Real Compact Surfaces

by Alexandru Oancea [1]

To Mona

Abstract

The classification of real compact surfaces is a main result which is at the same time easy to understand and non-trivial, simple in formulation and rich in consequences. The aim of this paper is to explain the theorem by means of many drawings. It is an invitation to a visual approach of mathematics.

1 First definitions and examples

We assume that the reader has already encountered the notion of a *topology* on a set X. Roughly speaking this enables one to recognize points that are "close" to a given point $x \in X$.

Continuous functions between topological spaces are functions that send "close" points to "close" points. We can also see this on their graph:

- the graph of a continuous function $f : \mathbb{R} \to \mathbb{R}$ can be drawn without raising the pencil from the drawing paper
- the graph of a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ can be thought of as a tablecloth which is *not torn*
- the graph of a continuous map between topological spaces $f : X \to Y$ is closed in the product space $X \times Y$.

The most familiar topological space is n-dimensional euclidean space $\mathbb{R}^n$. It will be the hidden star of the present paper, which is devoted to a rich generalization of this kind of space, namely manifolds. In the class of manifolds, surfaces are well understood and already provide a sufficiently rich set of examples. We will focus on them.

First let us recall that two spaces X and Y are said to be *topologically equivalent* or *homeomorphic* if there is a bicontinuous transformation between them[2]. How should we think of the homeomorphism relation? Here we indicate two ways:

- the abstract one: all topological statements which are valid for X are also valid for Y and the converse
- a visual one: Y should be thought of as a realization of X in a different "shape".

  There is a thing worth noticing when speaking of the "shape" of an object (here, object=topological space): this notion presupposes the possibility of "contemplating" the shape, that is an *ambient space* S in which our object should live. Thus one has to be aware of the limitations of the visual recipe, as the notion of homeomorphism is independent of any ambient space. Nevertheless we can formalize the notion of *equivalent shape* as being the final stage of a continuous deformation of X in S through which X remains homeomorphic to itself i.e. a *continuous isotopy*.

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[1] Ecole Normale Supérieure, 45 Rue d'Ulm, 75 005 Paris, FRANCE *and*
Université Paris Sud, Laboratoire de Mathématiques, Bât.425, 91 405 Orsay, FRANCE.
E-mail address: `alexandru.oancea@math.u-psud.fr`

[2] This is obviously an equivalence relation on the class of topological spaces.

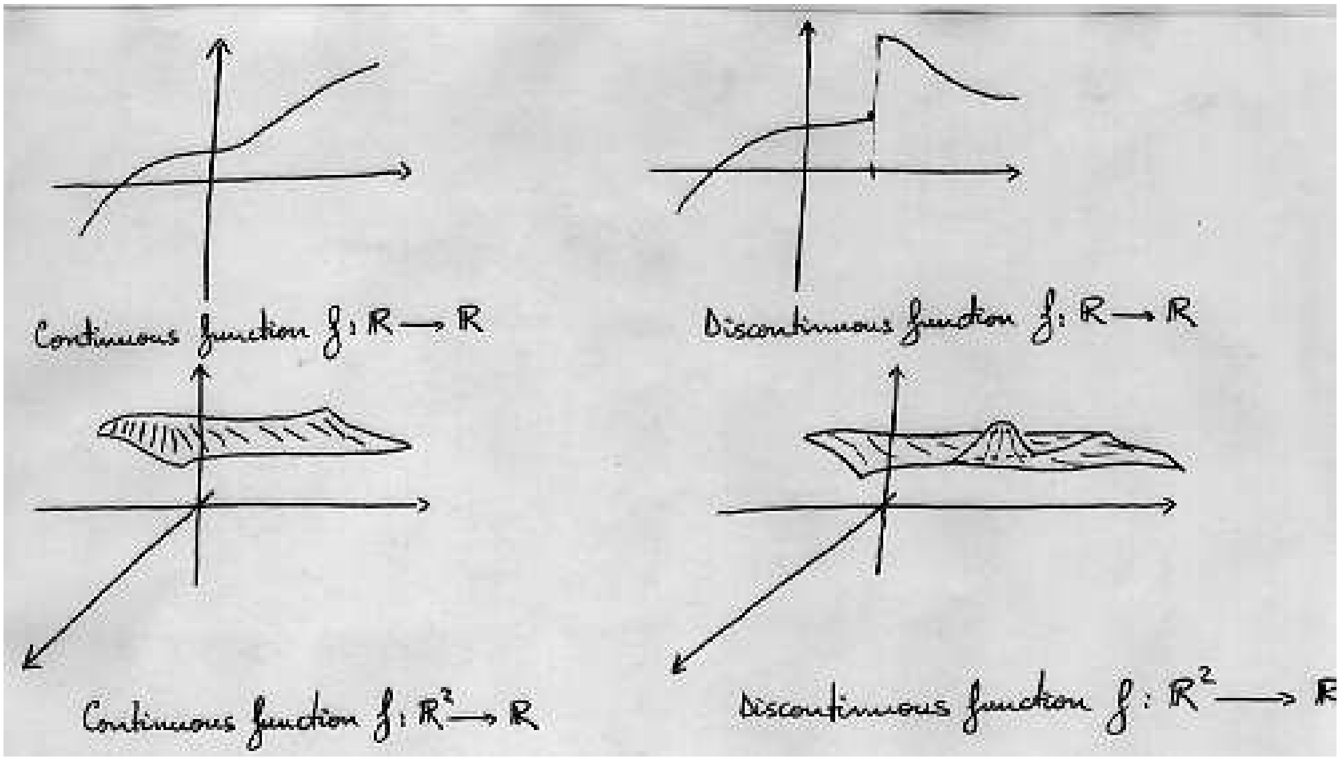

Figure 1: FUNCTIONS

In this paper, we are not interested (but one should be!) in the rigorous definitions of these notions, but rather in getting some feeling (or testing the one that is already present!) about them.

Let us now take a look at some homeomorphic spaces.

Example 1. ([Sa]) The letters of the latin alphabet fall into 9 homeomorphism classes (if we use the sans-serif font):

{A,R},{B},{C,G,I,J,L,M,N,S,U,V,W,Z},
{D,O},{E,F,T,Y},{H,K},{P},{Q},{X}

Example 2. Simple closed curves (i.e. without self-intersection points) in euclidean space $\mathbb{R}^n$. By definition, they are all homeomorphic images of the circle hence homeomorphic to each other.

