AN INTRODUCTION TO THE LERAY-SERRE SPECTRAL SEQUENCE AND ITS CONSTRUCTION IN MORSE HOMOLOGY

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Abstract. This short paper is intended as an advertisement for spectral sequences. We present a few classical spectacular computations of cohomological rings and construct the Leray-Serre spectral sequence for a smooth fibration in the setting of Morse homology.

Contents

1. Introduction

Spectral sequences are famous for being an exotic and difficult tool. The purpose of this short paper is to advertise for them and show that they are amazingly efficient and easy to use (Sections 2 and 3). The idea of presenting spectral sequences from the user's point of view is borrowed from McCleary and this paper is also a means to express my gratitude for his wonderful book [6] and the smooth and encouraging introduction it provided to a field that seemed hostile at first sight. I give a construction of the Leray-Serre spectral sequence within the setting of Morse homology (Sections 4 to 6). This constituted part of my doctoral dissertation [8]. Although this approach is somewhat longer than the classical construction of Serre because it asks for the additional presentation of Morse homology in Section 5, I chose to explain it because it serves as a motivation for similar constructions in Floer homology, which are beyond the scope of this paper.

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2. The Tool

2.1. **Literary definition.** A spectral sequence should be viewed as a book consisting of pages E_r, $r \geq 0$. The term E_0 is the *cover*, the term E_1 is the *foreword*, while the true story begins at term E_2. This is because the first two pages are usually not intrinsic to the given problem, whereas the subsequent ones are (one speaks in this case of *functoriality* of the given spectral sequence). The pages are algebraic objects (differential complexes) and there is a notion of *turning the page*, which means computing the homology of the corresponding complex. Just like the number of pages in a book is finite, the sequence E_r stabilizes (in a suitable sense) for r big enough. Rather than a novel, in which the scene becomes more and more crowded as the action evolves, our book is more like an "Elizabethan drama [...] in which the business of each character is to kill at least one other character, so that at the end of the play one has the stage strewn with corpses and only one actor left alive" (J.F. Adams, quoted in [6, §6]). We shall see instances of this pattern on the very next page of this paper.

2.2. **Half-rigorous definition.** Each term of a spectral sequence is a bigraded differential module $(E_r^{p,q}, d_r)$, with differential d_r of bidegree $(r, 1-r)$, such that

$$E_{r+1} \simeq H^*(E_r) \ .$$

The Leray-Serre spectral sequence is supported in the first quadrant and therefore $E_r^{p,q}$ stabilizes for $r \geq \max\{p, q\} + 1$. We denote the stable limit of $E_r^{p,q}$ by $E_\infty^{p,q}$. We say that the spectral sequence converges to the graded module H^* if there exists a decreasing filtration

$$H^k = F_0^k \supset \cdots \supset F_p^k \supset \cdots \supset 0$$

such that

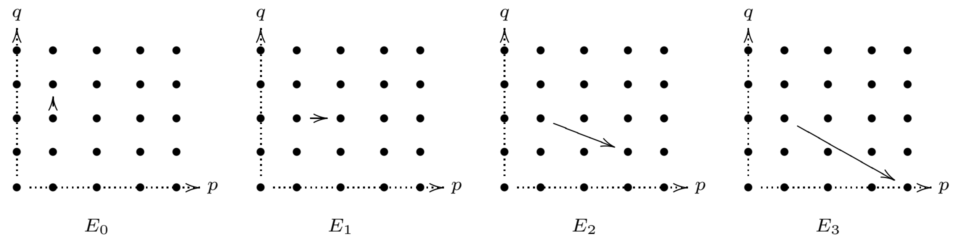
$$E_\infty^{p,q} \simeq F_p^{p+q} / F_{p+1}^{p+q} \ . \tag{1}$$

One subtlety of this notion of convergence is that E_∞ does not uniquely determine the limit H^*. As an example, the two filtrations $0 \subset \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z}$ and $0 \subset \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ have the same successive quotients, although the groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are not isomorphic. Nevertheless, it is true that E_∞ determines H^* in the category of vector spaces, in which case we have

$$H^k \simeq \bigoplus_{p+q=k} E_\infty^{p,q} \ . \tag{2}$$

Formulas (1) and (2) should be understood as follows: the index p stands for the *depth of the filtration*, the sum $p+q$ stands for the *total degree*, and the limit H^* is read on the diagonals of the page E_∞. We shall come back to this in Section 4, where we shall also explain why the bidegree of the differential is $(r, 1-r)$.

The successive pages of a spectral sequence are usually depicted as grids whose points represent the terms $E_r^{p,q}$. The following diagrams show the first differentials.



Spectral sequences were first described by Leray [4], but the algebraic formalism and the homotopical setting were laid down by Serre [11] in his Ph.D. dissertation.

