LAGRANGIAN SKELETONS, PERIODIC GEODESIC FLOWS AND SYMPLECTIC CUTTINGS

by

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A remote goal of this work could be the determination of the symplectic manifolds which contain a sphere as a Lagrangian submanifold. As this is out of reach, I will seek after a more modest achievement, namely to try to understand which polarized symplectic manifolds (in a sense derived from [**4**], see [**6**]) have a Lagrangian sphere or a Lagrangian projective space as their isotropic skeleton.

The contents of the paper are as follows. In § 1, I remind the readers of all the known examples (from [**4**]) of polarized symplectic manifolds having a Lagrangian submanifold as isotropic skeleton. Then, in § 2, I look at the topological constraints that the fact of having such a skeleton imposes on the symplectic manifold, proving that they must have the same cohomology as in the examples listed in § 1; this can be considered as a continuation of [**2**]. In § 3, I remind the readers of the construction of the symplectic cutting (of [**7**]), stressing its relevance to the subject of the present paper. I eventually (in § 4) show that all the examples listed in § 1 are symplectic cuts of cotangent bundles of Riemannian manifolds with a periodic geodesic flow.

1. Polarized symplectic manifolds

1.1. Definition. I will use in this paper a very restrictive notion of polarized symplectic manifold. This will be a symplectic manifold, usually called W, endowed with a Morse-Bott function

$$f : W \longrightarrow \mathbf{R}$$

having exactly two critical submanifolds,

- one of which, usually called V, is a symplectic submanifold,
- the other one, usually called L, is a Lagrangian submanifold.

The reason why this is far more restrictive than the definition of polarized Kähler manifolds given in [**6**] is that we require that the isotropic CW-complex on which the complement of V retracts is a Lagrangian submanifold.

As shown in [**2**], the symplectic submanifold V must in general have codimension 2 (we did not insist *a priori* on V satisfying this condition) and the Lagrangian submanifold L must have a rather small fundamental group (a quotient of $\mathbf{Z}$).

1.2. Examples. In Biran's paper [**4**], there is a list of examples of polarized Kähler manifolds. Some of these examples are indeed polarized symplectic manifolds in our restricted sense:

- the symplectic manifold is $\mathbf{CP}^n$, the Lagrangian submanifold L is its "real part" $\mathbf{RP}^n$, the symplectic submanifold V is a projective quadric;
- the symplectic manifold is a projective quadric (of complex dimension n), the Lagrangian submanifold L is a sphere S^n and the symplectic submanifold V is again a quadric;
- the symplectic manifold is the product $\mathbf{CP}^n \times \mathbf{CP}^n$, the Lagrangian submanifold L is a complex projective space $\mathbf{CP}^n$ and the symplectic submanifold V is a bidegree $(1, 1)$-hypersurface.

Notation. The complex projective quadric of equation

$$\sum_{i=0}^{n+1} z_i^2 = 0$$

in $\mathbf{CP}^{n+1}$ is denoted by Q^n (its complex dimension is n). It is classical that this quadric can be identified with the Grassmannian of real oriented 2-planes in $\mathbf{R}^{n+2}$,

$$Q^n = \widetilde{G}_2(\mathbf{R}^{n+2}) = \mathrm{SO}(n+2)/\mathrm{SO}(n) \times \mathrm{SO}(2).$$

For instance, an oriented plane may be described by an orthonormal basis (x, y); consider the mapping form the Stiefel manifold of such bases in $\mathbf{CP}^{n+1}$

$$\begin{aligned} V_2(\mathbf{R}^{n+2}) &\longrightarrow \mathbf{CP}^{n+1} \\ (x,y) &\longmapsto [x+iy] \end{aligned}$$

(denoting with brackets the classes in $\mathbf{CP}^{n+1}$ of the nonzero vectors in $\mathbf{C}^{n+2}$); the equations $\|x\|^2 = \|y\|^2$ and $x \cdot y = 0$ become

$$\sum_{j=1}^{n+1}(x_j^2 - y_j^2) = 0 \text{ et } \sum_{j=1}^{n+1} x_j y_j = 0, \text{ namely } \sum_{j=1}^{n+1} z_j^2 = 0,$$

Moreover the map descends to quotient (under the SO(2)-action) and becomes an isomorphism of the Grassmannian onto its image, the quadric Q^n.

