define the CR-distance between two points x and y of X by :

$$d_X^{\mathrm{CR}}(x,y) := \inf\{\ell(\gamma),\ \gamma \text{ complex path between } x \text{ and } y\}.$$

We will not write explicitly the dependance of d^{CR} in X in non-ambiguous situations. The corresponding balls of center x and radius δ are denoted $B_X^{\text{CR}}(x,\delta)$.

The CR and euclidean distances are highly related in minimal hypersurfaces (*i.e.* which does not contain pieces of holomorphic hypersurfaces). In particular, large CR-balls contain large euclidean balls. For strictly pseudoconvex hypersurfaces, this property is a very simple particular case of theorem 4 in [14].

Proposition 3.1. *The topology induced by the CR-distance coincides with the usual topology for smooth strictly pseudoconvex hypersurfaces in $\mathbb{C}^k$. In particular, any bounded open set of such an hypersurface is d^{CR}-bounded.*

In view of this proposition, the principle stated above implies that images of non-equicontinuous sequences of CR-maps contain "large" euclidean balls. It relies on Hopf's lemma, which bounds from below the boundary derivatives of proper holomorphic maps.

Lemma 3.2 (Hopf). *Let D be a domain in $\mathbb{C}^k$ and $\chi \in \mathcal{C}^1(\overline{D})$ a p.s.h function, negative on D and which vanishes at $q \in bD$. Let ε such that $B_\varepsilon(q) \subset D$ and $M := \min\{|\chi(z)|, z \in B_\varepsilon^+(q)\}$. Then, $\chi(q_t) \leq -\dfrac{Mt}{8\varepsilon}$ for $t \leq \varepsilon$ and thus $\|\vec{\nabla}\chi(q)\| \geq \dfrac{M}{8\varepsilon}$.*

We now combine Hopf's lemma to a transfer of estimations on normal derivatives to complex tangential derivatives in order to get the precise version of the above principle which will prove usefull in our context. We refer the reader to p.3 for the notations used in the following statement.

Lemma 3.3. *Let Ω, $\widetilde{\Omega}$ be smoothly bounded domains in $\mathbb{C}^k$ and $U \Subset \mathcal{SPC}(b\Omega)$ be an open set. Let W be an open set of $\mathcal{SPC}(b\widetilde{\Omega})$ such that Λ_x is positive on $\widetilde{\Omega}$ for all $x \in W$. We denote κ_1 and κ_2 the smallest and largest eigenvalue of the Levi form on U and W respectively. Let $p \in U$ and ε, τ two positive real numbers such that $B^{CR}(p,\tau) \Subset U$ and $B_\varepsilon(q) \subset \Omega$ for all $q \in B^{CR}(p,\tau)$.*

Let $F : \Omega \longrightarrow \widetilde{\Omega}$ a U-proper holomorphic map such that $F(p) \in W$. We define :

$$M := \min\big\{|\Lambda_x(z)|,\ x \in W,\ z \in F\big(\bigcup_{q \in B^{CR}(p,\tau)} B_\varepsilon^+(q)\big)\big\}.$$

Then

$$F\big(B^{CR}(p,\tau)\big) \supset B_W^{CR}\left(F(p), \left[\frac{\kappa_1}{8\kappa_2}\frac{M}{\varepsilon}\right]^{\frac{1}{2}}\tau\right).$$

The proof is composed of two steps. In the first one, Hopf's lemma is used to bound from below the complex tangential derivatives of F on the inverse image of W. The second step consists in integrating these estimates.

