

Outline

Part I: A
representation of
the braid group

Part II: Lefschetz
fibrations and
Morse theory

A representation of the braid group from Lefschetz fibrations

Alexandru Oancea (Strasbourg)

joint with Gwénaél Massuyeau (Strasbourg)
and Dietmar Salamon (Zurich)

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Outline of Part I

- 1 Generalities on the framed braid group
- 2 Group-ring valued cocycles
- 3 Linear representations from cocycles

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Part II: Lefschetz
fibrations and
Morse theory

Outline of Part II

- 4 Lefschetz fibrations and vanishing cycles
- 5 The Picard–Lefschetz formula
- 6 Morse theory: Real vs. Complex

A
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group from
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Generalities

1-cocycles

Linear repre-
sentations

Part I

A Representation of the Braid Group

The (framed) braid group

Z finite set $\subset \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, $z_0 \in \partial\mathbb{D}$, $m = \text{card}(Z)$

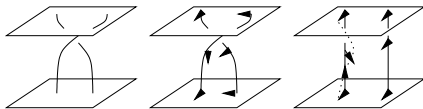
Definition (braid group)

$$\mathcal{B} := \pi_1(\text{Sym}_m(\overset{\circ}{\mathbb{D}}) \setminus \Delta, [Z])$$

Choose *nonzero* tangent vectors $v_z \in T_z\mathbb{D}$, $z \in Z$

Definition (framed braid group)

$$\tilde{\mathcal{B}} = \pi_1(\text{Sym}_m(\overset{\circ}{T\mathbb{D}} \setminus 0) \setminus \Delta, [Z, \{v_z\}])$$



Mapping class groups

$$\mathcal{G} := \{\phi \in \text{Diff}_0(\mathbb{D}, z_0) : \phi(Z) = Z\}$$

$$\mathcal{G}_0 := \{\phi \in \mathcal{G} : \exists \phi_t \in \mathcal{G} \text{ s.t. } \phi_0 = \text{Id}, \phi_1 = \phi\}$$

Proposition

There is an isomorphism $\mathcal{G}/\mathcal{G}_0 \simeq \mathcal{B}$.

$$\tilde{\mathcal{G}} := \{\phi \in \text{Diff}_0(\mathbb{D}, z_0) : \phi(Z) = Z, d\phi(z)v_z = v_{\phi(z)}, \forall z \in Z\}$$

$$\tilde{\mathcal{G}}_0 := \{\phi \in \tilde{\mathcal{G}} : \exists \phi_t \in \tilde{\mathcal{G}} \text{ s.t. } \phi_0 = \text{Id}, \phi_1 = \phi\}$$

Proposition

There is an isomorphism $\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_0 \simeq \tilde{\mathcal{B}}$.

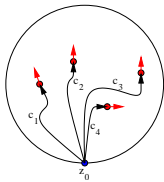
Distinguished configurations

Recall the choice of Z , z_0 , $\{v_z\}_{z \in Z}$, and $m := \text{card}(Z)$

Definition

A (*marked*) *distinguished configuration* is an m -tuple $c = (c_1, \dots, c_m)$ of smooth embedded paths $c_i : [0, 1] \rightarrow \mathbb{D}$ s.t.

- i. $c_i(0) = z_0$, and c_i intersects c_j only at z_0 for all $i \neq j$;
- ii. $\{c_1(1), \dots, c_m(1)\} = Z$;
- iii. the vectors $\dot{c}_1(0), \dots, \dot{c}_m(0)$ are pairwise non-collinear and ordered clockwise at $T_{z_0}\mathbb{D}$;
- iv. (*marked*) $\dot{c}_i(1) = -v_{c_i(1)}$.



Distinguished configurations

$\mathcal{C} = \{ \text{homotopy classes of distinguished configurations} \}$

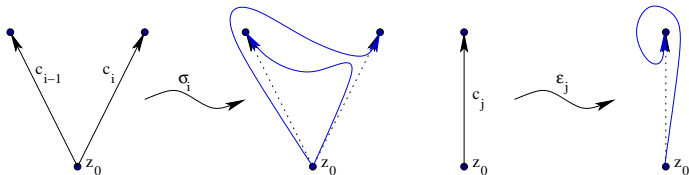
Proposition

The braid group $\mathcal{G}/\mathcal{G}_0$ acts freely and transitively on \mathcal{C} .

