Enhanced convergence estimates for semi-Lagrangian schemes with application to the Vlasov-Poisson system

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Semi-Lagrangian estimates with Strang splitting

\[ O(\Delta t^2 + \frac{\Delta x^{p+1}}{\Delta t}) \]

Optimality of the estimate often raised, not much documented
more likely \( O(\Delta t^2 + \Delta x^p) \) for small time step (CFL \(< \ll 1)\)
cf Tutorial I, Ferretti
see also Bermejo talk, Morton, Süli, 1995 for similar results

⇒ Here such enhanced estimate for the Vlasov Poisson system
⇒ Also refined estimate for the linear advection
⇒ Completed with some overview and other results of SL schemes for Vlasov Poisson
Outline

- Introduction
- Vlasov-Poisson system $1D \times 1D$
- Time discretization for VP
- Space discretization for VP
GYSELA code (GYrokinetic SEmi LAgrangian), CEA Cadarache

(Courtesy V. Grandgirard)

5D mesh of $272 \cdot 10^9$ points. 31 days on 8192 processors
SeLaLib

SEmi LAgrangian Librarry

Goal
modular library for the gyrokinetic simulation model by a
semi-Lagrangian method

Support
- Large scale Initiative Fusion of INRIA
- ANR Project GYPSI (2010-2014)
- INRIA CALVI Project
- Collaboration with CEA Cadarache
**Vlasov equation**

**Distribution function** $f(t, x, \nu)$ solution of the Vlasov equation

$f(t, x, \nu)dx\,d\nu$ represents the probability of finding particles in a volume element $dx\,d\nu$ at time $t$ at point $(x, \nu)$ (position, velocity)

$$\partial_t f + \nu \cdot \nabla_x f + F(t, x) \cdot \nabla_\nu f = 0$$

- Transport equation
- Non linearity through the field $F$ which depends on $f$ (Poisson, Maxwell)
- Description of the dynamic of charged particles in a plasma
Vlasov-Poisson system

\[ \partial_t f(t, x, \nu) + \nu \partial_x f(t, x, \nu) + E(t, x) \partial_\nu f(t, x, \nu) = 0, \]

where the field \( E \) is solution of the Poisson equation

\[ \partial_x E(t, x) = \int_{\mathbb{R}} f(t, x, \nu) d\nu - 1 \]

with zero mean condition \( \int_0^L E(t, x) dx = 0 \)

⇒ Simplified model; first plasmas test cases
⇒ Smooth solution but development of small scales
Time semi-discretization : Strang Splitting

Transport in $x$ over $\Delta t/2$

$$\partial_t f(t, x, v) + v \partial_x f(t, x, v) = 0$$

Update of $E$ through the Poisson equation

Transport in $v$ over $\Delta t$

$$\partial_t f(t, x, v) + E(t, x) \partial_v f(t, x, v) = 0$$

Transport in $x$ over $\Delta t/2$

$$\partial_t f(t, x, v) + v \partial_x f(t, x, v) = 0$$
About time discretizations

- Strang splitting often used (since Cheng-Knorr [1976]), leads to $O(\Delta t^2)$ error
- Higher order splitting possible (Yoshida[1990], Blanes et al [2000,2008], Schaeffer [2009], Watanabe-Sugama [2004])

  splitting steps : $a_0, \ldots, a_{2s}$
  Strang splitting : $s = 1, \ a_0 = 1/2, a_1 = 1, a_2 = 1/2$

- An alternative : Integral Deferred Corrections (Qiu-Christlieb-Morton[2011])
Benefit of high order time discretizations

Strong Landau damping testcase

Energy conservation

strang
strang v-x
triple jump
order 4 s=6
order 6 s=11
strang v-x dt/20

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Order conditions

Theorem (N. Crouseilles, E. Faou, M. M.)