Proof : Estimations on the derivatives of F. Let q be a point in U such that $F(q) \in W$. Let us show that :

$$\|F'(q)u\| \geq \left[\frac{\kappa_1}{8\kappa_2}\frac{M}{\varepsilon}\right]^{1/2}\|u\|, \qquad \forall u \in T_q^{\mathbb{C}}b\Omega. \tag{9}$$

Applying lemma 3.2 for the fonction $-\Lambda_{F(q)} \circ F$ lead to the estimate $\|\ \vec{\nabla}\ [\ \Lambda_{F(q)} \circ F](q)\ \| \ \geq\ \frac{M}{8\varepsilon}$. As $\vec{\nabla}\Lambda_{F(q)}\big(F(q)\big) = \vec{N}(F(q))$, we get :

$$n_q(F) := \langle F'(q)\vec{N}(q), \vec{N}(F(q))\rangle = d\Lambda_{F(q)}[F'(q)\vec{N}(q)] = \|\vec{\nabla}[\Lambda_{F(q)} \circ F](q)\| \geq \frac{M}{8\varepsilon}. \tag{10}$$

Let $\mathcal{L}(\varphi, a, u)$ denote the Levi form of a fonction φ at a point a, and applied to a vector u. Namely,

$$\mathcal{L}(\varphi, a, u) = \sum_{1\leq i,j\leq k} \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z_j}} u_i \overline{u_j}.$$

The boundary being smooth along W, there exists a defining function ψ of $\widetilde{\Omega}$ in a neighbourhood of $F(q)$, whose gradient can be assumed to be of norm 1 at $F(q)$: $\vec{\nabla}\psi(F(q)) = \vec{N}(F(q))$. As for (10), we see that $\|\vec{\nabla}\psi \circ F(q)\| = n_q(F)$. Moreover, $\psi \circ F$ is a defining function around q, in the neighbourhood of which $b\Omega$ is strictly pseudoconvex. In view of (10), we get for any vector $u \in T_q^{\mathbb{C}} b\Omega$:

$$\mathcal{L}(\psi \circ F, q, u) \geq \kappa_1 \|\vec{\nabla}\left[\psi \circ F\right](q)\| \|u\|^2 = \kappa_1 n_q(F) \|u\|^2 \geq \frac{\kappa_1 M}{8\varepsilon} \|u\|^2. \tag{11}$$

By the functoriality of the Levi form and the definition of κ_2, we have :

$$\mathcal{L}(\chi \circ F, q, u) = \mathcal{L}(\chi, F(q), F'(q)u) \leq \kappa_2 \|F'(q)u\|^2. \tag{12}$$

The inequalities (11) and (12) imply (9).

Conclusion. We now integrate the inequality (9). As $F(p)$ belongs to W it is enough to check that :

$$bF\left(B^{\mathrm{CR}}(p,\tau)\right) \cap B^{\mathrm{CR}}_{\widetilde{W}}\left(F(p), \left[\frac{\kappa_1}{8\kappa_2}\frac{M}{\varepsilon}\right]^{1/2} \tau\right) = \emptyset.$$

Or equivalently,

$$d^{\mathrm{CR}}_{W}(F(p), x) \geq \left[\frac{\kappa_1}{8\kappa_2}\frac{M}{\varepsilon}\right]^{\frac{1}{2}} \tau \ , \forall x \in bF(B^{\mathrm{CR}}(p,\tau)) \cap W. \tag{13}$$

Let x be such a point and $\widetilde{\gamma}$ a complex path of W between $F(p)$ and x, with an arc-length parametrization. Since F is a local CR-diffeomorphism at every point of $B^{\mathrm{CR}}(p,\tau)$, the connected component of $F(p)$ in $\widetilde{\gamma} \cap F(B^{\mathrm{CR}}(p,\tau))$ can be lifted to a complex path γ as long as γ does not get out of $B^{\mathrm{CR}}(p,\tau)$. Precisely, there exists $l \leq \ell(\widetilde{\gamma})$ and $\gamma : [0,l] \longrightarrow \overline{B^{\mathrm{CR}}(p,\tau)}$ connecting p to $bB^{\mathrm{CR}}(p,\tau)$ and such that $F \circ \gamma(t) = \widetilde{\gamma}(t)$ for all $t \in [0,l]$. As $\gamma(t)$ belongs to U and $F(\gamma(t))$ to W for all t, the estimate (9) obtained in the previous step shows that :

$$\|F'(\gamma(t))u\| \geq \left[\frac{\kappa_1}{\kappa_2}\frac{M}{8\varepsilon}\right]^{\frac{1}{2}} \|u\| \quad \forall t \in [0,l],\ \forall u \in T^{\mathbb{C}}_{\gamma(t)} b\Omega.$$