- $n = 2$: any two simple closed and rectifiable curves are deformation equivalent
- $n = 3$: this is the most interesting case - we call them *knots*. Deciding whether two given knots are equivalent or not is a difficult matter. See Fig.2. See also [Sh], §14.
- $n \geq 4$: any curve is unknotted and any two rectifiable curves are deformation equivalent

Example 3. A doughnut is homeomorphic to a tea-cup with one handle and to a torus. Moreover, they can be deformed one into the other in 3-space. See Fig.3.

2 Surfaces

The rest of the paper will consist mainly of examples of compact surfaces and remarkable constructions that can be performed with them, as an invitation to further study in this direction. The

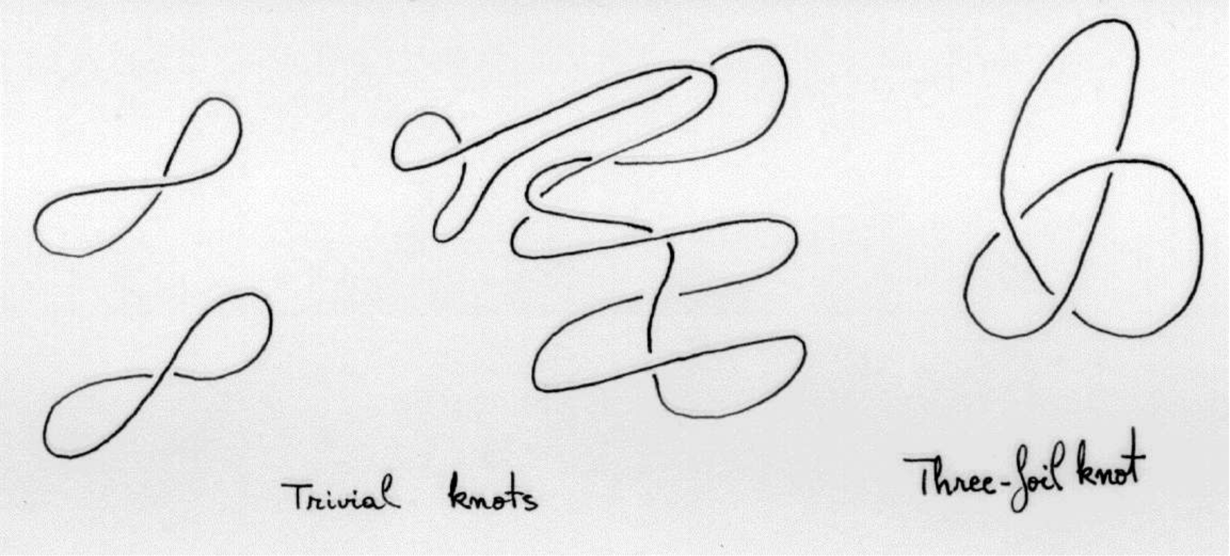

Figure 2: KNOTS

final aim is to state the classification theorem for compact surfaces, which we consider a perfect motivation to the further study of algebraic and differential topology.

Definition 1. *A surface is a topological space X with the property that any point $x \in X$ has an open neighbourhood U homeomorphic to the open unit ball in $\mathbb{R}^2$. Surfaces are also called 2-dimensional manifolds* [3].

Otherwise stated, a surface is made out of 2-dimensional patches. If the number of patches can be chosen to be finite, we say that the surface is compact (and this is equivalent to the standard definition of compacity). The reader is advised to verify that the constructions below produce surfaces in the sense of the definition.

Warning: In the sequel it is important to make the distinction between the intrinsic definition of a surface and its extrinsic realization in an ambient space. The last one can provide *intuitive* understanding of various properties of the surface.

The first better known surface is the 2-dimensional sphere (see Fig.4b)

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$$

and the second one is the torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. An immediate consequence of the definition is the embedding

$$T^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$$

but we shall view it in $\mathbb{R}^3$ as obtained by rotating a small circle orthogonal to the xy-plane around the radius 1 circle in the xy-plane. See Fig.4a.

The series begun by the sphere and the torus can be continued by the orientable genus g surfaces Σ_g, or *tori with g holes* (see Fig.4c)

This last type of surface can be also viewed as represented by (i.e. homeomorphic to!) a sphere with g handles (see Fig.5). A *handle* is a cylinder attached to the sphere along the boundaries of two disks which are cut out of it. Thus the torus is the genus 1 surface i.e. the sphere with one handle.