Theorem 2.1. *Let $F \hookrightarrow X \longrightarrow B$ be a fibration, with F connected and B path-connected and simply connected. There exists a spectral sequence E_r converging to the singular cohomology $H^*(X)$ (with arbitrary coefficients) such that*

$$E_2^{p,q} \simeq H^p(B;\, H^q(F)) \ . \tag{3}$$

Moreover, the spectral sequence is compatible with the product structure on cohomology, i.e. there exist bilinear products

$$E_r^{p,q} \times E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$

which are induced by the cup-product on E_2 and such that d_r, $r \geq 2$ is a derivation:

$$d_r(xy) = d_r(x)y + (-1)^{p+q} x d_r(y) \ .$$

One of the main issues in the statement of the theorem is that of *the right definition for a fibration*. In this paper we shall refer only to (smooth) locally trivial fibrations, but one of Serre's insights was that the definition should be stated in homotopical terms alone. The statement of the theorem holds for *Serre fibrations*, which are spaces satisfying a certain homotopy lifting property. One of the most important examples is that of *the path-space fibration* $\Omega B \hookrightarrow \mathcal{P}B \xrightarrow{\pi} B$, where $(B, *)$ is a pointed space, $\mathcal{P}B$ is the space of paths starting at $*$, ΩB is the space of loops based at $*$ and π associates to every path its endpoint.

The theorem admits a formulation for bases which are not simply connected (cohomology with local coefficients), as well as a formulation in homology without the multiplicative property and with arrows going in the reverse direction. We shall prove in Section 6 the existence (but not the multiplicativity) of the spectral sequence for a smooth fibration.

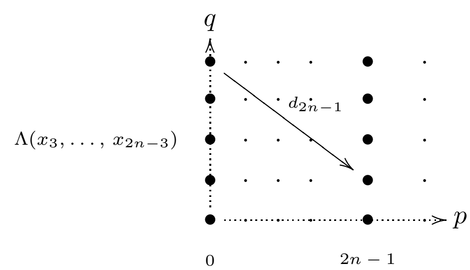
3. The Craft

This section features some computations of cohomology rings which use the spectral sequence tool. It is worth emphasizing that the latter can be used efficiently even without going into the algebraic details of its construction, just like the long exact sequence for the (co)homology of a pair is used without referring to the diagram chasing argument involved in its construction.

(I) *Computation of $H^*(U(n);\mathbb{Z})$.* We show that the cohomology ring of the unitary group $U(n)$ is isomorphic to the exterior algebra $\Lambda(x_1,\, x_3, \ldots,\, x_{2n-1})$ with generators x_{2k-1}, $1 \leq k \leq n$ of degree $\deg(x_{2k-1}) = 2k-1$.

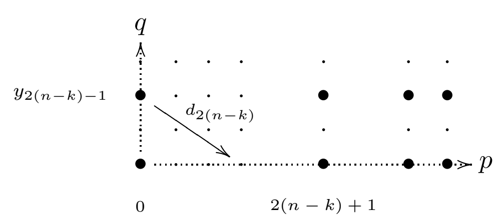
Because $U(n) \simeq \mathbb{S}^1 \times SU(n)$ and because of the Künneth formula, it is enough to show that $H^*(SU(n))$ is isomorphic to $\Lambda(x_3,\, x_5, \ldots,\, x_{2n-1})$. We prove this by induction. For $n = 2$ we have $SU(2) = \mathbb{S}^3$ hence $H^*(SU(2)) \simeq \Lambda(x_3)$. Assume now $H^*(SU(n-1)) \simeq \Lambda(x_3, x_5, \ldots, x_{2n-3})$. There is a (locally trivial) fibration $SU(n-1) \hookrightarrow SU(n) \xrightarrow{\pi} \mathbb{S}^{2n-1}$, where π associates to a matrix its first column vector. By the universal coefficient theorem we have $E_2^{p,q} \simeq H^p(\mathbb{S}^{2n-1}) \otimes H^q(SU(n-1))$ and the induction hypothesis gives $E_2^{*,*} \simeq \Lambda(x_3, \ldots, x_{2n-3}, y_{2n-1})$. For bidegree reasons the only differential that can be nonzero is d_{2n-1}, but the latter clearly verifies $d_{2n-1}(x_{2k-1}) = 0$, $1 \le k \le n-1$. By the multiplicativity property of the spectral sequence we have $d_{2n-1} = 0$ on $\Lambda(x_3, \ldots, x_{2n-3})$, hence $d_{2n-1} = 0$. This shows that $E_2 \simeq E_\infty$ and finishes the proof in the case of field coefficients. On the other hand, the fact that E_∞ is a free graded-commutative bigraded algebra ensures that the conclusion holds even with integer coefficients [6, 1.K].

$$\begin{array}{ccc} SU(n-1) & \hookrightarrow & SU(n) \\ & & \downarrow \pi \\ & & \mathbb{S}^{2n-1} \end{array}$$