Henceforth,

$$\ell(\widetilde{\gamma}) \geq l = \int_0^l \|\dot{\widetilde{\gamma}}(t)\| dt = \int_0^l \|F'(\gamma(t))\dot{\gamma}(t)\| dt \geq \left[\frac{\kappa_1}{\kappa_2}\frac{M}{8\varepsilon}\right]^{\frac{1}{2}} \int_0^l \|\dot{\gamma}(t)\| dt \geq \left[\frac{\kappa_1}{\kappa_2}\frac{M}{8\varepsilon}\right]^{\frac{1}{2}} \ell(\gamma).$$

This last inequality shows (13) because γ connects p to $bB^{\mathrm{CR}}(p,\tau)$ and $\widetilde{\gamma}$ stands for any complex path from $F(p)$ to x. □

Lemma 3.3 enables to estimate the size of the image of a ball $B^{\mathrm{CR}}(p,\tau)$ by a proper holomorphic map in terms of its behaviour on

$$A_\varepsilon(p,\tau) := \bigcup_{q \in B^{\mathrm{CR}}(p,\tau)} B^+_\varepsilon(q).$$

The following proposition specifies lemma 3.3 to the three concrete configurations we will have to face when proving $\mathcal{K}(F_n)$, $\mathcal{K}(G_n)$, $O(F_n)$ and $O(G_n)$. Figure 3 illustrates these configurations and constitutes together with lemma 3.3 the proof of this proposition.

Proposition 3.4. *Let Ω, $\widetilde{\Omega}$ be domains in $\mathbb{C}^k$. Let U and W be open relatively compact sets of $\mathcal{SPC}(b\Omega)$ and $\mathcal{SPC}(b\widetilde{\Omega})$. We assume that $\Lambda_x \geq 0$ on $\widetilde{\Omega}$ for all $x \in W$.*

Fix points $p \in U$, $a \in W$ and real numbers $\varepsilon_0, \tau > 0$ such that $B^{CR}(p,\tau) \Subset U$ and $A_{\varepsilon_0}(p,\tau) \subset \Omega$.

Let $\mathcal{F}_{p,a}$ the family of U-proper holomorphic mappings $F : \Omega \longrightarrow \widetilde{\Omega}$ for which $F(p) = a$.

1. *To any compact set K of $\widetilde{\Omega}$ is associated a constant $c(K) > 0$ such that*

$$\forall F \in \mathcal{F}_{p,a} \;:\; F(A_{\varepsilon_0}(p,\tau)) \subset K \implies F(B^{CR}(p,\tau)) \supset B^{CR}_W(F(p), c\tau).$$

2. *Fix $\lambda > 0$ and $0 < \eta < \lambda$. There exists a constant $c(\eta) > 0$ such that*

$$\forall \varepsilon \leq \varepsilon_0, \;\; \forall F \in \mathcal{F}_{p,a} \;: \\ F(A_\varepsilon(p,\tau)) \subset \{\Lambda_a \geq \lambda\} \Longrightarrow F(B^{CR}(p,\tau)) \supset B^{CR}_W\big(a, \tfrac{c(\eta)\tau}{\sqrt{\varepsilon}}\big) \cap \{\Lambda_a \leq \lambda - \eta\}.$$

3. *Let $y \in W \backslash \{a\}$ and $\eta < d(a,y)$. There exists a constant $c(\eta) > 0$ such that :*

$$\forall \varepsilon \leq \varepsilon_0, \;\; \forall F \in \mathcal{F}_{p,a} \;: \\ F(A_\varepsilon(p,\tau)) \subset B(y,\eta) \Longrightarrow F(B^{CR}(p,\tau)) \supset B^{CR}_W\big(a, \tfrac{c(\eta)\tau}{\sqrt{\varepsilon}}\big) \backslash B(y, 2\eta).$$

Moreover, the constants $c(K)$ and $c(\eta)$ are only dependant on values of the Levi form on U and W. The points 1, 2 and 3 remain true for half constants when $\Omega, \widetilde{\Omega}$ are replaced by small perturbations (in Hausdorff sense) as long as U and W are slightly perturbed in the $\mathcal{C}^2$-sense.