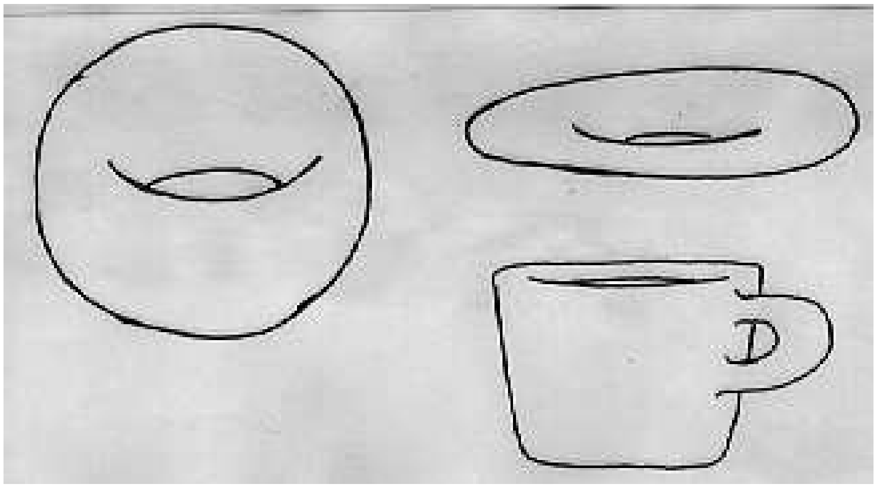

Figure 3: TORI

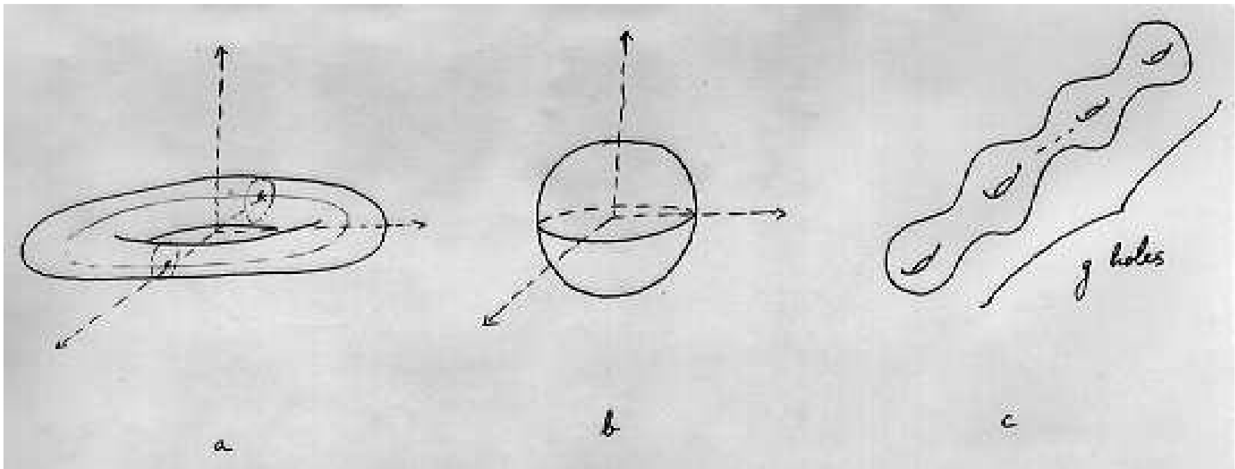

Figure 4: COMPACT ORIENTABLE SURFACES

Here is now a new method of construction for the genus g surfaces using the *connected sum* of two surfaces. Denote them by S_1 and S_2. Cut out of each of them the interior of a small disk $D_i,\ i = 1,2$ and identify the two circles which stand as boundaries after having fixed a running direction on each of them. We obtain a new surface $S_1\#S_2$, whose homeomorphism type is independent of the orientations of the circles. (prove this!) (see Fig.6)

- look at a genus g surface not in $\mathbb{R}^3$, but in its one-point compactification $\mathbb{S}^3$. While in $\mathbb{R}^3$ the Jordan-Brouwer theorem tells us that the complement of Σ_g defines two connected components, one bounded and the other one unbounded, in $\mathbb{S}^3$ both become bounded (compact). Moreover, they are *homeomorphic* with common boundary Σ_g.

As an example we have
$\mathbb{S}^2\#\Sigma = \Sigma$, $\Sigma_{g_1}\#\Sigma_{g_2} = \Sigma_{g_1+g_2}$, $\Sigma_g = \Sigma_1\#\Sigma_1\#\ldots\#\Sigma_1 = T^2\#T^2\#\ldots\#T^2$ (*g terms*)

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[3] Replacing allover 2 by n gives the definition of an n-dimensional topological manifold

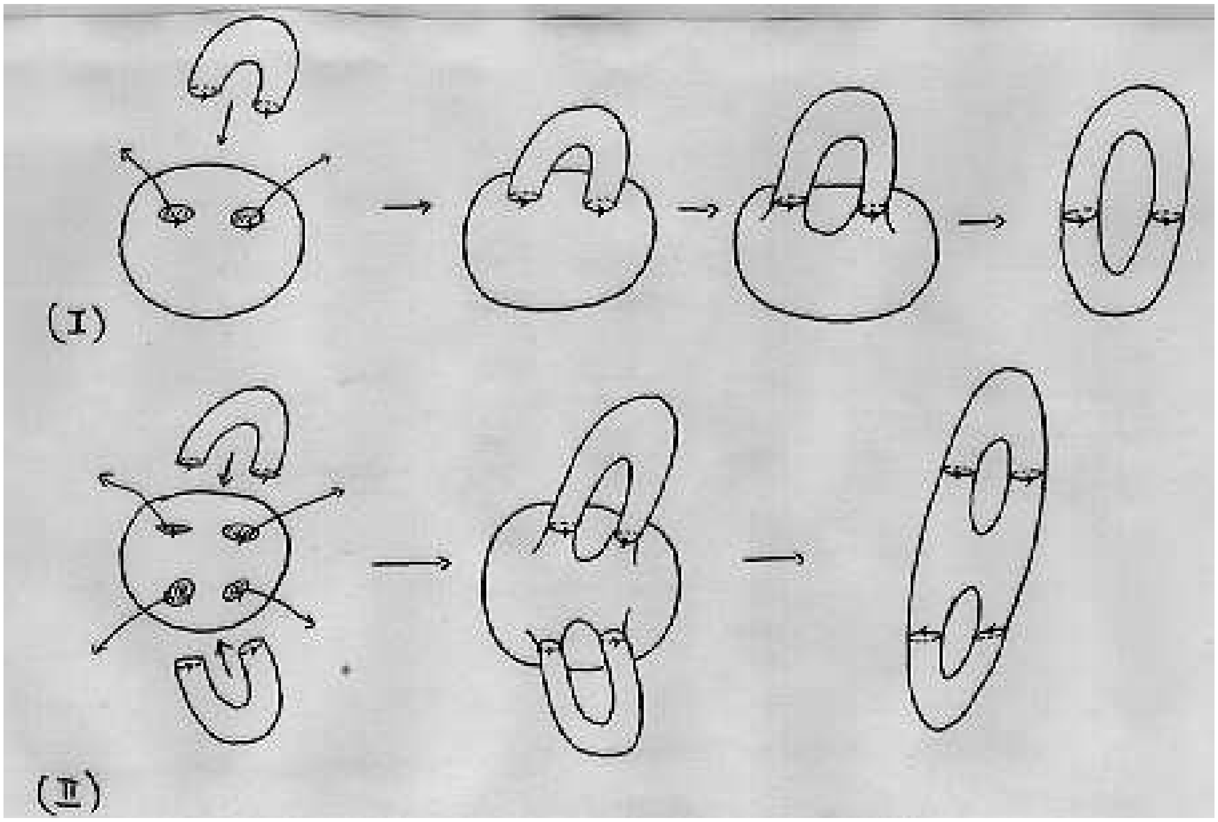

Figure 5: ATTACHING HANDLES

- Notice that the connected-sum operation is associative, commutative and has as a unit element the homeomorphism class of the sphere $\mathbb{S}^2$. It thus defines a monoid structure on the homeomorphism classes of (compact) surfaces. The classification theorem will describe this monoid explicitly.

Now look at another way of describing the genus g surfaces. Start with the torus T^2. We claim that it can be described as a square in which we identify the opposite edges as indicated in the figure below (see Fig.7). Indeed, by identifying the edges designed by a we obtain a cylinder and we must identify its boundaries *respecting orientations*. This is done by suitably bending the cylinder. In the end we obtain the torus.

Let us now look how the connected sum of two tori translates in this new representation. First remark that a torus with a cut-out disk is described by a pentagon in which the boundary edges are identified as in Fig.8. Indeed, in the process of pasting, the c edge becomes a boundary loop on the torus.