(II) *Computation of* $H^*(V_k(\mathbb{C}^n))$. Here $V_k(\mathbb{C}^n)$ is the space of orthonormal k-frames in $\mathbb{C}^n$, or the *Stiefel variety*. By an argument entirely similar to the previous one we show that $H^*(V_k(\mathbb{C}^n)) \simeq \Lambda(x_{2(n-k)+1}, \ldots, x_{2n-1})$. The proof goes by induction on k. For $k = 1$ we have $V_1(\mathbb{C}^n) = \mathbb{S}^{2n-1}$, hence $H^*(V_1(\mathbb{C}^n)) \simeq \Lambda(x_{2n-1})$. The proof of the induction step uses the Leray-Serre spectral sequence of the fibration $\mathbb{S}^{2(n-k)-1} \hookrightarrow V_{k+1}(\mathbb{C}^n) \xrightarrow{\pi} V_k(\mathbb{C}^n)$, where π associates to a $k+1$-frame the k-frame consisting of its first k vectors. We have $E_2^{*,*} \simeq \Lambda(y_{2(n-k)-1}, x_{2(n-k)+1}, \ldots, x_{2n-1})$ and the only differential which might be nonzero is $d_{2(n-k)}$. On the other hand, for dimensional reasons we have $d_{2(n-k)}(y_{2(n-k)-1}) = 0$, hence $d_{2(n-k)} = 0$ by multiplicativity. Therefore $E_2 \simeq E_\infty$ and we conclude in the same way as for (I).

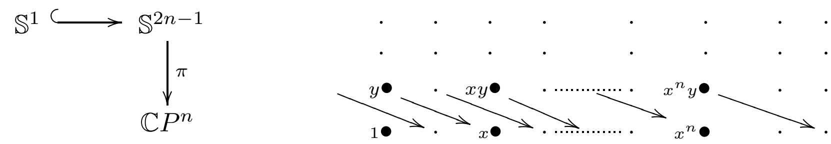
$$\begin{array}{ccc} \mathbb{S}^{2(n-k)-1} & \hookrightarrow & V_{k+1}(\mathbb{C}^n) \\ & & \downarrow \pi \\ & & V_k(\mathbb{C}^n) \end{array}$$



(III) *Computation of* $H^*(\mathbb{C}P^n)$. We show the well known fact that $H^*(\mathbb{C}P^n) \simeq \mathbb{Z}[x]/(x^{n+1})$, $\deg(x) = 2$.

We use the Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \longrightarrow \mathbb{C}P^n$. Because $H^*(\mathbb{S}^1) \simeq \Lambda(y)$, $\deg(y) = 1$ the groups $E_2^{p,q}$ can be nontrivial only for $q = 0, 1$. On the other hand we know that the spectral sequence converges to $H^*(\mathbb{S}^{2n+1}) \simeq \Lambda(z)$, $\deg(z) = 2n+1$ which implies that $E_\infty^{p,q} = 0$ except for $(p, q) = (0, 0)$ and $(p, q) = (2n, 1)$. For bidegree reasons, the differentials d_r, $r \ge 3$ vanish and therefore $E_3 \simeq E_\infty$. We infer that all differentials d_2 are isomorphisms, except those having $E_2^{0,0}$ and $E_2^{2n,1}$

as source or target. As a consequence the groups $E_2^{2k+1,0}$ vanish, while the groups $E_2^{2k,0}$, $0 \leq k \leq n$ are free of rank one. Moreover, if the generator of $E_2^{2,0}$ is $x = d_2(y)$, $\deg(x) = 2$, the generator of $E_2^{4,0}$ is $d_2(xy) = d_2(x)y + xd_2(y) = x^2$ and, by induction, the generator of $E_2^{2k,0}$ is x^k for $k \leq n$. The differential d_2 has to vanish on $E_2^{2n,1}$ which is generated by $x^n y$ i.e. $d_2(x^n y) = x^{n+1} = 0$. We therefore have $H^*(\mathbb{C}P^n) \simeq E_2^{*,0} \simeq \mathbb{Z}[x]/(x^{n+1})$.



(IV) *Computation of* $H^*(\mathrm{Gr}_2(\mathbb{C}^\infty))$. Here $\mathrm{Gr}_2(\mathbb{C}^\infty)$ is the space of 2-planes in $\mathbb{C}^\infty$, or the rank 2 complex infinite Grassmannian. We show that $H^*(\mathrm{Gr}_2(\mathbb{C}^\infty)) \simeq \mathbb{Z}[x_2, x_4]$, $\deg(x_2) = 2$, $\deg(x_4) = 4$.

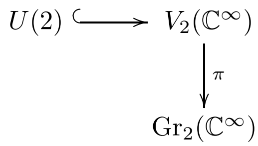
Note. The computation in (III) shows that the cohomology of the rank 1 complex infinite Grassmannian $\mathrm{Gr}_1(\mathbb{C}^\infty) = \mathbb{C}P^\infty$ is $H^*(\mathbb{C}P^\infty) \simeq \mathbb{Z}[x]$, $\deg(x) = 2$. In general, the cohomology of the rank k complex infinite Grassmannian is $H^*(\mathrm{Gr}_k(\mathbb{C}^\infty)) \simeq \mathbb{Z}[x_2, x_4, \ldots, x_{2k}]$, $\deg(x_{2i}) = 2i$, $1 \leq i \leq k$ (cf. [1]). This is the fundamental computation which establishes the existence of Chern classes (cf. [7]).