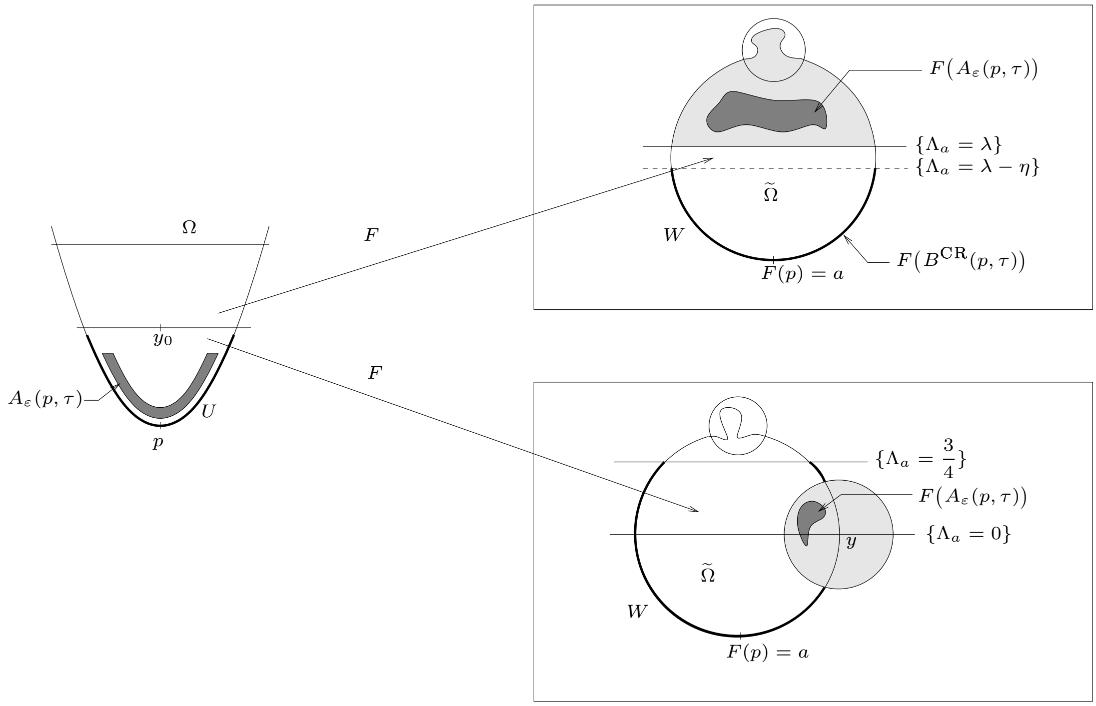

Figure 3: Situation 2 (top), situation 3 (bottom)

4 Proof of theorem 1.

We now use proposition 3.4 in order to check that the properties $\mathcal{K}(F_n)$, $O(F_n)$, $\mathcal{K}(G_n)$ and $O(G_n)$ hold. It will conclude the proof of theorem 1 *via* proposition 2.7. The whole proof is in fact based on the injectivity of F_n on the boundary and the particular choices we made for the renormalization.

Recall that $(\widetilde{\Omega}_n)$ converges to B in the $\mathcal{C}^2$-sense on any compact sets of $\mathbb{C}^2\backslash\{(1,0)\}$ (see proposition 2.2). Hence, for a large enough integer $n \geq n_0$, $[b\widetilde{\Omega}_n]_{3/4}$ is pseudoconvex and Λ_x is positive on $\widetilde{\Omega}$ for $x \in [b\widetilde{\Omega}_n]_{3/4}$. The domain Ω itself being fixed and strictly convex, proposition 3.4 then applies when $\{U, W\} := \{[b\Omega]_1, [b\widetilde{\Omega}_n]_{3/4}\}$.