Then the pasting of two tori along the boundaries of two such cut-out disks is the same as pasting two such pentagons along the c edges, resulting in a polygon with 8 edges which have to be identified as in Fig.9a.

In the same way, the genus g surface is obtained from a $4g$ polygon where edges are identified as in Fig.9b.

We keep on representing surfaces using polygons in the plane and search for new ones. In the square representation of the torus (Fig.7) we see that there are two more ways to identify the

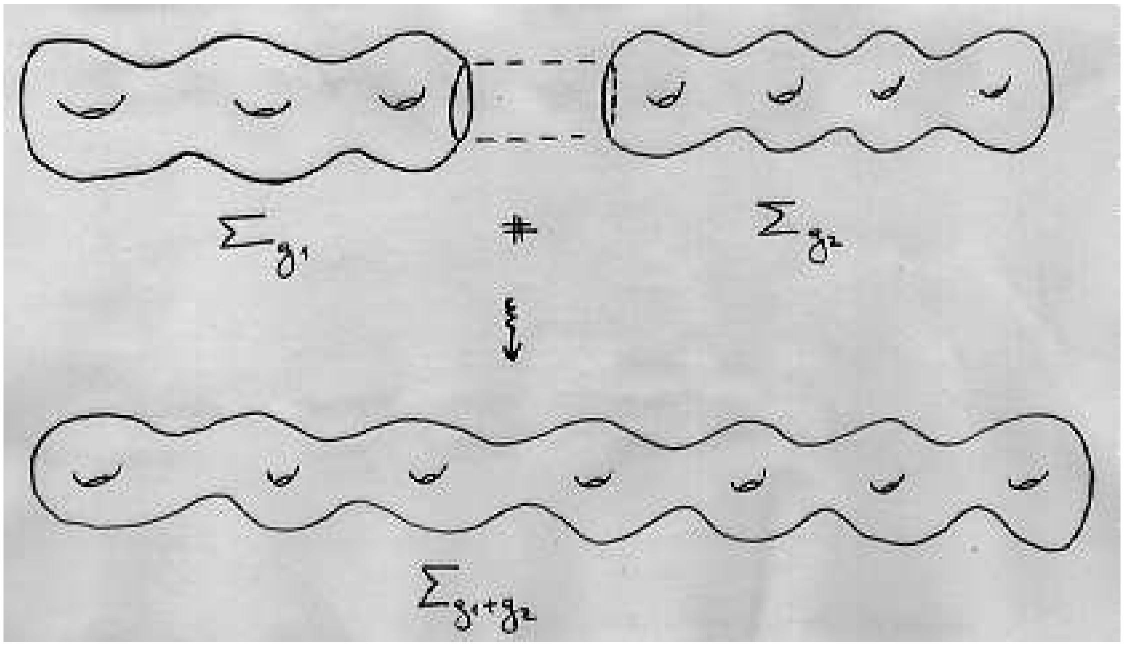

Figure 6: CONNECTED SUM

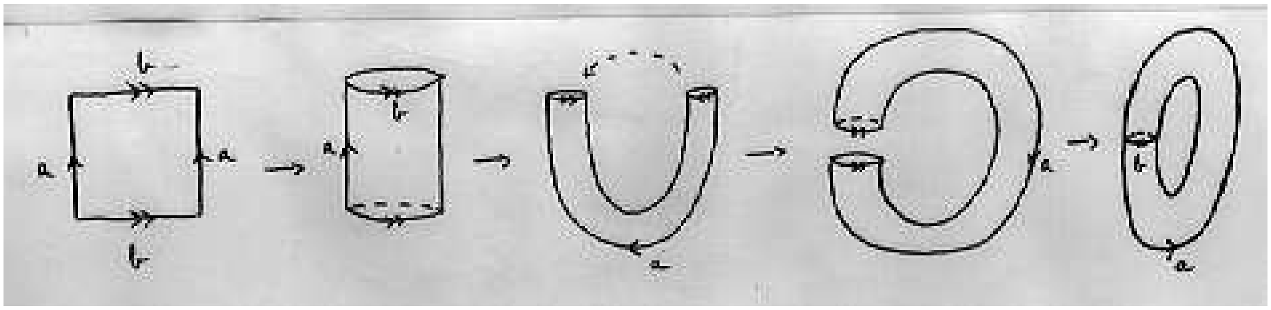

Figure 7: CONSTRUCTING THE TORUS

opposite edges. The first one gives the *real projective plane* $\mathbb{R}P^2$ (Fig.10) while the second one gives the so-called *Klein bottle* K (Fig.11).

The Klein bottle K resembles very much the torus as to its construction: after having identified the edges denoted by a and having obtained a cylinder, we must identify its boundary circles *reversing* the orientation. We mention two important differences with respect to the torus:

- it will necessarily have self-intersections if we want to represent it in $\mathbb{R}^3$ (see Fig.11).
- it is *non-orientable* i.e. if one tries to walk along the bottle there is a way to come at the same point but upside-down. We may say that the Klein bottle has only one side. This phenomenon would not happen with the Σ_g's.

By now we have seen only closed surfaces (i.e. compact without boundary). As an intermediate step in the visualization of the projective plane $\mathbb{R}P^2$ take a look at a famous compact surface *with*