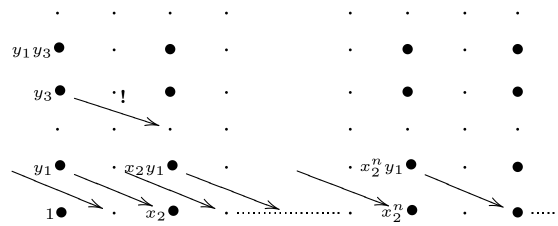
There is a fibration $U(2) \hookrightarrow V_2(\mathbb{C}^\infty) \xrightarrow{\pi} \mathrm{Gr}_2(\mathbb{C}^\infty)$, where π associates to 2-frame the plane which it spans. The point now is that the spaces $V_k(\mathbb{C}^\infty)$, $k \geq 1$ are contractible. This is intuitively clear, at least for $k = 1$: the sphere $\mathbb{S}^{2n-1}$ is contractible in $\mathbb{S}^{2n+1}$ and, by letting n go to infinity, we see that $\mathbb{S}^\infty = V_1(\mathbb{C}^\infty)$ is contractible. We deduce that the limit page of the spectral sequence satisfies $E_\infty^{p,q} = 0$ if $(p, q) \neq (0, 0)$ and $E_\infty^{0,0} \simeq \mathbb{Z}$. We have seen that $H^*(U(2))$ is free of rank one in degrees $0, 1, 3, 4$ and zero otherwise, with generators 1, y_1, y_3 and $y_1 y_3$, $\deg(y_i) = i$. The page E_2 is therefore supported by the lines $q = 0, 1, 3, 4$ and, for bidegree reasons, the differentials d_r, $r \geq 6$ vanish. By multiplicativity the differential d_5 also vanishes, hence $E_5 \simeq E_\infty$.

We first show by induction on k that $E_2^{2k+1,0} = 0$, $k \geq 0$ hence $E_2^{2k+1,*} = 0$ due to the special form of the E_2 term in the Leray-Serre spectral sequence. The claim is true for $k = 0$ because $E_2^{1,0}$ remains unchanged as r increases. In order to die in E_∞, a nontrivial element in $E_r^{2k+1,0}$ has to lie in the image of d_2 or d_4 (the only possibly nontrivial differentials with target $E_r^{*,0}$). The preimage necessarily lies in $E_2^{2k-1,1}$ or $E_4^{2k-3,3}$, which both vanish by the induction hypothesis. Therefore $E_2^{2k+1,0}$, $k \geq 1$ does not change with r and has to vanish.

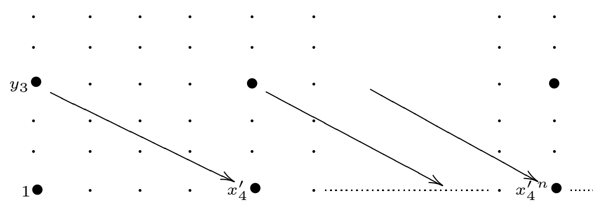
Because $E_r^{3,*} = 0$, $r \geq 2$ we infer that $d_3(y_3) = 0$, hence $d_3 = 0$ by multiplicativity. In particular $E_3 \simeq E_4$.

The generator $y_1 \in E_2^{0,1}$ can die in E_∞ only if it is sent to a nontorsion element $x_2 = d_2(y_1) \in E_2^{2,0} \simeq H^2(\mathrm{Gr}_2(\mathbb{C}^\infty))$. This element dies in E_3 and, by multiplicativity, the same is true for all monomials of the form $x_2^n y$, $n \geq 1$, $y \in \Lambda(y_1, y_3)$. The generator $y_3 \in E_2^{0,3}$ lives to E_4 and has to die in E_5 by being sent to a nontorsion element $x_4' = d_4(y_3) \in E_4^{4,0} \simeq E_2^{4,0}/\mathbb{Z}x_2^2$. By multiplicativity, the elements ${x_4'}^n$ are all nonzero. Let x_4 be a representative of x_4' in $E_2^{4,0} \simeq H^4(\mathrm{Gr}_2(\mathbb{C}^\infty))$. The last thing which remains to be proved is that there is no algebraic relation between x_2 and x_4. Assume this is the case i.e. there exists in $E_2^{*,0}$ a homogeneous algebraic relation of degree $2k$, $k \geq 3$ between x_2 and x_4. Because $E_2^{*,0}/(x_2) \simeq \mathbb{Z}[x_4]$, the relation is of the form $x_2^\ell z = 0$, $\ell \geq 1$. But then $y_1 x_2^{\ell-1} z \in E_2^{2k-2,1}$ is sent to zero by d_2, hence lives to E_3, to E_4 and also to $E_5 \simeq E_\infty$. This is a contradiction with $E_\infty^{2k-2,1} = 0$ and shows that $E_2^{*,0} \simeq H^*(\mathrm{Gr}_2(\mathbb{C}^\infty)) \simeq \mathbb{Z}[x_2, x_4]$.