$\tilde{\mathcal{C}} = \{ \text{homotopy classes of marked distinguished configurations} \}$

Proposition

The framed braid group $\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_0$ acts freely and transitively on $\tilde{\mathcal{C}}$.



Generators and relations

$$\mathcal{B}_m := \left\langle \sigma_2, \dots, \sigma_m \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2 \end{array} \right\rangle$$

$$\tilde{\mathcal{B}}_m := \left\langle \begin{array}{l} \sigma_2, \dots, \sigma_m \\ \varepsilon_1, \dots, \varepsilon_m \end{array} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \varepsilon_i \sigma_i = \sigma_i \varepsilon_{i-1} \\ \varepsilon_{i-1} \sigma_i = \sigma_i \varepsilon_i \\ \text{other generators commute} \end{array} \right\rangle$$

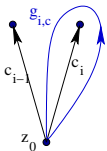
Proposition

$$\mathcal{B}_m \simeq \mathcal{B}, \quad \tilde{\mathcal{B}}_m \simeq \tilde{\mathcal{B}}.$$

The above isomorphisms depend on the choice of a (marked) distinguished configuration c !

The Monodromy cocycle

Fix $c \in \tilde{\mathcal{C}} \rightsquigarrow$ ordering $Z = \{z_1, \dots, z_m\}$



distinguished loops $g_{i,c} \in \Gamma = \pi_1(\mathbb{D} \setminus Z; z_0)$

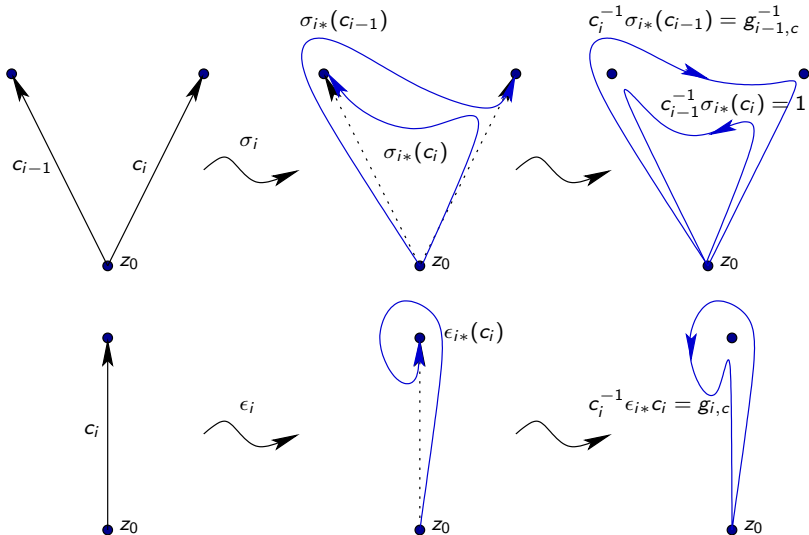
$\sigma \in \tilde{\mathcal{B}} \rightsquigarrow$ permutation $\pi_{\sigma,c} \in \mathfrak{S}_m$ via $\sigma(z_j) = z_{\pi_{\sigma,c}(j)}$

Definition (Massuyeau, O., Salamon)

Define $S_c : \tilde{\mathcal{B}} \rightarrow GL_m(\mathbb{Z}[\Gamma])$ by

$$S_c(\sigma)_{ij} := \begin{cases} c_i^{-1} \cdot \sigma_* c_j, & i = \pi_{\sigma,c}(j), \\ 0, & \text{otherwise.} \end{cases}$$

The Monodromy cocycle



Point of view: non-abelian cohomology

Definition

Let G and A be (non-abelian) groups, such that G acts on A on the left $(g, a) \mapsto g_* a$. A map $s : G \rightarrow A$ is a 1-cocycle if

$$s(gh) = s(g)g_*s(h).$$

Two cocycles $s_0, s_1 : G \rightarrow A$ are *cohomologous* if $\exists a \in A$ s.t.