For the Vlasov-Poisson system, with splitting steps $a_0, \ldots, a_{2s}$, we get the same order conditions as RKN conditions derived for ODE up to the fourth order:

\[ p_{2s+1} = 0, \quad B_1 = 1, \quad B_2 = 0, \quad B_{3a} = B_{3b} = 0, \quad B_{4a} = -4B_{4b} = 4B_{4c}, \]

as soon as we assume that the following functions inside the brackets are independent:

- $\{vl_0(x) - 1, l_1(x) - \bar{l}_1\}$,
- $\{\partial_x l_2(x) + v^2\partial_x l_0(x) - 2v\partial_x l_1(x), E(0, x)\}$ and
- $\{-\partial_x^2 l_3(x) + 3\partial_x^2 l_2(x)v - 3v^2\partial_x^2 l_1(x) + v^3\partial_x^2 l_0(x),
(l_0(x) - 1)(l_1(x) - vl_0(x)), (l_0(x) - 1)((l_0(x) - 1)v - \bar{l}_1)\}$.
Order conditions

We have set

\[ p_0 = 0, \quad p_{j+1} = a_j - p_j, \quad j = 0, \ldots, 2s, \]

together with \[ B_1 = \sum_{j=1}^{2s} p_j, \quad B_2 = \sum_{j=1}^{2s} (-1)^j p_j^2 \]

\[ B_{4a} = \sum_{j=1}^{2s} (-1)^j p_j^4, \quad B_{4b} = \sum_{j=1}^{s} (p_{2j}^3 + p_{2j-1}^3) \sum_{k=1}^{2j-1} p_k, \]

\[ B_{4c} = \sum_{j=1}^{s} (p_{2j}^2 - p_{2j-1}^2) \left( \sum_{k=1}^{j-1} p_{2k} \sum_{\ell=1}^{2k-1} p_{\ell} + \sum_{k=1}^{j} p_{2k-1} \sum_{\ell=1}^{2k-1} p_{\ell} \right). \]

\[ I_k(x) = \int_{\mathbb{R}} v^k f_0(x, v) dv, \quad k = 0, 1, 2, \quad \bar{I}_1 = \frac{1}{L} \int_0^L I_1(y) dy \]
Some remarks

- Long but elementary computations: backward characteristics at each substep $\rightarrow$ electric field $\rightarrow$ forward
- Arbitrary number of steps
- Negative coefficients automatically for degree $\geq 3$
- Minimal number of stages not always the best
- Problem of time data representation for too large time steps
- No good order for intermediate electric fields; CK procedure possible for having it (cf Rossmanith, Seal, 2011)
- Link with Poisson structure

\[
\{\{T, U\}_f, U\}_f = 2U,
\]

which implies the RKN type relation

\[
\{\{\{T, U\}_f, U\}_f, U\}_f = 0
\]

$\Rightarrow$ hope to have larger choice of coefficients for degree $\geq 5$. 
Linear transport equations

\[ \partial_t f(t, x, v) + v \partial_x f(t, x, v) = 0, \quad \partial_t f(t, x, v) + E(t, x) \partial_v f(t, x, v) = 0 \]

Some schemes and history

- SL with cubic splines (Cheng-Knorr, 1976)
  - good compromise between cost and accuracy
  - general framework (Sonnendrücker et al., 1998), used in GYSELA
- Fourier-Hermite (see e.g. Schumer, Holloway, 1998)
- SL with Hermite interpolation (Nakamura, Yabe, 1999)
- SL PFC method: positive and conservative (Filbet et al., 2001)
- SL on unstructured grids (Besse, Sonnendrücker, 2003)
- Forward SL (Crouseilles, Respaud, Sonnendrücker, 2009)
- Conservative SL WENO schemes (Qiu, Christlieb, Shu, 2011)
- Discontinuous Galerkin SL (Qiu, Shu; Rossmanith, Seal, 2011; CEMRACS 2010, vladg project)
Conservative formulation