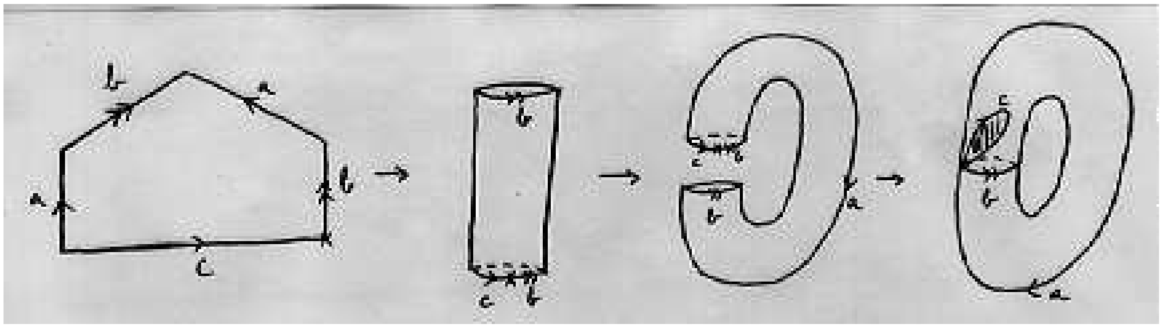

Figure 8: DISK REMOVED FROM THE TORUS

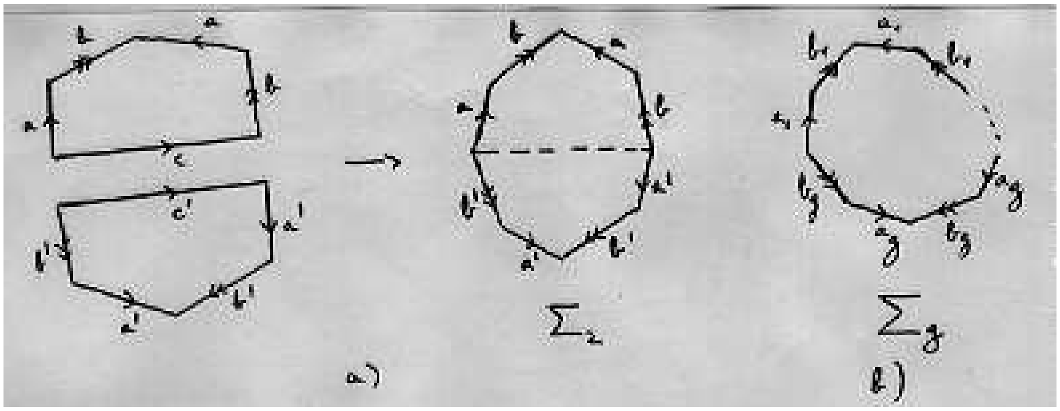

Figure 9: IDENTIFYING SIDES OF POLYGONS

boundary, namely the *Möbius band.* This is obtained as in Fig.12, by identifying only two opposite sides of a square.

The boundary of the Möbius band consists of *one* circle. If we try to construct the projective plane by two successive pastings as we have done with the torus or the Klein bottle according to Fig.10a, we see that the Möbius band appears as the first intermediate step. The b sides of Fig.10a will constitute two half-circles on the boundary of this Möbius band and $\mathbb{R}P^2$ will be obtained by identifying them accordingly.

There is another intimate link between the Möbius band and the projective plane: the former is identified with a neighborhood of the circle ab in $\mathbb{R}P^2$ (in a more technical language we say that the unit normal bundle of $\mathbb{R}P^1$ in $\mathbb{R}P^2$ is the Möbius band[4]). This means that if we cut out a disk from $\mathbb{R}P^2$ we again obtain the Möbius band. See Fig.14: as usual, a cut-out disk with boundary c is represented by inserting a segment c on the boundary of the square. Draw an auxiliary segment d and identify squares $a - b - \frac{1}{2}c - d$ in $\mathbb{R}P^2 \setminus disk$ with triangles $ab - \frac{1}{2}c - d$ in the Möbius band.

We can rephrase the construction above by saying that the projective plane is obtained by pasting a disk and a Möbius band along their boundaries.

Let us now focus on two concrete ways of visualizing the projective plane: the cross-cap and Boy's surface ([H-CV], pp.314-320).

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[4] In projective geometry we call $\mathbb{R}P^1 \subset \mathbb{R}P^2$ the "line at infinity", in connection with the geometrical construction of projective plane as the set of directions in 3-space $\mathbb{R}^3$.

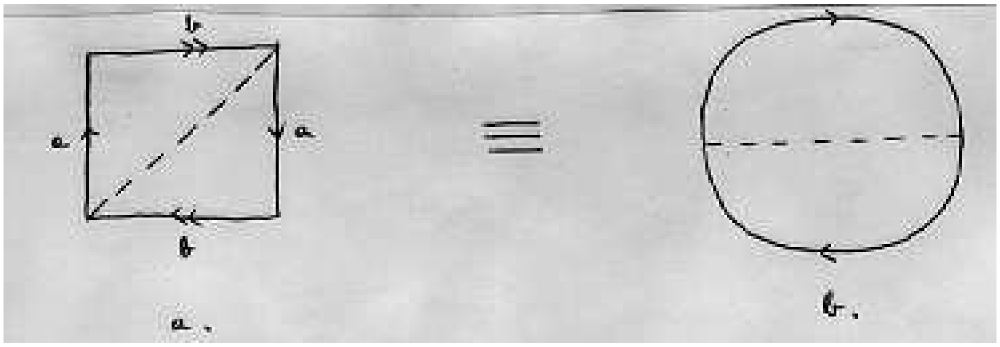

Figure 10: PROJECTIVE PLANE. FIRST DRAFT.

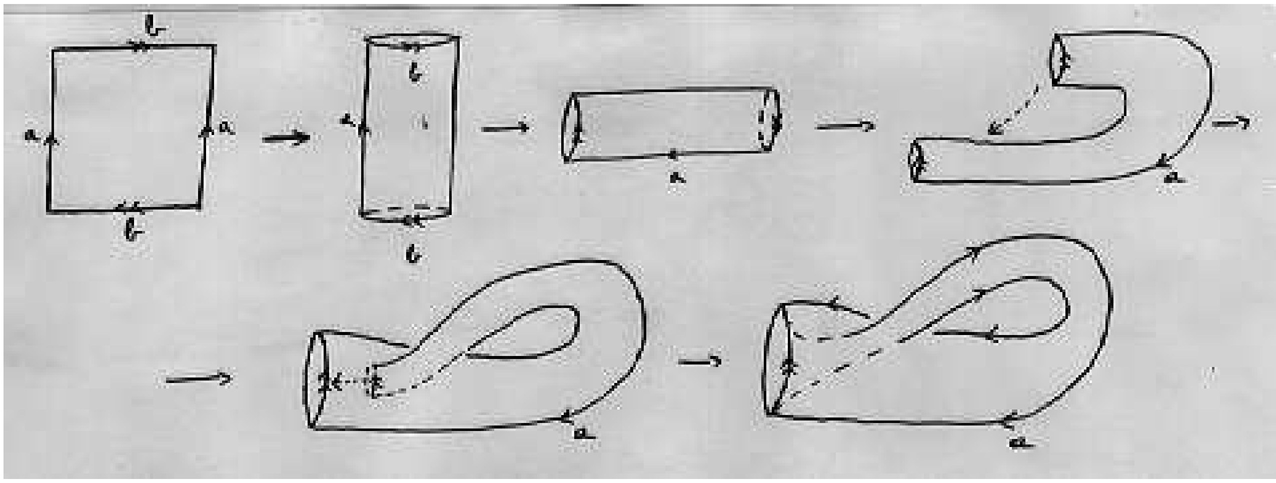

Figure 11: KLEIN BOTTLE

In order to construct the *cross-cap* (Fig.15) start with the square $ABCD$, in which we have to identify the edges $AD - CB$ and $AB - CD$ respectively. Think of it as being a flexible cap, bend it and lower the points B, D, then identify the edges of the square. The extremities of the "crossing-line" $A, C - B, D$ are special points, called "singular". We shall not explain this term here.