4.1 Proof of "$\mathcal{K} \Longrightarrow O$".

$\underline{\mathcal{K}(F_n) \Longrightarrow O(F_n)}$: Apply point 1 of proposition 3.4 to F_n, with $U := b\Omega_1$ and $W_n := [b\widetilde{\Omega}_n]_{3/4}$. Since $F_n(p) = a$ and $F_n(A_{\varepsilon_0}(p,\tau))$ converges to $F(A_{\varepsilon_0}(p,\tau)) \Subset B$, in virtue of the hypothesis $\mathcal{K}(F_n)$, we get :

$$F_n\big(B^{\mathrm{CR}}(p,\tau)\big) \supset B^{\mathrm{CR}}_{W_n}(a, c\tau).$$

Recalling lemma 2.5, it is enough to check that there exists $\lambda > 0$ such that $B^{\mathrm{CR}}_{W_n}(a, c\tau) \supset [b\widetilde{\Omega}_n]_\lambda$ for all $n \in \mathbb{N}$. This last point results immediately from the fact that W_n converges to $[bB]_{3/4}$ in the $\mathcal{C}^2$-sense. □

In the next paragraph (4.2), we prove $\mathcal{K}(F_n)$ independantly of the rest of this paragraph. It will also establish $O(F_n)$. Once this property is known to hold, lemma 2.5 guarantees the existence of a right inverse G_n for F_n, defined on $\widetilde{\Omega}_{n,\lambda}$. The sequence of these applications is normal because it takes values in Ω_1. After possible extraction, it can be assumed to be converging to a map $G : B_\lambda \longrightarrow \overline{\Omega}_1$. As behind, we are going to show that if G takes value in Ω_2 then $O(G_n)$ holds.

$\underline{\mathcal{K}(G_n) \Longrightarrow O(G_n)}$: Apply point 1 of proposition 3.4 to G_n with $U_n := [b\widetilde{\Omega}_n]_\lambda$ and $W := [b\Omega]_2$. Since $G_n(p) = a$ and $G_n(A_{\varepsilon_0}(a,\tau))$ converges to $G(A_{\varepsilon_0}(p,\tau)) \Subset \Omega$ in virtue of the hypothesis $\mathcal{K}(G_n)$, we get :

$$G_n\big(B^{\mathrm{CR}}_{U_n}(a,t)\big) \supset B^{\mathrm{CR}}_W(p,ct) \qquad \forall t \leq \tau. \tag{14}$$

Since the CR-distance is bigger that the euclidean one, we have $B^{\mathrm{CR}}_{U_n}(a,t) \subset [b\widetilde{\Omega}_n]_t$. Moreover, $B^{\mathrm{CR}}(p,t) \supset [b\Omega]_{At^2}$ where A is the minimum of the eigenvalues of the Levi form on $[b\Omega]_1$ (it is the most simple case of theorem 4 in [14]). The property $O(G_n)$ then results from (14) and lemma 2.5. □

Replacing p by an arbitary point $q \in [b\Omega]_1$ in the proof of $\mathcal{K}(F_n) \Longrightarrow O(F_n)$, we get a slightly stronger property than $O(F_n)$ alone. It will also be usefull to show $\mathcal{K}(G_n)$.