**Linear advection** \( \partial_t f(t, x) + a \partial_x f(t, x) = 0 \)

From \( f^n_j \approx f(t_n, x_j), \ j = 0, \ldots, N - 1 \), reconstruct \( f^n_h \) such that

\[
\frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} f^n_h(x) \, dx = f^n_j
\]

and update through

\[
f^{n+1}_j = \frac{1}{\Delta x} \int_{x_{(j-1/2)^*}}^{x_{(j+1/2)^*}} f^n_h(x) \, dx, \ x_{(j-1/2)^*} = x_{j-1/2} - a \Delta t / \Delta x.
\]

**Framework of uniform grid and periodic conditions**

Equivalence with classical SL point based methods and possibility to add filters (Crouseilles, M., Sonnendrücker, 2010)

Adaptation for the WENO context (Qiu-Shu, 2011)
Lagrange reconstructions : LAG-2d+1

Cell based formulation
Search $f^n_h$ of degree $\leq 2d$ on $]x_{j-1/2}, x_{j+1/2}[$ such that

$$\frac{1}{\Delta x} \int_{x_{k-1/2}}^{x_{k+1/2}} f^n_h(x) dx = f^n_k, \; k = j - d, \ldots, j + d.$$ 

Point based formulation $f^{n+1}_j = P^n_h(x_j - a\Delta t/\Delta x)$, where $P^n_h$ is of degree $\leq 2d + 1$ on $[x^*_j, x^*_j + 1]$, with

$$P^n_h(x_k) = f^n_k, \; k = j^* - d, \ldots, j^* + d + 1$$

- For $d = 0$, upwind scheme (under $CFL \leq 1$)
- $d = 1$, PFC no limiter (Laprise, Plante, 1995; Filbet et al, 2001)
- Strang schemes of odd order (see Després, 2008, 2009) $CFL \leq 1$
- Shifted odd Strang schemes (compact, explicit)
Spline interpolation Use of $B$-splines

$$B_d(x) = \int_{\mathbb{R}} B_{d-1}(t)B_0(x-t)dt, \quad B_0(x) = 1_{[-1/2,1/2]}(x).$$

⇒ spline interpolation on primitive function
⇒ take $f_j - \frac{1}{N} \sum f_k$ for keeping periodic boundary conditions
⇒ FFT like implementation possible

- $d = 3$ generally used
- Classical cubic spline interpolation (point based formulation)
- Also PSM scheme (cell based formulation) Zerroukat et al, 2006
- Possible use of local splines, Crouseilles, Latu, Sonnendr., 2007
Hermite formulation

We consider on the cell \([x_{j-1/2}, x_{j+1/2}], f^n_h\) of degree \(\leq 2\) satisfying

\[
\frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} f^n_h(x) \, dx = f^n_j, \quad f^n_h(x_{j-1/2}^+) = f^n_{(j-1/2)+}, \quad f^n_h(x_{j+1/2}^-) = f^n_{(j+1/2)-}
\]

Centered reconstructions

- stencil \(j - 1 - d, \ldots, j + d\) for \(f^n_{(j-1/2)} = f^n_{(j-1/2)+} = f^n_{(j-1/2)-}\)
  - \(d = 0\) PPM0: \(f^n_{(j-1/2)} = \frac{f^n_{j-1} + f^n_j}{2}\)
  - \(d = 1\) PPM1: Colella, Woodward, 1984
  - \(d = 2\) PPM2: Colella, Sekora, 2008
- PSM with Simpson approximation of \(\int_{x_{j-3/2}}^{x_{j+1/2}} f(x) \, dx\)

Upwind reconstructions

- stencil \(j - d, \ldots, j + d\) for \(f^n_{(j-1/2)+}\)
  - \(d = 1\) : LAG-3
- similar to LAG-2d+1 for small \(\Delta t\); finite volume limit (FOV project, CEMRACS 2011)
Theorem (Charles, Després, M.)