Boy's construction of an immersion of the projective plane in 3-space [B] starts with a hexagon divided in three equal pieces which are pasted together like in Fig.16a.

Deform the hexagon to a spherical one like in Fig.16b and take one of the three pieces of sphere which appeared. Start deforming it like in Fig.17 (taken from [H-CV]): join the points A, B, C to a point N and create the loops a, b. Bend the surface by a rotation of the loop a and after the final stage of Fig.17a start moving the loop b till obtaining the surface in Fig.17b. This surface is symmetric in the following sense: a $\frac{2\pi}{3}$ rotation moves c onto d and b onto a. Make the same operations with the two other pieces of sphere (where the loops are denoted $a', b', \ldots$ and $a'', b'', \ldots$), then paste them like in Fig.16a. The final result is depicted in Fig.17c.

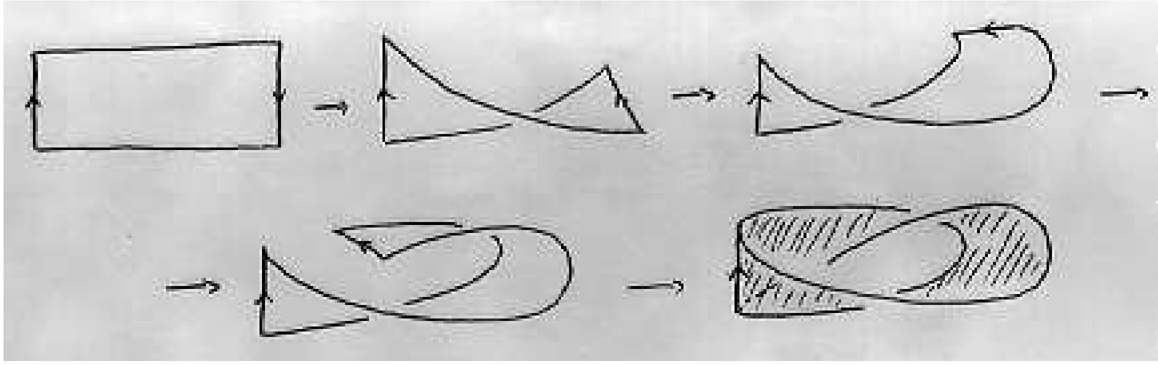

Figure 12: MÖBIUS BAND

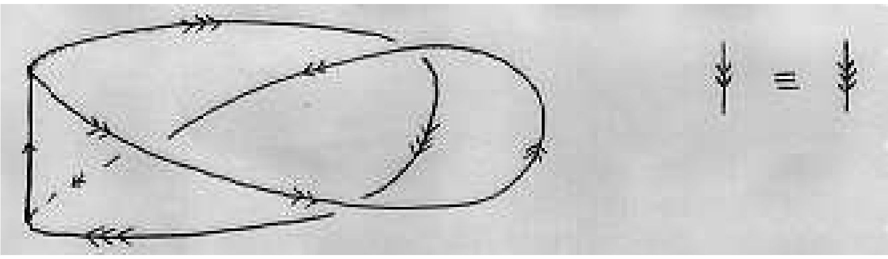

Figure 13: PROJECTIVE PLANE OUT OF MÖBIUS BAND

3 The classification theorem

We have seen by now various closed surfaces. They form together with the operation of connected sum a monoid, whose structure can be described and constitutes the fundamental theorem of topological classification of surfaces. The proof requires some tools from algebraic or differential topology and can be found for example in [Gr], §5 or [D-F-N], §1.3.

Notation: For $q \geq 1$ denote by U_q the connected sum of q copies of $\mathbb{R}P^2$:

$$U_q = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2, \quad q \; times$$

The same argument as for the Σ_g's shows that the U_q's can be represented by $2q$-polygons with the egdges identified as in Fig.18.

Theorem 1. *(Classification of compact surfaces)([Gr]) Any compact surface is homeomorphic to a Σ_g, $g \geq 0$ or to a U_q, $q \geq 1$. No two surfaces in these two families are homeomorphic.*

This theorem can also be stated in terms of the structure of the unitary monoid of homeomorphism classes of compact surfaces.

Theorem 2. *The unitary monoid of homeomorphism classes of compact surfaces is isomorphic to the free abelian unitary monoid generated by the symbols P and T subject to the single relation*

$$P \# T = P^3$$

where P stands for the class of projective space $\mathbb{R}P^2$ and T stands for the class of the torus Σ_1.

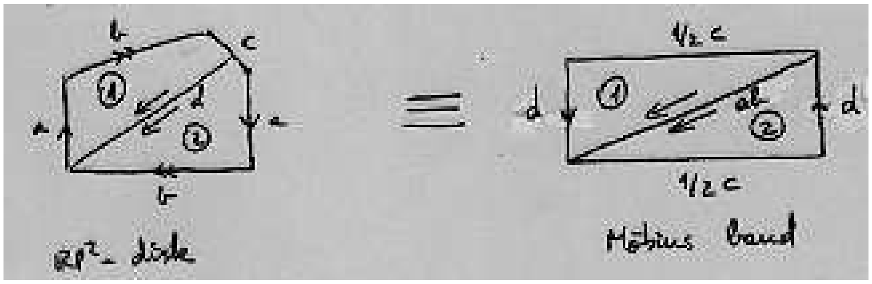

Figure 14: MÖBIUS BAND AS PROJECTIVE PLANE WITH CUT-OUT DISK

Let's verify that we have indeed $\mathbb{R}P^2 \# T^2 \simeq (\mathbb{R}P^2)^{\#3}$. First we notice that $\mathbb{R}P^2 \# \mathbb{R}P^2 \simeq K$ by Fig.19, using the fact that $\mathbb{R}P^2 \setminus$ disk =Möbius band.

Now we follow Fig.20. Starting with a hexagonal representation of $(\mathbb{R}P^2)^{\#3}$ we first replace an $(\mathbb{R}P^2)^{\#2}$ with a Klein bottle K, then cut along the segment $A - D$ in order to exhibit the pasting of a handle (torus) on a copy of $\mathbb{R}P^2$.

The surface classification theorem is a result of central importance in mathematics. Surfaces arise in ALL its branches and it is a good exercise for the reader to trace the implications of this theorem in the various fields that he or she encounters.

Acknowledgements: The moral debt owned to the beautiful book [Fr] is enormous. I would also like to thank H.R.Patapievici and some other participants in the Open Society Foundation summer school "Science and Religion", Tescani, Romania, 1997 for having focused my attention on visualization problems.