E_2



E_4

4. The Algebra

Spectral sequences are the natural algebraic structure arising whenever one deals with a *filtered* differential complex (C^*, d), which means that one is given a decreasing sequence of *subcomplexes* $C^* = C_0^* \supset C_1^* \supset \cdots \supset C_p^* \supset \cdots \supset 0$. This induces in homology the filtration

$$H(C) \supset \ \mathrm{im}\big(H(C_1) \to H(C)\big) \supset \cdots \supset \ \mathrm{im}\big(H(C_p) \to H(C)\big) \supset \cdots \supset 0\ .$$

We denote $F_pH = \mathrm{im}\big(H(C_p) \to H(C)\big)$. The algebraic result behind the construction of the Leray-Serre spectral sequence is the following.

Theorem 4.1. *Let (C^*, d) be a filtered differential complex and $(C_p^*)_p$ the filtration. There exists a spectral sequence $E_r^{*,*}$, $r \geq 0$ converging to $H(C^*)$ filtered by $(F_pH)_p$ such that*

$$E_0^{p,q} = C_p^{p+q}/C_{p+1}^{p+q}$$

and $d_0 : E_0^{p,q} \longrightarrow E_0^{p,q+1}$ is induced by d. In particular

$$E_1^{p,q} \simeq H(C_p^{p+q}/C_{p+1}^{p+q})\ .$$

One particular instance of this theorem must be already familiar to the reader that got to read so far: the case of a two term filtration $C = C_0 \supset C_1 \supset C_2 = 0$. The short exact sequence $0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C_0/C_1 \longrightarrow 0$ induces a long exact sequence in homology, written as an exact triangle where the [+1] indicates a shift in the degree by 1.

(4)
$$\begin{array}{ccc} H(C_1) = H(C_1/C_2) & \longrightarrow & H(C_0) \\ & \nwarrow^{\delta}_{[+1]} \quad \swarrow & \\ & H(C_0/C_1) & \end{array}$$

The page E_1 is supported by the columns $p = 0, 1$. For bidegree reasons, the differentials d_r, $r \geq 2$ vanish and the spectral sequence stabilizes at E_2, with

$$\begin{aligned} E_2^{0,q} &\simeq \ker\left(E_1^{0,q} \xrightarrow{d_1} E_1^{1,q}\right) \\ &\simeq \ker\left(H^q(C_0/C_1) \xrightarrow{\delta} H^{q+1}(C_1)\right), \\ E_2^{1,q} &\simeq E_1^{1,q}/\ \mathrm{im}\left(E_1^{0,q} \xrightarrow{d_1} E_1^{1,q}\right) \\ &\simeq H^{q+1}(C_1)/\ \mathrm{im}\left(H^q(C_0/C_1) \xrightarrow{\delta} H^{q+1}(C_1)\right) . \end{aligned}$$

The fact that d_1 coincides with δ is an easy consequence of the construction that we explain below. The exactness of the triangle (4) implies that E_r converges to $H^*(C)$ filtered by the F_pH. Indeed, this last fact would mean that

$$\begin{aligned} E_2^{0,q} &\simeq H^q(C_0)/\ \mathrm{im}\left(H^q(C_1) \longrightarrow H^q(C_0)\right) \\ &\simeq H^q(C_0)/\ker\left(H^q(C_0) \longrightarrow H^q(C_0/C_1)\right) \\ &\simeq \mathrm{im}\left(H^q(C_0) \longrightarrow H^q(C_0/C_1)\right), \end{aligned}$$

where we used exactness of (4) at $H(C_0)$ for the second isomorphism, and

$$\begin{aligned} E_2^{1,q} &\simeq \mathrm{im}\left(H^{q+1}(C_1) \longrightarrow H^{q+1}(C_0)\right) \\ &\simeq H^{q+1}(C_1)/\ker\left(H^{q+1}(C_1) \longrightarrow H^{q+1}(C_0)\right) . \end{aligned}$$

Exactness at $H(C_0/C_1)$ and $H(C_1)$ imply the desired conclusion. We see that exactness of (4) is strictly stronger than the convergence of the spectral sequence, but this last phenomenon is the one that generalizes correctly to a filtration having more than two terms.

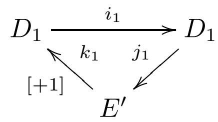
Let us now describe the construction of the spectral sequence in Theorem 4.1. We use exact couples as introduced by Massey [5] (see also [6, §2.2]). An *exact couple* is an exact triangle of the form (5.1). The *derived exact couple* is the exact (!) triangle of the form (5.2)

(5)
$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & \nwarrow^{k}_{[+1]} \quad \swarrow^{j} & \\ & E & \end{array} \qquad \qquad \begin{array}{ccc} D' & \xrightarrow{i'} & D' \\ & \nwarrow^{k'}_{[+1]} \quad \swarrow^{j'} & \\ & E' & \end{array}$$

1. 2.