$$s_1(g) = a^{-1}s_0(g)g_*a.$$

Example: $\tilde{\mathcal{B}}$ acts on Γ , hence on $GL_m(\mathbb{Z}[\Gamma])$ componentwise

Proposition

- (i) Each map $S_c : \tilde{\mathcal{B}} \rightarrow GL_m(\mathbb{Z}[\Gamma])$ is a 1-cocycle.
- (ii) Any two cocycles S_c and S_{τ_*c} are cohomologous:

$$S_{\tau_*c}(\sigma) = S_c(\tau)^{-1}S_c(\sigma)\sigma_*S_c(\tau), \quad \forall \sigma \in \tilde{\mathcal{B}}.$$

$[S_c]$ is canonically defined: the “Lefschetz monodromy class”

Proof of the cocycle properties

(i) We prove $S_c(\sigma\tau) = S_c(\sigma)\sigma_*S_c(\tau)$.

Set $j := \pi_{\tau,c}(k)$, $i := \pi_{\sigma,c}(j) = \pi_{\sigma\tau,c}(k)$. Then

$$c_i^{-1} \cdot \sigma_*c_j \cdot \sigma_*(c_j^{-1} \cdot \tau_*c_k) = c_i^{-1} \cdot \sigma_*\tau_*c_k = c_i^{-1} \cdot (\sigma\tau)_*c_k$$

(ii) We prove $S_c(\tau)S_{\tau_*c}(\sigma) = S_c(\sigma\tau) = S_c(\sigma)\sigma_*S_c(\tau)$.

Set $\ell := \pi_{\sigma,\tau_*c}(k)$ and $i := \pi_{\tau,c}(\ell) = \pi_{\sigma\tau,c}(k)$. Then

$$c_i^{-1} \cdot \tau_*c_\ell \cdot (\tau_*c)_\ell^{-1} \cdot \sigma_*(\tau_*c)_k = c_i^{-1} \cdot \sigma_*\tau_*c_k = c_i^{-1} \cdot (\sigma\tau)_*c_k$$

□

Proposition

For any $c \in \tilde{\mathcal{C}}$, the map $S_c : \tilde{\mathcal{B}} \rightarrow \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ is injective.

Proof.

$\tilde{\mathcal{B}}$ acts on $\tilde{\mathcal{C}}$ freely and transitively.

□

Digression: the Magnus cocycle

$\tilde{\mathcal{B}}$ acts on $\Gamma = \pi_1(\mathbb{D} \setminus Z; z_0)$ free on m generators $g_{1,c}, \dots, g_{m,c}$

Definition

Given $c \in \tilde{\mathcal{C}}$, the Magnus cocycle $M_c : \tilde{\mathcal{B}} \rightarrow \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ is

$$M_c(\sigma) := \left(\frac{\overline{\partial \sigma_* g_{j,c}}}{\partial g_{i,c}} \right)_{i,j=1,\dots,m}$$

Fox calculus for a free group Γ with basis g_1, \dots, g_m

- derivation $d : \mathbb{Z}[\Gamma] \rightarrow \mathbb{Z}[\Gamma]$ is additive homomorphism s.t.

$$d(gh) = d(g) \cdot \varepsilon(h) + g \cdot d(h), \quad \varepsilon : \mathbb{Z}[\Gamma] \rightarrow \mathbb{Z} \text{ augmentation}$$

- $d(1) = 0$, $d(g^{-1}) = -g^{-1}d(g)$

- $\exists!$ derivation $\frac{\partial}{\partial g_i}$ such that $\frac{\partial g_j}{\partial g_i} = \delta_i^j$

- example: $\frac{\partial (g_{i-1} g_i g_{i-1}^{-1})}{\partial g_{i-1}} = 1 + g_{i-1} \frac{\partial (g_i g_{i-1}^{-1})}{\partial g_{i-1}} = 1 - g_{i-1} g_i g_{i-1}^{-1}$