For Vlasov Poisson, with time splitting and LAG-2d+1 reconstruction, we have the convergence estimate

$$\| f^n - f(t_n) \|_2 \leq C \left( \min \left( \frac{\Delta x}{\Delta t}, 1 \right) \Delta x^{2d+1} + \Delta t^2 \right).$$

Error minimisation ($d \geq 1$) for $\Delta t \simeq \Delta x^{(2d+1)/2}$ and thus

$$\min \left( \frac{\Delta x}{\Delta t}, 1 \right) = 1$$

$\Rightarrow$ displacement smaller than a cell, for a semi-lagrangian scheme!

Former result (Besse, M. 2008) : $\Delta x^{2d+2}/\Delta t + \Delta t^2$
Stability of the scheme in $L^2$ norm

- known since Strang, 1962; Strang-Iserles, 1983
- new proofs: Undsdorfer-Verwer, 2003, Besse, M., 2008, Després 2009...
- Després shows also stability on $L^p$, $p \geq 1$ norm for odd Lagrange schemes of the linear advection
- see also work of Falcone-Ferretti, 1998, Ferretti, 2010

**Lemma (Strang, 1962)**

Let $d \in \mathbb{N}^*$, $\theta \in \mathbb{R}$.

Let $P$ polynomial of degree $\leq 2d$ satisfying

$$P(k) = \exp(i\theta k), \ k = -d, \ldots, d.$$ 

Then we have

$$|P(x)| \leq 1, \ -1 \leq x \leq 1.$$
Improved estimates for the linear advection

Lemma (Charles, Després, M.)

Considering the linear advection \( \partial_t f(t, x) + a \partial_x f(t, x) = 0 \) and writing

\[
x_j - a \Delta t = x_{j+r} + \alpha \Delta x, \quad 0 \leq \alpha < 1,
\]

the error for \( n \) steps, \( n \Delta t \leq T \) satisfies

\[
\| (f(t_n, x_j) - f_j^n) \|_2 \leq C_d T \frac{(1 - \alpha) \alpha \Delta x^{2d+2}}{\Delta t} \| u_0^{(2d+2)} \|_{L^2}.
\]

with

\[
C_d = O \left( \frac{(d + 1)!d!}{(2d + 2)!(2d + 2)^{3/4}} \right) = O \left( \frac{1}{2^{2d} d^{1/4}} \right).
\]
Some remarks

- Proof based of fine estimation of the Fourier kernel (Desprès)
- Other proof thanks to results on B-splines maximum norm (Meinardus et al. 1995)
  - use of kernel representation error with B-splines
  - sharp estimate thanks to uniform grid
- Easy proof when no care of sharp constant against $d$ is searched
- Not valid for all interpolation schemes (see e.g. Lax Friedrichs: no convergence for small $\Delta t$!)
- Enables convergence of exponential integrators; link with finite volume schemes (fov project, CEMRACS2011)
- Adaptation to Vlasov-Poisson with electric field via error decomposition (see details in HAL)
Lemma (De Boor, 1976)

Let \( m \in \mathbb{N}^* \), \( \theta \in \mathbb{R} \).

Let \( B_m \) the B-spline of order \( m \) defined by convolution of the characteristic function \( 1_{[-\frac{1}{2}, \frac{1}{2}]} \).

Then the quantity

\[
\Phi_m(\alpha) := \left| \sum_{k \in \mathbb{Z}} B_m(k + \alpha) e^{ik\theta} \right|^2
\]

admit its maximum on the integers.

Used in Besse, M. 2008

Other stability lemma for other reconstructions; generally affordable with computer for given degree; proof more difficult for arbitrary degree.
Error estimation for cubic splines

From Hermite representation, the error writes

\[(\Delta x)^4 \alpha^2 (1 - \alpha)^2 \frac{\max_\xi |f^{(4)}(\xi)|}{4!} + \Delta x \max |f'_j - f'(x_j)| \alpha (1 - \alpha),\]

with cubic splines, we have

\[\max |f'_j - f'(x_j)| \leq C \max_\xi |f^{(5)}(\xi)| (\Delta x)^4,\]