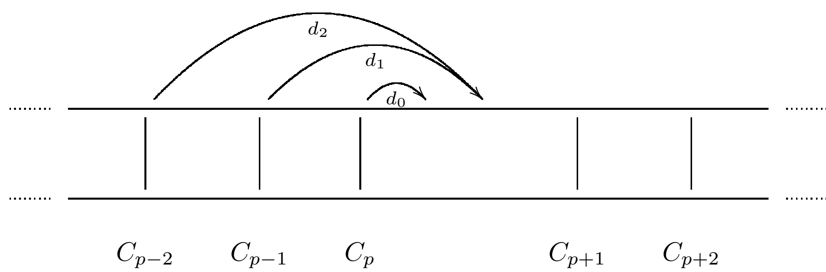
with $D' = i(D) \subset D$, $E' = H(E, d)$, $d = j \circ k$ and i', j', k' induced by i, j, k. Given a differential complex (C, d) with filtration (C_p) we define $D_0 = \oplus_p C_p$, $E_0 = \oplus_p C_p/C_{p+1}$ and $i_0 : D_0 \longrightarrow D_0$ to be equal on each factor with the inclusion $C_{p+1} \hookrightarrow C_p$. The short exact sequence of complexes $0 \longrightarrow D_0 \xrightarrow{i_0} D_0 \longrightarrow E_0 \longrightarrow 0$

gives rise to a long exact sequence in homology which we denote



with $D_1 = \oplus_p H(C_p)$, $E_1 = \oplus_p H(C_p/C_{p+1})$. These carry the natural bigrading $D_1^{p,q} = H^{p+q}(C_p)$, $E_1^{p,q} = H^{p+q}(C_p/C_{p+1})$ with respect to which i_1, j_1, k_1 respectively have bidegree $(-1,1)$, $(0,0)$ and $(1,0)$. The spectral sequence $(E_r,\, d_r)$ is constructed by taking successive derived couples $(D_r,\, E_r,\, i_r,\, j_r,\, k_r)$, with $d_r = j_r \circ k_r$. The maps i_r, j_r, k_r respectively have bidegree $(-1,1)$, $(r-1, -r+1)$ and $(1,0)$, hence the differential d_r has bidegree $(r,\, -r+1)$.

I hope that by now this half-page definition appears palatable. One can find a proof of Theorem 4.1 in [6, §2.2], so I will only give some additional heuristic remarks. The spectral sequence associated to a filtration describes the homology of a complex C from the point of view of the relative quotients C_p/C_{p+1}. Computing the homology $H(C)$ means modding out elements that lie in the image $d(C)$. The spectral sequence mods out elements in C_p/C_{p+1} that lie in the "image" of C_p/C_{p+1}, C_{p-1}/C_p, C_{p-2}/C_{p-1} etc. The groups $E_r^{p,q}$ should be thought of as the "good" quotients where the "image" of C_{p-r+1}/C_{p-r+2} gets modded out. In the limit, all homological relations are taken into account and the spectral sequence reconstructs in a weak sense $H(C)$. The picture below suggests a vague analogy with physical energy spectra, which most likely prompted the name of *spectral sequence*.



5. Sidetrip: Morse Homology

We recall in this section the construction of Morse homology, which is a refinement of Morse theory allowing one to compute the homology of a closed manifold M in terms of critical points and gradient trajectories of a Morse function $f : M \longrightarrow \mathbb{R}$. This section mostly intends to fix notations, rather than to give an introduction to Morse homology. One possible reference on this topic is [10].

A function f is said to be Morse if all its critical points are nondegenerate, which is equivalent to the fact that, in suitable coordinate charts around any critical point x, the function can be written as a nondegenerate quadratic form $f(x_1, \ldots, x_n) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2$. The number k of its negative eigenvalues is called the *Morse index* of x and is denoted $\mathrm{ind}_f(x)$. The set of critical points of index k is denoted $\mathrm{Crit}_k(f)$. We put $\mathrm{Crit}(f) = \bigcup_k \mathrm{Crit}_k(f)$.

A vector field $\mathcal{X}$ on M is said to be *gradient-like* if $\mathcal{X} \cdot f \leq 0$, with equality only at critical points of f. We call it *admissible* if, in a neighbourhood of the critical points, it is equal to the gradient of f with respect to some metric. Let $\varphi^t_{\mathcal{X}}$ be the flow of $\mathcal{X}$. We call $\mathcal{X}$ *transverse* if, for any $x, y \in \mathrm{Crit}(f)$, the unstable and stable manifolds $W^u(x)$, $W^s(y)$ of x and y are transverse, where

$$W^u(x) = \{z \in M \ : \ \varphi^t_{\mathcal{X}}(z) \longrightarrow x, \ t \to -\infty\},$$
$$W^s(y) = \{z \in M \ : \ \varphi^t_{\mathcal{X}}(z) \longrightarrow y, \ t \to \infty\} \ .$$