The Magnus cocycle – continued

Proposition

Any two cocycles M_c and M_{τ_*c} are cohomologous:

$$M_{\tau_*c}(\sigma) = M_c(\tau)^{-1} M_c(\sigma) \sigma_* M_c(\tau), \quad \forall \sigma \in \tilde{\mathcal{B}}.$$

Proof.

$$[M_c(\tau) M_{\tau_*c}(\sigma)]_{ik} = \sum_j \frac{\overline{\partial_{\tau_*} g_{j,c}}}{\partial g_{i,c}} \cdot \frac{\overline{\partial_{\sigma_* \tau_*} g_{k,c}}}{\partial_{\tau_*} g_{j,c}} = \frac{\overline{\partial_{(\sigma\tau)_*} g_{k,c}}}{\partial g_{i,c}} = M_c(\sigma\tau)_{ik}$$



Interesting (?) **question**: study $H^1(\tilde{\mathcal{B}}, \mathrm{GL}_m(\mathbb{Z}[\Gamma]))$.

Magnus vs. Lefschetz monodromy cocycles - first comparison

$$M_C(\sigma_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}, \quad S_C(\sigma_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}$$

$$\square = \begin{pmatrix} 1 - g_{i-1,c}^{-1} g_{i,c}^{-1} g_{i-1,c}^{-1} & 1 \\ g_{i-1,c}^{-1} & 0 \end{pmatrix}, \quad \square = \begin{pmatrix} 0 & 1 \\ g_{i-1,c}^{-1} & 0 \end{pmatrix}$$

$$M_C(\varepsilon_{i,c}) = \mathbb{1}_m, \quad S_C(\varepsilon_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-1} & 0 & 0 \\ 0 & g_{i,c} & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}$$

View $\mathcal{B} \hookrightarrow \tilde{\mathcal{B}}$ using framing determined by trivialization of $T\mathbb{D}$
 \rightsquigarrow **restriction of the Magnus cocycle $M_C|_{\mathcal{B}}$ is injective**

The Burau representation

$$\text{Recall action of } \tilde{\mathcal{B}} \text{ on } \Gamma: \quad (\sigma_i)_* : \begin{cases} g_{i-1} \mapsto g_{i-1}g_i g_{i-1}^{-1}, \\ g_i \mapsto g_{i-1}, \\ g_j \mapsto g_j, \quad j \neq i-1, i \end{cases}$$
$$(\varepsilon_i)_* = \text{Id}$$

Definition

For $c \in \tilde{\mathcal{C}}$, the Burau representation $\overline{M}_c : \tilde{\mathcal{B}} \rightarrow \text{GL}_m(\mathbb{Z}[t, t^{-1}])$ is obtained by reducing the Magnus cocycle

$$\overline{M}_c(\sigma) := M_c(\sigma) \Big|_{g_{1,c}=\dots=g_{m,c}=t^{-1}}$$

Remarks on \overline{M}_c :

- independent of $c \in \tilde{\mathcal{C}}$ up to conjugation
- reducible $1 + (m-1)$, with eigenvector $(1, t, t^2, \dots, t^{m-1})$
- defines polynomial link invariant: **Alexander polynomial**
(invariance under Markov moves)

The Monodromy representation

Definition (MOS)

For $c \in \tilde{\mathcal{C}}$, the Lefschetz monodromy representation

$$\bar{S}_c : \tilde{\mathcal{B}} \rightarrow \mathrm{GL}_m(\mathbb{Z}[t, t^{-1}])$$

is obtained by reducing the Lefschetz monodromy cocycle

$$\bar{S}_c(\sigma) := S_c(\sigma)|_{g_{1,c}=\dots=g_{m,c}=t^{-1}}$$

Note: independent of $c \in \tilde{\mathcal{C}}$ up to conjugation

does not define polynomial link invariant

Comparison of the Burau and Monodromy representations

$$\overline{M}_c(\sigma_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}, \quad \overline{S}_c(\sigma_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-2} & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}$$

$$\square = \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}, \quad \square = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

$$\overline{M}_c(\varepsilon_{i,c}) = \mathbb{1}_m, \quad \overline{S}_c(\varepsilon_{i,c}) = \begin{pmatrix} \mathbb{1}_{i-1} & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & \mathbb{1}_{m-i} \end{pmatrix}$$

$$\begin{array}{ccccccc} m & \neq & m-2 & \neq & m-1-t & \neq & m \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{tr}(\text{Id}) & & \text{tr}(\overline{S}_c(\sigma_i)) & & \text{tr}(\overline{M}_c(\sigma_i)) & & \end{array}$$

Corollary

The Magnus class $[M_c]$ and the Lefschetz monodromy class $[S_c]$ are non-trivial and distinct.