The error then writes

\[\text{error} \leq \min \left(1, \left(\frac{\Delta t}{\Delta x}\right)^2 O\left(\frac{\Delta x^4}{\Delta t}\right) + \min \left(1, \frac{\Delta t}{\Delta x}\right) O\left(\frac{\Delta x^5}{\Delta t}\right)\right).\]

For example: \(\Delta t = \Delta x^2\) leads to \(\text{error} \leq O(\Delta x^4)\).
Discontinuous Galerkin method

C. Steiner, Master report; preliminary results

- Characteristic Galerkin like method

\[ \int_{x_{j-1/2}}^{x_{j+1/2}} f^{n+1}(x) \phi(x) \, dx = \sum_{j=0}^{d} \omega_j f_{i,j} \]

- use of Gauss points
- simplicity of constant advection is helpful
- super convergence: order \(2d + 1\) for mean value
Equivalent equation $d = 2$ and order 5 and 6 terms
Order for $d = 3$
Lemma

We have the relation

\[
\sum_{j,j'=0}^{d} (\alpha_{j'} - \alpha_{j} - 1)^k \int_0^1 \phi'(s)\phi(j(s - \alpha))ds
\]

\[
+ (\alpha_{j'} - \alpha_{j})^k \int_0^\alpha \phi''(s)\phi(j(s + 1 - \alpha))ds = (\alpha - 1)^k, \ k = 1, \ldots, 2d + 1
\]

- We use \(\sum_{j=0}^{d} P(\alpha_{j})\phi(j(s)) = P(s), \deg P \leq d\) and alternatively
  \[
  \int_0^1 P(x)dx = \sum_{j=0}^{d} \omega_j P(\alpha_j), \deg P \leq 2d + 1
  \]
- Related results Lowrie, 1996 ; Cheng, Shu 2008
TSI Lagrange 17 Nx=Nv=2048
SOME NUMERICAL RESULTS

TSI Lagrange 17 Nx=128 Nv=256

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Estimates for SL schemes
TSI PSM $N_x=128$ $N_v=256$
TSI Lagrange 3 $N_x=128$ $N_v=256$
SOME NUMERICAL RESULTS

Fourier $N_x=128$ $N_v=256$

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Estimates for SL schemes
SOME NUMERICAL RESULTS

Temps (sec. vs Nx=Nv) SLD dt=0.1 1000 iterations
Conclusion/Perspectives

- Study of the behavior of SL schemes for small $\Delta t$
- Improvement of convergence estimates of SL schemes for VP
- Fine estimation of error in Lagrange schemes
- To combine with high order in time
- Quality criteria for choosing "best" interpolation ( ?)
- Convergence for such high order SL schemes when no splitting is done or when splitting implies non constant advection : widely open ?
- Convergence on non uniform meshes ?
- Study/Convergence on mapped meshes ?
- Mix of SL (for linear fast dynamic) and FV schemes ?
About filters

- Extrema definition: positive, global, local (Umeda, 2006)
- Extrema limitation: adaptation of Hyman, 1983
- Oscillation limitations
  - tests in Vlasov-Poisson
  - adapted in GYSELA code Braeunig et al 2009, other filters
- useful for Vlasov-Poisson?
- maybe more mandatory for more difficult situations
- loss of symmetry or gain in stability?
- how to prove some mathematical properties?
- Other strategies: WENO (Qiu, Shu)...

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About conservative update in 2D

- CEMRACS 2011, FOV project
  - exploring quadrature based method for the flux
    - CFL condition
    - $2D$ different from $1D$
  - Banks, Hittinger, 2010 methodology
    - better upwind schemes
    - CFL condition

- $2D$ remapping, as in Lauritzen, 2010 (P. Glanc, PhD thesis)
  - no CFL condition a priori
  - needs mesh intersection
  - simplifications as one mesh is cartesian
  - first straight lines approximations
  - coupling with specific time integrator (Crouch-Grossman...)

- Other strategies...