Gradient-like admissible vector fields which are transverse are generic. From now on $\mathcal{X}$ will be assumed to have these properties, which guarantee that the space

$$\mathcal{M}(x^-, x^+) = \{z \in M \ : \ \varphi^t_{\mathcal{X}}(z) \longrightarrow x^\pm, \ t \to \pm\infty\} = W^u(x^-) \pitchfork W^s(x^+)$$

of parameterized trajectories connecting two critical points $x^\pm$ is a manifold of dimension

$$\dim \mathcal{M}(x^-, \ x^+) = \mathrm{ind}_f(x^-) - \mathrm{ind}_f(x^+) \ .$$

The additive group $\mathbb{R}$ acts by translation on $\mathcal{M}(x^-, x^+)$ and we denote the quotient (the space of unparameterized trajectories from x^- to x^+) by $\widetilde{\mathcal{M}}(x^-, x^+)$. This space has dimension $\dim \mathcal{M}(x^-, x^+) - 1$. A fundamental compactness result states the following:

(i) if $\mathrm{ind}_f(x^-) - \mathrm{ind}_f(x^+) = 1$ the 0-dimensional manifold $\widetilde{\mathcal{M}}(x^-, x^+)$ consists of a *finite* set of points;

(ii) if $\mathrm{ind}_f(x^-) - \mathrm{ind}_f(x^+) = 2$, the "ends" of the noncompact components of the 1-dimensional manifold $\widetilde{\mathcal{M}}(x^-, x^+)$ consist *exactly* of the pairs

$$(\gamma_1, \gamma_2) \in \widetilde{\mathcal{M}}(x^-, y) \times \widetilde{\mathcal{M}}(y, x^+), \ \mathrm{ind}_f(x^-) - \mathrm{ind}_f(y) = 1 \ .$$

Otherwise stated, a noncompact family of index two trajectories "breaks" in a unique way at its ends in two pairs of index one trajectories.

This geometric picture allows one to build a differential complex which encodes the variational properties of f, which are in turn determined by the topology of M. The Morse complex is defined as follows.

$$\begin{aligned} C^k(f, \mathcal{X}) &= \bigoplus_{\mathrm{ind}_f(x)=k} \mathbb{Z}\, x, \\ d &: C^k(f, \mathcal{X}) \longrightarrow C^{k+1}(f, \mathcal{X}), \\ d(x) &= \sum_{\mathrm{ind}_f(y)=k+1} \#\widetilde{\mathcal{M}}(y, x)\, y \ . \end{aligned}$$

Two remarks have to be made at this point. First, the notation $\#\widetilde{\mathcal{M}}(y, x)$ stands for an algebraic count of the elements of $\widetilde{\mathcal{M}}(y, x)$. This can be done by choosing an orientation of the unstable manifolds of $\mathcal{X}$, which induce a coorientation of the stable manifolds and therefore associate a sign to each intersection trajectory when the difference of the indices is equal to one. Second, the differential satisfies $d \circ d = 0$ due to condition (ii), which is a straightforward computation at least in the case of mod 2 coefficients. The fundamental result of the theory is the following.

Theorem 5.1. *For every choice of admissible generic pairs $(f_i, \mathcal{X}_i)$, $i = 0, 1$ there is an isomorphism depending only on the choice of orientations of the unstable manifolds*

$$H^*(C^*(f_0, \mathcal{X}_0), d) \simeq H^*(C^*(f_1, \mathcal{X}_1), d) \ .$$

Moreover, for any choice of an admissible generic pair $(f, \mathcal{X})$ *we have*

$$H^*(C^*(f, \mathcal{X}), d) \simeq H^*(M; \mathbb{Z}) \ .$$

One can best understand this isomorphism through the intermediate of *cellular homology*. Indeed, for vector fields which can be written in the neighbourhood of the critical points as the gradient of a quadratic form with respect to a Euclidean metric, the unstable manifolds form a cellular decomposition of M and the incidence numbers of the cells are given by the coefficients $\#\widetilde{\mathcal{M}}(y, x)$. Morse homology is therefore equal to cellular homology, which in turn is isomorphic to singular homology. This is summarized in [8] following results of Laudenbach [3] and Pozniak [9].

6. The Construction

We explain in this section the construction of the Leray-Serre spectral sequence for a smooth locally trivial fibration $F \hookrightarrow X \xrightarrow{\pi} B$, where F, X and B are closed connected manifolds. The reader can find a proof of Theorem 2.1 in [6, §5] or [11]. Full details of the approach presented here can be found in [8, §3], and a variant of it is the subject of [2].

Let us choose a Morse function $f : B \longrightarrow \mathbb{R}$ and denote $\widetilde{f} = f \circ \pi$. Let $\mathrm{Crit}(f) = \{x_1, \ldots, x_N\}$. We choose mutually disjoint trivializing open sets $U_i \ni x_i$ with trivializations $\pi^{-1}(U_i) \xrightarrow{\Psi_i} U_i \times F$. Let $\varphi : F \longrightarrow \mathbb{R}$ be a Morse function and $\varphi_i = \varphi \circ \mathrm{pr}_2 \circ \Psi_i$. Let $\rho_i : B \longrightarrow [0, 1]$ be cut-off functions supported in U_i and constant equal to 1 in the neighbourhood of x_i.