Irreducibility of the monodromy representation

Proposition

The representation \overline{S}_c is irreducible.

Proof.

Step 1. \overline{S}_c cannot fix a subspace of dimension 1.

$$\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{cases} b = \lambda a \\ ta = \lambda b \end{cases} \Rightarrow \begin{cases} ta = \lambda^2 a \\ a, b = 0 \end{cases}$$

Step 2. \overline{S}_c cannot be reducible $m = k + \ell$, $k, \ell \geq 2$. Assuming this to be true, we further specialize $t = 1$, and \overline{S}_c descends to the canonical representation $\mathfrak{S}_m \rightarrow \mathrm{GL}_m(\mathbb{Z})$ which is reducible $m = 1 + (m - 1)$. The $(m - 1)$ -dim. factor is the restriction to $x_1 + \cdots + x_m = 0$ and is irreducible, a contradiction. \square

The monodromy representation and linking numbers

View $\mathcal{B} \hookrightarrow \tilde{\mathcal{B}}$ via trivialization of $T\mathbb{D} \rightsquigarrow S_c : \mathcal{B} \rightarrow \mathrm{GL}_m(\mathbb{Z}[\Gamma])$

Proposition

Let $\sigma \in \mathcal{PB} \subset \mathcal{B}$ be a pure braid. Then

$$S_c(\sigma) = \mathrm{Diag}(t^{\ell_1}, \dots, t^{\ell_m}), \quad \ell_i = \sum_{j>i} \mathrm{lk}(i, j) + \sum_{j<i} \mathrm{lk}(j, i),$$

$\mathrm{lk}(i, j) =$ linking number of components i, j of the closed braid.

Proof: direct computation using presentation of \mathcal{PB} and

$$S_c(\sigma_{i,c}^2) = \mathrm{Diag}(1, \dots, \underset{i-1}{t}, \underset{i}{t}, \dots, 1)$$



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Lefschetz
fibrations

Picard–
Lefschetz
formula

Morse theory:
 \mathbb{R} vs. \mathbb{C}

Part II

Lefschetz fibrations and Morse theory

Lefschetz fibrations - local model

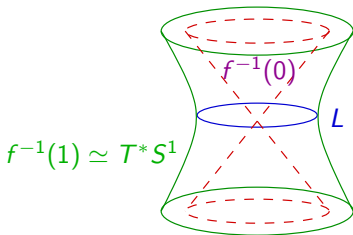
$$f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}, f(z) = z_1^2 + \cdots + z_{n+1}^2$$

$$f^{-1}(1) = \{x + iy : |x|^2 - |y|^2 = 1, \langle x, y \rangle = 0\}$$

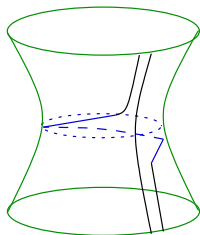
$$\parallel (x, y) \mapsto (x/|x|, |x|y) \quad \text{symplectic}$$

$$T^*S^n = \{(\xi, \eta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |\xi| = 1, \langle \xi, \eta \rangle = 0\}$$

Lefschetz fibrations - local model



$$f(z_1, z_2) = z_1^2 + z_2^2$$



Dehn twist along L

Monodromy around $t \mapsto e^{2\pi it} =$ **Dehn twist**.

Vanishing cycle at 0 in the direction $\mathbb{R}^{\geq 0}$ is the 0-section S^n

self-intersection number is $2(-1)^{n/2}$ for even n , 0 for odd n

Lefschetz fibrations

Definition

A **Lefschetz fibration** over the disc $\mathbb{D} \subset \mathbb{C}$ is a holomorphic map $f : X \rightarrow \mathbb{D}$ with nondegenerate critical points, and which correspond to distinct critical values in $\overset{\circ}{\mathbb{D}}$.