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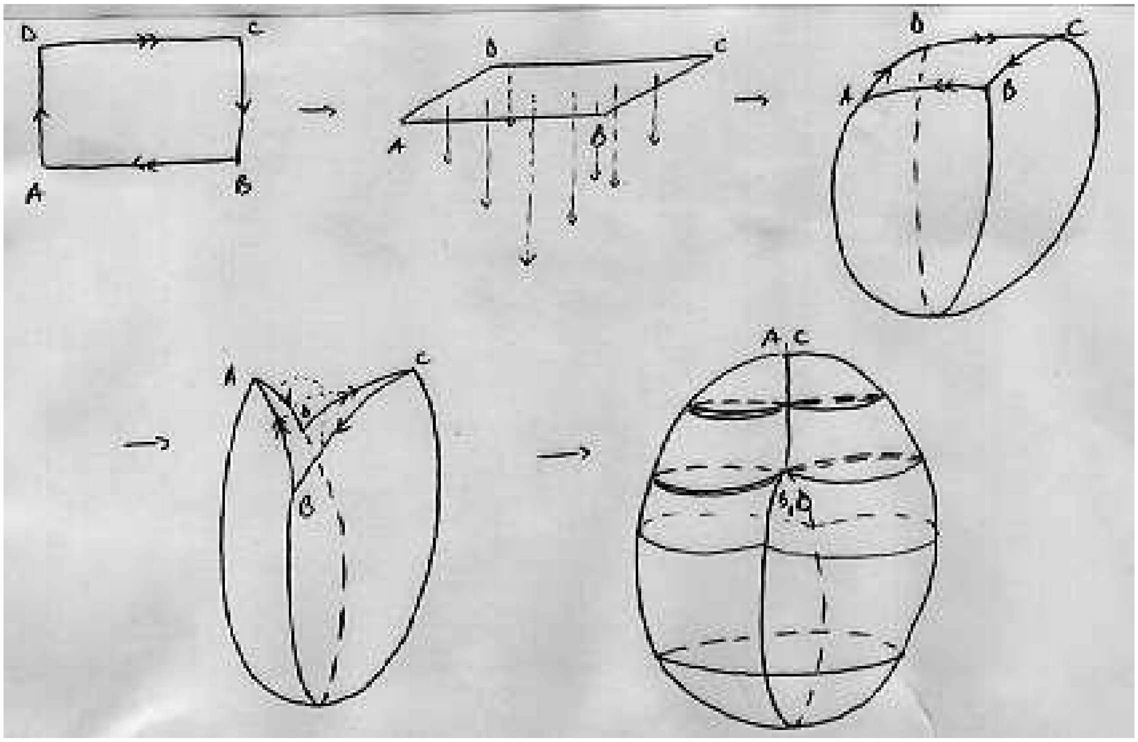