To deal with the third of Biran's examples above, it will be more convenient to consider that $\mathbf{CP}^n \times \mathbf{CP}^n$ is endowed with the symplectic form $\omega \oplus -\omega$ and to look at the codimension-2 submanifold of equation

$$\sum_{i=0}^{n} z_i \overline{w}_i = 0,$$

that we will denote M^{4n-2}.

2. Polarized symplectic manifolds the Lagrangian skeleton of which is a sphere or a real projective space

2.1. The sphere. From the point of view of the cohomology rings, the example of the complex quadric containing a Lagrangian sphere and a codimension 1 quadric is the only possible case.

Proposition 2.1. *Let W be a polarized symplectic manifold the Lagrangian skeleton of which is diffeomorphic to S^n. Then W and its symplectic submanifold V are simply connected, V has codimension 2 in W, the integral cohomology groups of W are isomorphic to those of the complex quadric Q^n and the integral cohomology groups of V to those of the complex quadric Q^{n-1}.*

Remark 2.2. Notice first that, if s is the dual class to $[S^n] \in H_n(W; \mathbf{Z})$, we have

$$s \smile s = e(\nu_{S^n}) = e(T^\star S^n) = \begin{cases} -2 & \text{if } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

As in [**2**], we use a regular level $\mathcal{E}$ of the Morse-Bott function, which is both an oriented S^{n-1}-bundle over S^n, the sphere of the cotangent bundle $T^\star S^n$, and an S^{2k-1}-bundle over V if V has codimension $2k$.

Having proved that $k = 1$, consider a fiber $S^{n-1} \subset \mathcal{E}$ and the homology class of the composition

$$S^{n-1} \subset \mathcal{E} \xrightarrow{\ \pi\ } V.$$

The self-intersection of this class is zero if $n-1$ is odd, call it λ if $n-1$ is even. Denote by $t \in H^{n-1}(V; \mathbf{Z})$ the dual class. We shall prove a more precise statement (giving the ring structure of the integral cohomology of V and W):

Proposition 2.3. *If* $n = 2k$, *that is,* $\dim W = 4k$, $\dim V = 4k-2$,

$$H^\star(V;\mathbf{Z}) = \begin{cases} 0 & \star \text{ odd} \\ \mathbf{Z} \text{ generated by } e^i & \star = 2i \leq 2(k-1) \\ \mathbf{Z} \text{ generated by } u \text{ such that } e^k = 2u & \star = 2k \\ \mathbf{Z} \text{ generated by } ue^i & \star = 2k+2i \geq 2(k+1) \end{cases}$$

and

$$H^\star(W;\mathbf{Z}) = \begin{cases} 0 & \star \text{ odd} \\ \mathbf{Z} \text{ generated by } f^i & \star = 2i \neq 2k \\ \mathbf{Z}\oplus\mathbf{Z} \text{ generated by } s, \text{ the dual class to } S^n \subset W, \text{ and } f^k & \star = 2k \end{cases}$$

and $s \smile s = 2f^{2k}$.