Lemma 4.1. *If $\mathcal{K}(F_n)$ holds then there exist constants $c, \tau > 0$ such that for all $q \in [b\Omega]_1$ and n big enough,*

$$F_n(q) \in [b\widetilde{\Omega}_n]_{3/4} \Longrightarrow F_n\big(B^{CR}(q,\tau)\big) \supset B^{CR}_{[b\widetilde{\Omega}_n]_{3/4}}(F_n(q), c\tau).$$

4.2 Proof of $\mathcal{K}(F_n)$.

Because of the alternative which governs the convergence of (F_n) (see corollary 2.3), it is enough to show that in the situation described in section 2, $\widetilde{y}_n = F_n(y_0)$ remains in a compact subset of B. Let us proceed by contradiction and assume that $\widetilde{y}_n$ converges to $y_\infty \in bB$. The sequence (F_n) then converges locally uniformly to y_∞ (corollary 2.3). To any $\varepsilon > 0$ thus corresponds an integer n_ε such that $F_{n_\varepsilon}(A_\varepsilon(p,\tau)) \subset B(y_\infty, 1/4)$. Applying point 3 of proposition 3.4 to F_{n_ε} with $U := [b\Omega]_1$, $W_{n_\varepsilon} := [b\widetilde{\Omega}_{n_\varepsilon}]_{3/4}$, $\eta = 1/4$ et $(a,y) := (a, y_\infty)$, we get :

$$F_{n_\varepsilon}([b\Omega]_1) \supset B^{\mathrm{CR}}_{W_{n_\varepsilon}}\left(a, \frac{c(\eta)\tau}{\sqrt{\varepsilon}}\right) \backslash B(y_\infty, \frac{1}{2}). \tag{15}$$

Since the sets $[b\widetilde{\Omega}_n]_{3/4}$ converge to $[bB]_{3/4}$ in the $\mathcal{C}^2$-sense, their CR-diameters are uniformly bounded. It follows that for sufficiently small ε, (15) gives :

$$F_{n_\varepsilon}([b\Omega]_1) \supset [b\widetilde{\Omega}_{n_\varepsilon}]_{3/4} \backslash B(y_\infty, \frac{1}{2}). \tag{16}$$

From now on ε is fixed and we write n instead of n_ε. Since the map $F_{n|[b\Omega]_2}$ is a diffeomorphism on its image, $F_n([b\Omega]_1)$ is contractible. Then (16) shows that $F_n([b\Omega]_1)$ must contain $[b\widetilde{\Omega}_n]_{3/4}$. We thus get :

$$b\big[F_n([b\Omega]_1)\big] = F_n(b\Omega \cap \{\mathrm{Re}\, z_1 = 1\}) \subset [b\widetilde{\Omega}_n]^+_{3/4}. \tag{17}$$

This inclusion leads to a contradiction because the maximum principle applied to the function $-\mathrm{Re}\, z_1 \circ F_n$ shows that $F_n(\Omega \cap \{\mathrm{Re}\, z_1 = 1\}) \subset \widetilde{\Omega}^+_{n,3/4}$, when by construction $F_n(y_0) \in \{\mathrm{Re}\, z_1 = 0\}$. □

4.3 Proof of $\mathcal{K}(G_n)$.

Define $\lambda_n := \sup\{t \in \mathbb{R} \mid F_n(\Omega_1) \supset \widetilde{\Omega}_{n,t}\}$. Since F_n is a local diffeomorphism on Ω_1 (lemma 2.5), the boundary of $F_n(\Omega_1)$ in $\widetilde{\Omega}_n$ is a subset of $F_n(\Omega \cap \{\mathrm{Re}\, z_1 = 1\})$. By definition of λ_n,

$$\inf\big\{\mathrm{Re}\, z_1 \circ F_n(x),\ x \in \Omega \cap \{\mathrm{Re}\, z_1 = 1\}\big\} = \lambda_n.$$

Applying the maximum principle to $-\mathrm{Re}\, z_1 \circ F_n$ on $\Omega \cap \{\mathrm{Re}\, z_1 = 1\}$ shows henceforth :

$$\exists z_n' \in \{\mathrm{Re}\, z_1 = 1\} \cap b\Omega, \qquad F_n(z_n') \in \{\mathrm{Re}\, z_1 \leq \lambda_n\}. \tag{18}$$

Since $\mathcal{K}(F_n)$ and $O(F_n)$ are already checked, we know that all limits of λ_n are bigger to $\lambda > -1$. After possible extraction, the sequence of maps (G_n) henceforth converges to $G : B_{\lambda_\infty} \longrightarrow \overline{\Omega_1}$, where λ_∞ refers to one of the limits of λ_n which is of course greater than λ.