X **Kähler**, $\dim X = n + 1$, $m := \text{card}(Z = \text{Crit.val.}(f))$

Fix $z_0 \in \partial\mathbb{D}$ and $\{v_z \in T_z\mathbb{D}\}_{z \in Z}$. Choice of $c \in \tilde{\mathcal{C}}$

\rightsquigarrow vanishing cycles $L_{1,c}, \dots, L_{m,c} \subset M := f^{-1}(z_0)$

(local model + parallel transport by **canonical connection**)

Orientations for $L_{i,c} \rightsquigarrow$ **monodromy character**

$$\mathcal{N}_c^X : \Gamma = \pi_1(\mathbb{D} \setminus Z; z_0) \rightarrow \mathbb{Z}^{m \times m}$$

$$\mathcal{N}_c^X(g)_{ij} := \langle L_{i,c}, g_* L_{j,c} \rangle$$

The Lefschetz monodromy cocycle

Proposition (MOS)

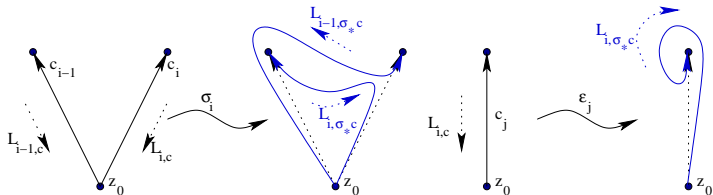
Given $\sigma \in \tilde{\mathcal{B}}$ we have

$$\mathcal{N}_{\sigma_*c}^X = S_c(\sigma)^t \mathcal{N}_c^X S_c(\sigma).$$

(conjugate transpose + convolution product)

Proof.

We have $L_{i,\sigma_*c} = (c_{i'}^{-1} \cdot \sigma_*c_i)^{-1} L_{i',c}$, $i' = \pi_{\sigma,c}(i)$ etc. □



The Picard–Lefschetz formula

Recall $f : X \rightarrow \mathbb{D}$ Lefschetz fibration, $z_0 \in \partial\mathbb{D}$, $M := f^{-1}(z_0)$
 $c \in \tilde{\mathcal{C}} \rightsquigarrow$ loops $g_{1,c}, \dots, g_{m,c} \in \Gamma \rightsquigarrow (g_{i,c})_* \in \text{Aut}(H_n(M))$

Proposition (Picard–Lefschetz formula)

Setting $\varepsilon = (-1)^{n(n+1)/2}$, we have

$$(g_{i,c})_* \alpha = \alpha - \varepsilon \langle L_{i,c}, \alpha \rangle L_{i,c}, \quad \forall \alpha \in H_n(M).$$

- $\mathcal{N}_c^X(g g_{i,c} h) = \mathcal{N}_c^X(gh) - \varepsilon \mathcal{N}_c^X(g) E_i \mathcal{N}_c^X(h)$
- since Γ is free on the $g_{i,c}$'s, the map $\mathcal{N}_c^X : \Gamma \rightarrow \mathbb{Z}^{m \times m}$ is uniquely (and explicitly) determined by the matrix

$$N_c^X := \mathcal{N}_c^X(1)$$

Nonlinear monodromy cocycles

The matrix $N_c = (n_{ij}) = N_c^X$ has the following properties.

- $N_c^T = (-1)^n N_c$
- $n_{ij} = \begin{cases} 0, & n \text{ odd,} & \text{(anti-symmetric)} \\ 2 \cdot (-1)^{n/2}, & n \text{ even.} & \text{(symmetric)} \end{cases}$

Let $\mathcal{S}_m = \mathcal{S}_m(n) =$ the set of such matrices $\subset \text{Mat}_m(\mathbb{Z})$.