If $n = 2k+1$, *that is,* $\dim W = 4k+2$, $\dim V = 4k$,

$$H^\star(V;\mathbf{Z}) = \begin{cases} 0 & \star \text{ odd} \\ \mathbf{Z} \text{ generated by } e^i & \star = 2i \neq 2k \\ \mathbf{Z}\oplus\mathbf{Z} \text{ generated by } t, \text{ the dual class to } S^{n-1} \subset \mathcal{E} \to V, \text{ and } e^k & \star = 2k. \end{cases}$$

and $t \smile t = \lambda e^{2k}$ *for some* $\lambda \in \mathbf{Z}$.

$$H^\star(W;\mathbf{Z}) = \begin{cases} 0 & \star \text{ odd} \\ \mathbf{Z} \text{ generated by } f^i & \star = 2i \leq 2k \\ \mathbf{Z} \text{ generated by } v \text{ such that } f^{k+1} = 2v & \star = 2k+2 \\ \mathbf{Z} \text{ generated by } vf^i & \star = 2k+2i \geq 2(k+2). \end{cases}$$

In the case where V and W are indeed complex quadrics, the integer λ is equal to -2, as a consequence of the next lemma, that will be proved in § 4 (this will be a consequence of Proposition 4.2).

Lemma 2.4. *Let* $x \in S^n$ *and let* S_x^{n-1} *denote the unit sphere in the cotangent space* $T_x^\star S^n$. *Then the composed mapping*

$$S_x^{n-1} \subset \mathcal{E} \longrightarrow Q^{n-1}$$

is a Lagrangian embedding.

Thus, this could also be considered as a way to compute the fundamental groups and cohomology rings of complex quadrics...

Corollary 2.5. *If* $n = 2k$, *that is,* $\dim_{\mathbf{R}} Q = 4k$,

$$H^\star(Q;\mathbf{Z}) = \begin{cases} 0 & \star \text{ odd} \\ \mathbf{Z} \text{ generated by } f^i & \star = 2i \neq 2k \\ \mathbf{Z}\oplus\mathbf{Z} \text{ generated by } s \text{ dual class to } S^n \subset W \text{ and } f^k & \star = 2k \end{cases}$$

and $s \smile s = -2f^{2k}$.

If $n = 2k+1$, *that is,* $\dim_{\mathbf{R}} Q = 4k+2$,

$$H^\star(Q;\mathbf{Z}) = \begin{cases} 0 & \star \text{ odd} \\ \mathbf{Z} \text{ generated by } f^i & \star = 2i \leq 2k \\ \mathbf{Z} \text{ generated by } v \text{ such that } f^{k+1} = 2v & \star = 2k+2 \\ \mathbf{Z} \text{ generated by } vf^i & \star = 2k+2i \geq 2(k+2). \end{cases}$$

Proof of Propositions 2.1, 2.3 and corollary 2.5. The proof is by soft techniques, namely by elementary algebraic topology. Let us call $\mathcal{U}$ a neighborhood of L and $\mathcal{V}$ a neighborhood of V. We consider $\mathcal{U}$ and $\mathcal{V}$ as disc bundles, they retract on L, V respectively, and their intersection retracts on $\mathcal{E}$.

In the case $n = 2$, [**2**, Propostion 2.3.1] gives that $V = S^2$ and W is diffeomorphic with $S^2 \times S^2$, so that the announced results are true. Let us thus assume that $n \geq 3$.

Let us prove first that V and W are simply connected. The homotopy exact sequence of the fibration

$$S^{n-1} \subset \mathcal{E} \longrightarrow S^n$$

gives that $\mathcal{E}$ is simply connected. That of the fibration

$$S^{2k-1} \subset \mathcal{E} \longrightarrow V$$

gives that V is simply connected as well and the Van Kampen theorem gives the fact that W also is simply connected.

Let us prove now that V has codimension 2. Otherwise, the exact homotopy sequence of the second fibration gives an isomorphism $\pi_2(\mathcal{E}) \to \pi_2(V)$, so that the projection map induces an isomorphism $H^2(V) \to H^2(\mathcal{E})$. The Mayer-Vietoris exact sequence for $W = \mathcal{U} \cup \mathcal{V}$ gives

$$\longrightarrow H^2(W) \longrightarrow H^2(L) \oplus H^2(V) \longrightarrow H^2(\mathcal{E}) \longrightarrow$$

hence the restriction map $H^2(W) \to H^2(V)$ must be zero. However (using real coefficients) the cohomology class of the symplectic form on W should be mapped to the class of the symplectic form on V, a contradiction. This shows that V has codimension 2.