When λ_∞ is positive, the sequence $\widetilde{y}_n$ is compact in B_{λ_∞} (because $\mathcal{K}(F_n)$ holds). Moreover, $G_n(\widetilde{y}_n) = y_0 \in \Omega_2$. Hence the limit map G takes value in Ω_2 : $\mathcal{K}(G_n)$ is checked.

When λ_∞ is non-positive, we prove below the existence of a sequence (x_n), relatively compact in Ω_1, and a constant $\eta_0 > 0$ such that $F_n(x_n) \in \widetilde{\Omega}_{n,\lambda_\infty - \eta_0}$ for any big enough integer n. Again, $\mathcal{K}(F_n)$ implies that $\{F_n(x_n)\} \Subset B_{\lambda_\infty}$. This time, the conclusion follows from $\big\{G_n[F_n(x_n)]\big\} = \big\{x_n\big\} \Subset \Omega_2$.

Fix $\tau > 0$ such that $B^{\mathrm{CR}}(p,\tau) \Subset [b\Omega]_1$. In order to show the existence of (x_n), it is sufficient to see that there exist two positive constants $\varepsilon_0, \eta_0 > 0$ such that :

$$F_n(A_{\varepsilon_0}(p,\tau)) \cap \widetilde{\Omega}_{n,\lambda_n - \eta_0} \neq \emptyset \qquad \forall n \gg 1.$$

Arguing by contradiction, we assume that for all $\varepsilon, \eta > 0$, there exist an integer n_ε such that $F_{n_\varepsilon}(A_\varepsilon(p,\tau)) \subset \{\mathrm{Re}\, z_1 \geq \lambda_{n_\varepsilon} - \eta\}$. Applying point 2 of proposition 3.4 to F_{n_ε} with $U := [b\Omega]_1$, $W := [b\widetilde{\Omega}_n]_{1/2}$ and $(\lambda, \eta) := (\lambda_{n_\varepsilon}, \eta)$ we get :

$$F_{n_\varepsilon}\big(B^{\mathrm{CR}}(p,\tau/2)\big) \supset B^{\mathrm{CR}}_{W_{n_\varepsilon}}\left(a, \frac{c(\eta)\tau}{2\sqrt{\varepsilon}}\right) \cap \{\mathrm{Re}\, z_1 \leq \lambda_{n_\varepsilon} - \eta\}. \tag{19}$$

Since the sets $[b\widetilde{\Omega}_n]_{3/4}$ converge to $[bB]_{3/4}$ in the $\mathcal{C}^2$-sense, their CR-diameters are uniformly bounded. It follows that for ε small enough, (19) gives :

$$F_{n_\varepsilon}\big(B^{\mathrm{CR}}(p,\tau/2)\big) \supset [b\widetilde{\Omega}_{n_\varepsilon}]_{\lambda_{n_\varepsilon} - \eta}.$$

This last inclusion being valid for arbitrarily small η, one obtains after possible extraction :

$$d^{\mathrm{CR}}\left(bF_n\big(B^{\mathrm{CR}}(p,\tau/2)\big), [b\widetilde{\Omega}_n]^+_{\lambda_n}\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{20}$$

Moreover, we see with the help of lemma 4.1 that the CR-distance between the points of $bF_n(B^{\mathrm{CR}}(p,\tau))$ and those of $F_n(B^{\mathrm{CR}}(p,\tau/2))$ is bounded from below by $c\tau/2$. We thus get from (20) that $F_n(B^{\mathrm{CR}}(p,\tau))$ contains $[b\widetilde{\Omega}_n]_{\lambda_n}$ for big enough n. Together with (18), this last point opposes to the injectivity of $F_{n|U}$. □

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