Figure 15: THE CROSS-CAP

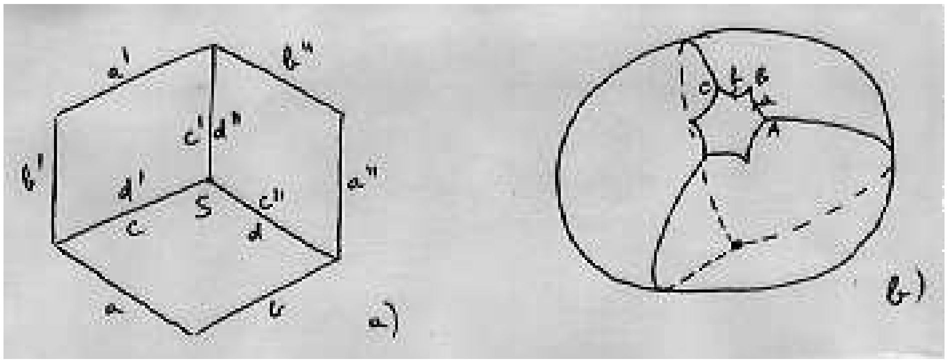

Figure 16: FIRST STEP IN BOY'S CONSTRUCTION OF PROJECTIVE PLANE

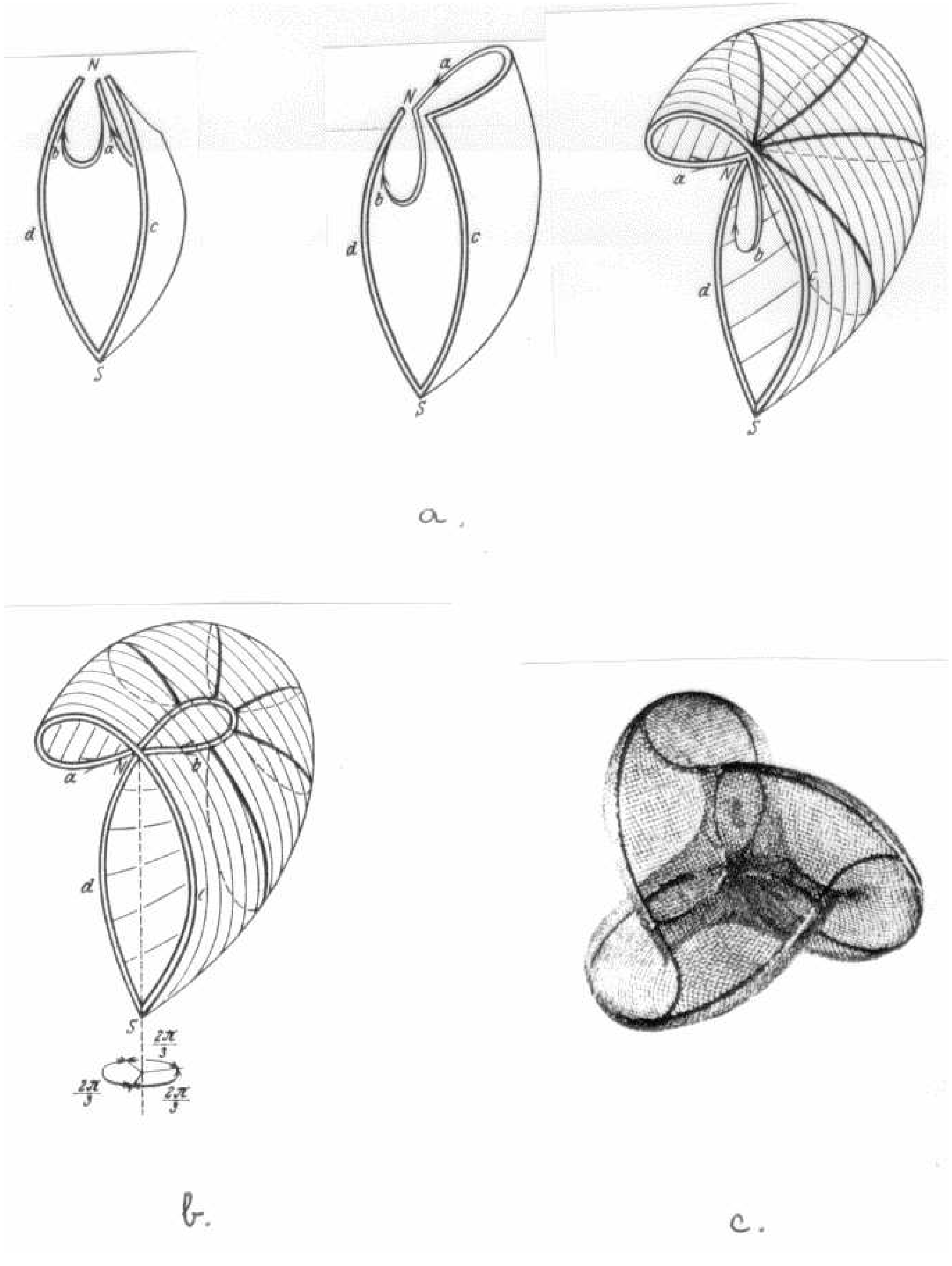

Figure 17: BOY'S MODEL OF PROJECTIVE PLANE (CF. [H-CV])

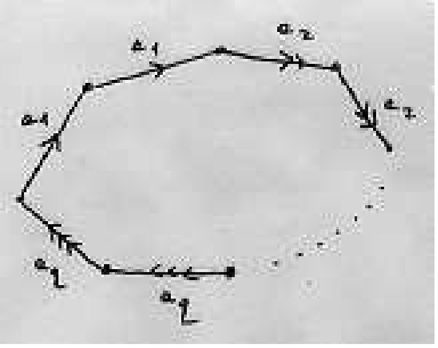

Figure 18: POLYGONAL REPRESENTATION OF $U_q,\ q \geq 1$

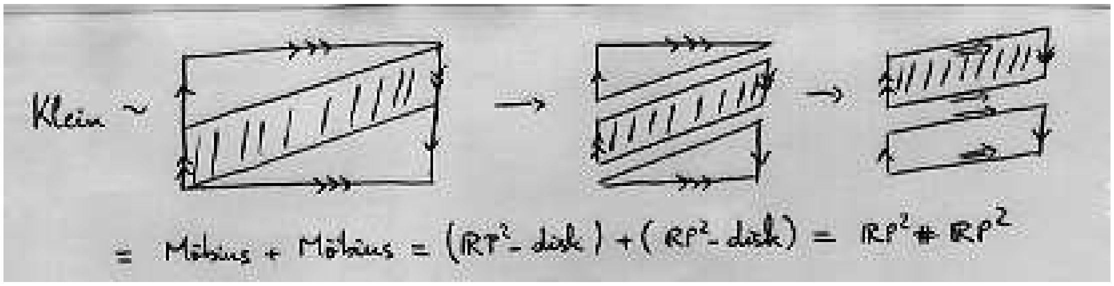

Figure 19: $P\#P = K$

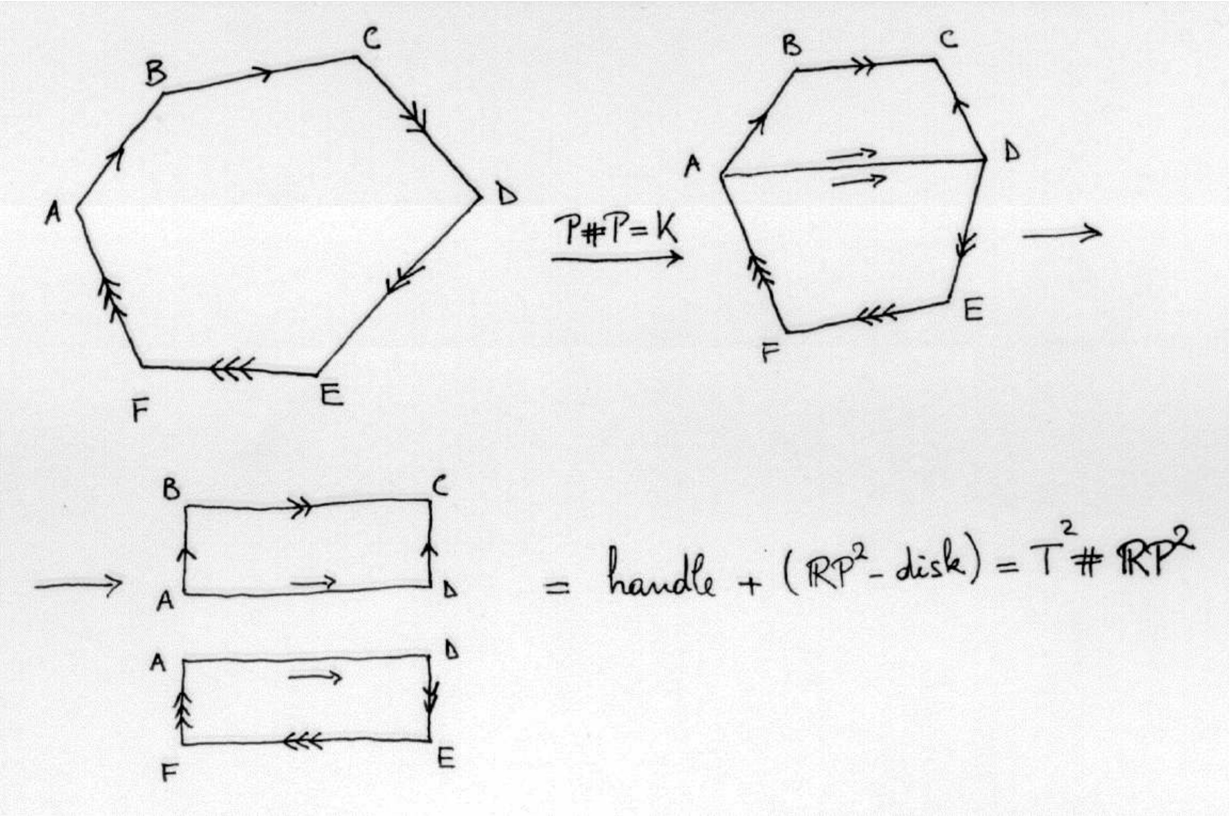

Figure 20: $P^3 = T\#P$