Let us use now the Gysin exact sequence of the cotangent bundle of S^n to compute the cohomology of $\mathcal{E}$,

$$\longrightarrow H^j(\mathcal{E}) \longrightarrow H^{j-n+1}(S^n) \xrightarrow{\smile \chi} H^{j+1}(S^n) \longrightarrow H^{j+1}(\mathcal{E}) \longrightarrow$$

with $\chi \in H^n(S^n; \mathbf{Z})$ the Euler class of the tangent bundle, namely 0 if n is odd and 2 times the generator if n is even. Hence we get

$$H^\star(\mathcal{E}; \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } \star = 0, 2n-1 \\ \mathbf{Z} & \text{if } \star = n-1, n \text{ and } n \text{ is odd} \\ \mathbf{Z}/2 & \text{if } \star = n \text{ and } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Let us then use the Gysin exact sequence of the circle bundle over V to compute the cohomology of V,

$$\longrightarrow H^j(\mathcal{E}) \longrightarrow H^{j-1}(V) \xrightarrow{\smile e} H^{j+1}(V) \longrightarrow H^{j+1}(\mathcal{E}) \longrightarrow$$

denoting by $e \in H^2(V; \mathbf{Z})$ the Euler class of this circle bundle.

Notice firstly that, starting from $j = 0$, an induction gives that the odd degree cohomology groups are zero. Let us look at the even degrees. Taking $j = 1$ gives that e generates $H^2(V; \mathbf{Z}) = \mathbf{Z}$, except if $n = 3$ in which case it generates a summand in $H^2(V; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$. Then, all the even groups are isomorphic with $\mathbf{Z}$, except for $H^{2k}(V; \mathbf{Z})$ if $n = 2k + 1$. Let us consider separately the cases where n is odd or even.

Assume first that $n = 2k + 1$ is odd. We get two short exact sequences

$$0 \longrightarrow H^{2k-2}(V) \xrightarrow{\smile e} H^{2k}(V) \xrightarrow{\pi^\star} H^{2k}(\mathcal{E}) \longrightarrow 0$$

and

$$0 \longrightarrow H^{2k+1}(\mathcal{E}) \xrightarrow{\pi_!} H^{2k}(V) \xrightarrow{\smile e} H^{2k+2}(V) \longrightarrow 0$$

which give $H^{2k}(V) = \mathbf{Z} \oplus \mathbf{Z}$, one of the summands being generated by e^k. Let $t \in H^{2k}(V)$ be the image of a generator of $H^{2k+1}(\mathcal{E})$, so that $\pi^\star t$ generates $H^{2k}(\mathcal{E})$ and $t \smile e = 0 \in H^{2k+2}(V)$. The higher groups $H^{2j}(V)$ are generated by the powers of e, so that we only need to compute $t^2 \in H^{2k}(V)$. By definition of $\pi_!$, the class t is dual to the image of the generator by

$$H_{n-1}(\mathcal{E}) \xrightarrow{\pi_\star} H_{n-1}(V)$$

which by Hurewicz is the class of a fiber of $\mathcal{E} \to L$

$$S^{n-1} \subset \mathcal{E} \xrightarrow{\pi} V.$$

By definition of λ, we have $t^2 = \lambda e^{2k}$, e^{2k} being the generator of the top dimensional cohomology group. This proves the assertions on the integral cohomology (groups and ring) of V if $n = 2k + 1$.