Proposition (Bondal, MOS)

For each $c \in \tilde{\mathcal{C}}$ there is a map $S_c : \tilde{\mathcal{B}} \times \mathcal{S}_m \rightarrow \text{GL}_m(\mathbb{Z})$ such that

$$N_{\sigma_* c} = S_c(\sigma, N_c)^T N_c S_c(\sigma, N_c), \quad \forall \sigma \in \tilde{\mathcal{B}},$$

and moreover $S_c(\sigma\tau, N_c) = S_c(\tau, N_c) S_c(\sigma, N_{\tau_* c})$.

Conclusion: the orbit of N_c under conjugation with $S_c(\cdot, N_c)$ is an invariant of f .

Straight lines

Previous procedure: consider intersection numbers of vanishing cycles in the distinguished fiber $M = f^{-1}(z_0)$.

Alternative procedure: consider intersection number of two vanishing cycles L_i, L_j along some **arbitrary** path γ_{ij} joining the corresponding critical values \rightsquigarrow **collection of numbers $m_{ij} \in \mathbb{Z}$**

Particular case: $\gamma_{ij} =$ straight line segment \rightsquigarrow numbers $\ell_{ij} \in \mathbb{Z}$.

Straight lines - continued

Proposition (MOS)

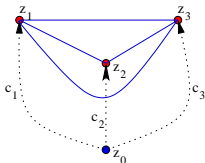
The collection (ℓ_{ij}) of intersection numbers along straight lines determines uniquely the matrices N_c , $c \in \tilde{\mathcal{C}}$ via the *Picard–Lefschetz formula*.

Proof.

In the figure, we have

$$\ell_{13} = \langle L_1, (g_2)_* L_3 \rangle = \langle L_1, L_3 - \varepsilon \langle L_2, L_3 \rangle L_2 \rangle = n_{13} - \varepsilon n_{12} n_{23}.$$

+ induction argument □

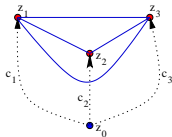


A dictionary

\mathbb{R}	\mathbb{C}
Morse function $f : M \rightarrow \mathbb{R}$	Lefschetz fibration $f : X \rightarrow \mathbb{C}$
$\{f \leq c - \rho\} \rightsquigarrow \{f \leq c + \rho\}$	monodromy of loop $c + \rho e^{2\pi i t}$
unstable mfd. $W^u(x, -\nabla f)$	unstable mfd. $W^u(x, -\nabla \operatorname{Re}(e^{-i\theta} f))$
connecting trajectories $\mathcal{M}(x, y; -\nabla f)$	connecting trajectories $\mathcal{M}(x, y; -\nabla \operatorname{Re}(e^{-i\theta} f))$ $\theta = \operatorname{Arg}(f(y) - f(x))$
$\#\mathcal{M}(x, y; -\nabla f)$	$L_x \cdot L_y$ along segment $[f(x), f(y)]$
Morse differential $\partial x = \sum_y \#\mathcal{M}(x, y)y$	monodromy character $\mathcal{N} \equiv \mathcal{N}(1) \equiv \text{collection } \{\ell_{xy}\}$
$\partial \circ \partial = 0$	Picard-Lefschetz formula $\ell_{13} = n_{13} - \varepsilon n_{12} n_{23}$

Homology

\mathbb{R}	\mathbb{C}
Morse homology $\simeq H_*(X)$	$\tilde{\mathcal{B}}$ -orbit of N
invariant under deformations f_t through smooth functions	invariant under deformations f_t through Lefschetz fibrations



Example of an invariant of the $\tilde{\mathcal{B}}$ -orbit of N :

$(\mathbb{Z}^m / \ker N, N)$ free abelian group + nondeg. bilinear form

Our hope of a dictionary

A
representation
of the braid
group from
Lefschetz
fibrations

A. Oancea,
joint with
G. Massuyeau
and
D. Salamon

Lefschetz
fibrations

Picard–
Lefschetz
formula

Morse theory:
 \mathbb{R} vs. \mathbb{C}

\mathbb{R} exact mfd. $(M, \omega = d\lambda)$

Floer theory = Morse th.
action functional on loop space

\mathbb{C} exact mfd. $(X, I, \omega^\theta = d\lambda)$

Floer theory = Picard–Lefschetz th.
holomorphic action functional

▶ The End

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Thank you for your attention and patience!