For $\varepsilon > 0$ small enough the function

$$f_\varepsilon = \widetilde{f} + \varepsilon \sum_{i=1}^{N} \rho_i \varphi_i$$

is Morse and its critical points are precisely the critical points of φ_i in the fibers lying over the critical points x_i. If we denote $\mathrm{Crit}(\varphi) = \{y_1, \ldots, y_\ell\}$ we therefore have

$$\mathrm{Crit}(f_\varepsilon) = \{(x_i, y_j) \ : \ 1 \le i \le N, \ 1 \le j \le \ell\} \ .$$

Before specifying the gradient-like vector field $\mathcal{X}$ on X, let us remark that there is a natural filtration on the Morse complex $C^*(f_\varepsilon)$ given by

$$C_p(f_\varepsilon) = \bigoplus_{\substack{\alpha \in \mathrm{Crit}(f_\varepsilon) \\ \alpha = (x, y) \ : \ \mathrm{ind}_f(x) \ge p}} \mathbb{Z}\,\alpha \ . \tag{6}$$

In order to build the spectral sequence we need that the Morse differential preserves the filtration, meaning that, whenever there is a trajectory in X between (x^-, y^-) and (x^+, y^+), we must have $\mathrm{ind}_f(x^-) \ge \mathrm{ind}_f(x^+)$. One way to achieve this is by choosing $\mathcal{X}$ such that its trajectories project on the gradient trajectories of f with respect to some metric on B.

Let us choose on B a metric g with respect to which ∇f is transverse. Let us choose on X a metric $\widetilde{g}$ such that π is a Riemannian submersion and which is of the form $g \oplus h$ on $\pi^{-1}(U_i)$, with h a metric on F such that $\nabla\varphi$ is transverse. We define

$$\mathcal{X}_\varepsilon = -\widetilde{\nabla^g f} - \epsilon \sum_{i=1}^{N} \rho_i \nabla^{\widetilde{g}} \varphi_i \ .$$

For ε small enough $\mathcal{X}_\varepsilon$ is gradient-like for f_ε, and because $\nabla^{\widetilde{g}}\varphi_i$ is vertical we have $\pi_* \mathcal{X}_\varepsilon = -\nabla^g f$ i.e. the trajectories of $\mathcal{X}_\varepsilon$ project on gradient trajectories of f on the base. Moreover, for a generic choice of $\widetilde{g}$ the vector field $\mathcal{X}_\varepsilon$ is transverse.

The next task is to compute the first two terms of this spectral sequence. I will carry over only the computation of E_1, which is already convincing as for the final outcome. By definition we have

$$E_0 = \bigoplus_p C_p/C_{p+1} = \bigoplus_p \bigoplus_{\mathrm{ind}_f(\pi(\alpha))=p} \mathbb{Z}\,\alpha \ .$$

The differential d_0 is induced by $d : C_p/C_{p+1} \longrightarrow C_p/C_{p+1}$ and acts as $d[\alpha] = d(\alpha) \pmod{C_{p+1}}$. We obtain

$$\begin{aligned} d(\alpha) \pmod{C_{p+1}} &= \sum_{\mathrm{ind}_f(\pi(\beta))=p} \#\widetilde{\mathcal{M}}(\beta,\, \alpha)\,[\beta] \\ &= \sum_{\pi(\beta)=\pi(\alpha)} \#\widetilde{\mathcal{M}}_{\mathrm{fiber}}(\beta,\, \alpha)\,[\beta] \\ &= \big[d_{\mathrm{fiber}}(\alpha)\big] \ . \end{aligned}$$

The second equality holds by transversality on B, which implies that there exists no trajectory of $\nabla^g f$ connecting two *distinct* critical points of the same index. As a consequence we obtain

$$\begin{aligned} E_1 &\simeq \bigoplus_p \bigoplus_{\mathrm{ind}_f(x_i)=p} H(C^*(\varphi_i,\, -\epsilon\nabla^{\widetilde{g}}\varphi_i)) \\ &\simeq \bigoplus_{x_i\in \mathrm{Crit}(f)} H(X_{x_i}), \end{aligned} \tag{7}$$

where X_{x_i} is the fiber lying over the critical point x_i.

Let us end with a heuristic explanation of why formula (7) gives the right answer for E_2. The reason is that it can be interpreted as a Morse complex based on the critical points of f *with coefficients* $H(F)$. The differential d_1 turns out to be the Morse differential on B and the term E_2 is therefore $H(B; H(F))$. We mention a slightly delicate point in the computation of d_1 related to the non-canonical identification of $H(X_{x_i})$ with $H(F)$. In the case of a base which is not simply connected this leads to the notion of a local system of coefficients. We refer to [8, §3] for further details.

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