When $n = 2k$, we need only consider

$$H^{n-1}(\mathcal{E}) \longrightarrow H^{n-2}(V) \xrightarrow{\smile e} H^n(V) \longrightarrow H^n(\mathcal{E}) \longrightarrow H^{n-1}(V),$$

namely

$$0 \longrightarrow \mathbf{Z}\langle e^{k-1} \rangle \xrightarrow{\smile e} H^n(V) \longrightarrow \mathbf{Z}/2 \longrightarrow 0$$

by induction. To determine $H^n(V)$, we notice that $H_{n-1}(V; \mathbf{Z}) \equiv H^{n-1}(V; \mathbf{Z})$ by Poincaré duality and that the latter is 0 since $n-1$ is odd, hence $H^n(V; \mathbf{Z})$ is isomorphic with $\operatorname{Hom}(H_n(V; \mathbf{Z}), \mathbf{Z})$, that is, by Poincaré duality, with $H^{n-2}(V;\mathbf{Z}) = \mathbf{Z}$. The Gysin exact sequence above shows that e^k is twice the generator in this group. This is what we wanted to prove on the cohomology of V when n is even.

Let us now look at the ambient symplectic manifold W itself. We use the decomposition $W = \mathcal{U} \cup \mathcal{V}$, the homotopy equivalences $\mathcal{U} \sim L$, $\mathcal{V} \sim V$ and $\mathcal{U} \cap \mathcal{V} \sim \mathcal{E}$ and the Mayer-Vietoris exact sequence. It gives that $H^2(W;\mathbf{Z})$ is isomorphic with $H^2(V;\mathbf{Z}) = \mathbf{Z}$ and is generated by a class f the restriction of which is the Euler class e already used. Notice that, since e is the Euler class of the normal bundle to V in W, f is the dual class to $[V] \in H_{2n-2}(W;\mathbf{Z})$.

Assume first that $n = 2k$ is even. All the cohomology groups of $\mathcal{E}$ are zero except for $H^n(\mathcal{E};\mathbf{Z}) = \mathbf{Z}/2$. Since we know that the natural map $H^n(V;\mathbf{Z}) \to H^n(\mathcal{E};\mathbf{Z})$ is onto, we get that $H^{n+1}(W;\mathbf{Z}) = 0$ (and all the odd-degree groups as well). At this level, Mayer-Vietoris gives

$$0 \longrightarrow H^n(W) \longrightarrow H^n(S^n) \oplus H^n(V) \longrightarrow H^n(\mathcal{E}) \longrightarrow 0$$

namely

$$0 \longrightarrow H^n(W) \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \longrightarrow \mathbf{Z}/2 \longrightarrow 0,$$

hence $H^n(W;\mathbf{Z}) \equiv \mathbf{Z} \oplus \mathbf{Z}$, generated by f^k and by the class dual to the inclusion of the Lagrangian sphere $S^n \subset W$, let us call it s, the self-intersection of v is $2f^{2k}$, since the normal bundle of S^n in W is isomorphic with its tangent bundle.

If $n = 2k+1$ is odd, we look again at Mayer-Vietoris,

$$0 \longrightarrow H^{n-1}(W) \longrightarrow H^{n-1}(V) \longrightarrow H^{n-1}(\mathcal{E}) \longrightarrow H^n(W) \longrightarrow H^n(S^n) \longrightarrow H^n(\mathcal{E}) \longrightarrow$$

in which we know that the two maps $H^n(S^n) \to H^n(\mathcal{E})$ and $H^{n-1}(V) \to H^{n-1}(\mathcal{E})$ are onto, hence $H^n(W)$ is zero (and so are all the odd-degree cohomology groups) and $H^{n-1}(W) = \mathbf{Z}$ and so are all the even-degree cohomology groups. Moreover $H^{2k+2}(W)$ is generated by f^{k+1} as $H^{2k+2}(V;\mathbf{Z})$ is generated by e^{k+1}. The group $H^{2k}(W)$ is generated by an element ℓ which is mapped to u by the restriction map to V. □

Proposition 2.6. *Let W be a polarized symplectic manifold the Lagrangian skeleton of which is diffeomorphic with S^n. Then W is monotone if and only if V is monotone.*

Proof. As we have just seen it, the second Betti numbers of W and V are 1 and the normal bundle of V in W has first Chern class e, hence we have

$$c_1(TV) = \mu e, \quad c_1(TW) = (\mu+1)f$$

and, since V is endowed with the symplectic form induced by that of W, we have

$$c_1(TW) = \alpha[\omega_W], \quad c_1(TV) = \frac{\mu}{\mu+1}\alpha[\omega_V].$$

This proves the assertion. □

2.2. The real projective plane and other quotients.

Proposition 2.7. *Assume that the closed manifold L is the quotient of S^n by the free action of a finite cyclic group $\mathbf{Z}/m$ and that this is the Lagrangian skeleton of some polarized symplectic compact manifold W. Let V be the symplectic submanifold of W on which the complement of V retracts. Then there exists a polarized symplectic manifold $\widetilde{W}$, the Lagrangian skeleton of which is S^n, the symplectic submanifold is V, and $\widetilde{W}$ is an m-fold covering of W, branched along V.*

Notice in particular that the symplectic submanifold is of the kind considered in § 2.1.

Corollary 2.8. *Assume the symplectic manifold W is polarized and its Lagrangian skeleton is he quotient of S^n by the free action of a finite cyclic group $\mathbf{Z}/m$. Then W and its symplectic submanifold V are simply connected.*

Corollary 2.9. *Assume the real projective space* $\mathbf{RP}^n$ *is the Lagrangian skeleton of a polarized symplectic manifold* W. *Then, there exists a class* $e \in H^2(V; \mathbf{Z})$ *such that, if* $n = 2k$,

$$H^\star(V;\mathbf{Z}) = \begin{cases} 0 & \star \textit{ odd} \\ \mathbf{Z} \textit{ generated by } e^i & \star = 2i \leq 2(k-1) \\ \mathbf{Z} \textit{ generated by } u \textit{ such that } e^k = 2u & \star = 2k \\ \mathbf{Z} \textit{ generated by } ue^i & \star = 2k + 2i \geq 2(k+1) \end{cases}$$

and, if $n = 2k+1$,

$$H^\star(V;\mathbf{Z}) = \begin{cases} 0 & \star \textit{ odd} \\ \mathbf{Z} \textit{ generated by } e^i & \star = 2i \neq 2k \\ \mathbf{Z} \oplus \mathbf{Z} \textit{ generated by } t \textit{ dual class to } S^{n-1} \subset \mathcal{E} \to V \textit{ and } e^k & \star = 2k. \end{cases}$$

and $t \smile t = \lambda e^{2k}$ *for some* $\lambda \in \mathbf{Z}$. □

Notice that e is not the Euler class of the circle bundle of a tubular neighborhood of V in W here. The Euler class of this bundle is $2e$.

Assume in general that V is a symplectic codimension-2 submanifold of a symplectic manifold W. There is a Darboux type theorem giving a normal form for the symplectic form of W in a neighborhood of V (the more general statement of Darboux type is that of [**1**]). Here is a well-known construction of such a form, that I include for completeness. Having chosen a calibrated almost complex structure on W, a tubular neighborhood of V is diffeomorphic with a disc bundle of a complex line bundle $\pi : E \to V$. The unit circle bundle $\pi : S(E) \to V$ can be endowed with a connection 1-form α, namely with a form α such that

$$\begin{cases} i_X\alpha \equiv 1 & \text{for } X \text{ the fundamental vector field of the action of } S^1 \text{ by rotations in the fiber} \\ d\alpha = \lambda\pi^\star\omega_V & \text{for some real constant } \lambda \text{ such that } \dfrac{\lambda}{2\pi}[\omega_V] = c_1(E) \in H^2(V;\mathbf{Z}). \end{cases}$$

The 2-form $\Omega_E = \pi^\star\omega_V + d(R^2\alpha)$ (using hopefully obvious notation where R is the square of the radius of an element in a fiber) is a symplectic form as long as $1 + \lambda R^2 > 0$, which is certainly the case on a small enough disc bundle (and is always the case for $\lambda > 0$). Let us now use this to prove Proposition 2.7.

Proof of Proposition 2.7. Call $F \to V$ the (complex) line bundle which is the normal bundle of the symplectic submanifold. Since $W - V$ retracts on L, we can consider the universal coverings

$$\begin{array}{ccc} S^n & \hookrightarrow & \widetilde{W - V} \\ \downarrow & & \downarrow p \\ L & \hookrightarrow & W - V \end{array}$$

and endow $\widetilde{W - V}$ with the symplectic form $p^\star\omega_W$. In restriction to the complement of V in a tubular neighborhood of V in W, this is isomorphic with the covering $F^{\otimes m} \to F$ (outside the zero section), so that the form $p^\star\omega_W$ is isomorphic with the pull-back of our form Ω_F, which the same as the form $\Omega_{F^{\otimes m}}$ if we choose the connection forms on $S(F)$ and $S(F^{\otimes m})$ in a compatible way. The manifold $\widetilde{W - V}$ can thus be completed in a compact symplectic manifold $\widetilde{W}$ by gluing to it a disc bundle of $F^{\otimes m}$. The latter is obviously an m-fold covering of W, branched along V, as expected. □

Proof of Corollary 2.8. The submanifold V is simply connected as we have seen it in § 2.1. Denoting as usual by $\mathcal{E}$ a regular level of the Morse-Bott function giving the polarization, the exact homotopy sequence of $\mathcal{E} \to L$ gives that $\pi_1\mathcal{E} \to \pi_1(L)$ is onto. Then the Van Kampen theorem gives that W is simply connected. □

3. Symplectic cuttings

A convenient way to produce symplectic manifolds having a codimension 2 symplectic submanifold is by symplectic cutting.

3.1. The symplectic cutting. Let W be a symplectic manifold and assume that

$$H : W \longrightarrow \mathbf{R}$$

is a periodic Hamiltonian, namely the moment map of a Hamiltonian circle action. In [**7**], Lerman showed that any sublevel

$$W_t = H^{-1}(]-\infty, t])$$

corresponding to a regular value t of H can be modified to produce a symplectic manifold (without boundary) $\widetilde{W_t}$ on which the function H is still a periodic Hamiltonian

$$H : \widetilde{W_t} \longrightarrow \mathbf{R}$$

the maximum of which is t. The level that corresponds to this value is a symplectic submanifold, which is nothing else that the reduced symplectic manifold $H^{-1}(t)/S^1$ of the given Hamiltonian. This is a simple and clever construction, which has already proven to be useful (see for instance [**8**]). We will use it here to give another construction of polarized symplectic manifolds.

The actual definition of W_t is indeed very simple: the S^1-action is extended to the product $W \times \mathbf{C}$ by

$$u \cdot (x, z) = (u \cdot x, uz),$$

which is a Hamiltonian action, the moment map of which is

$$\begin{aligned} \widetilde{H} : W \times \mathbf{C} &\longrightarrow \mathbf{R} \\ (x, z) &\longmapsto H(x) + \frac{1}{2}|z|^2. \end{aligned}$$

The value t is regular for $\widetilde{H}$ if and only if it is regular for H. The symplectic cutting W_t is simply the reduced symplectic manifold

$$W_t = \widetilde{H}^{-1}(t)/S^1.$$

Figure 1 shows the construction rather efficiently.

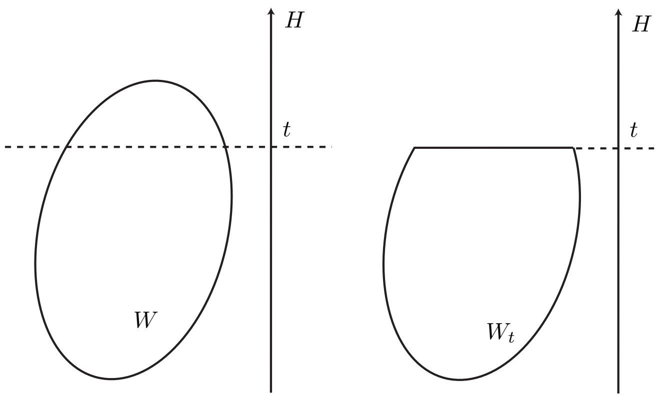

FIGURE 1

One usually cuts a compact symplectic manifold, but there is no reason why we should restrict to these. In the two constructions below, we cut a non compact symplectic manifold.

3.2. Symplectic cuttings of symplectizations of certain contact manifolds. Let $\mathcal{E}$ be a contact manifold of dimension $2n-1$. Assume the contact structure is defined by a 1-form α. The symplectization of $\mathcal{E}$ is the product manifold $\mathcal{E} \times \mathbf{R}$, endowed with the symplectic form $\omega = d(e^u\alpha)$.

Let us make now the crucial and restrictive assumption that the Reeb vector field $\mathcal{R}$ of α is periodic, in the sense that its flow is periodic. The extended vector field

$$\mathcal{R}(x,u) = \mathcal{R}(x) \text{ on } \mathcal{E} \times \mathbf{R}$$

is the Hamiltonian vector field of the function $H(x,u) = e^u$, a periodic Hamiltonian under our assumption. The symplectic cutting of $\mathcal{E} \times \mathbf{R}$ at the level $t > 0$ is a symplectic manifold which contains both the contact manifold $\mathcal{E}$ (as the 0-level) and the symplectic manifold V which is both the space of orbits of the Reeb field and the space of characteristics of the hypersurface $\mathcal{E}$.

There is no reason why such a symplectic manifold could in general be compactified to yield a polarized compact manifold. If this were the case, the contact manifold $\mathcal{E}$ would be the total space of a sphere bundle over the Lagrangian skeleton. The example of $\mathcal{E} = S^{2n-1}$ with the standard circle action shows that this is not always the case. It is easy to see that S^{2n-1} cannot be the sphere bundle of the cotangent bundle of a manifold L: L would be simply connected and in particular orientable, then the Gysin exact sequence would give that $H^\star(L;\mathbf{Z}) = H^\star(S^n;\mathbf{Z})$ so that the Euler characteristic of L is 0 or 2 according to the parity of n, and that the cup product with the Euler class is an isomorphism $H^0(L;\mathbf{Z}) \to H^n(L;\mathbf{Z})$, which is the desired contradiction.

The next step is to consider the case where $\mathcal{E}$ is indeed a sphere bundle on a manifold.

4. Symplectic cuttings of cotangent bundles

4.1. The construction. Let us now consider a manifold L of dimension n and its cotangent bundle $T^\star L$. Assume that L is endowed with a Riemannian metric. This is not a restrictive assumption. The restrictive assumption will be that the geodesic flow of this metric is periodic. The known examples of such a situation are (see [**3**])

– the "round" sphere S^n, namely the unit sphere in the Euclidean space $\mathbf{R}^{n+1}$ with the induced metric;
– the real projective space $\mathbf{RP}^n$ with the metric deduced from that of the round sphere;
– the complex projective space $\mathbf{CP}^n$ endowed with the "Fubini-Study" metric, namely the usual metric;
– the quaternionian projective space $\mathbf{HP}^n$ with the natural *ad hoc* metric.

Using the metric on L, we identify its tangent bundle with its cotangent bundle. The geodesic flow is defined on the complement of the zero section in $T^\star L$: a point of L and a nonzero tangent vector determine a unique geodesic. The corresponding Hamiltonian is the function

$$\begin{aligned} h : T^\star L - L &\longrightarrow \mathbf{R} \\ (x,y) &\longmapsto \|y\|. \end{aligned}$$

Fix a regular value of H (all the values are regular), say the value 1. The level $H^{-1}(1)$ is the unit sphere bundle $S(T^\star L)$. The symplectic reduction of this level is the quotient $S(T^\star L)/S^1$: two pairs consisting of a point in L and a unit vector tangent to L at this point are identified with each other when the define the same geodesic. The reduced symplectic manifold is simply the space of (unparametrized but oriented) geodesics of the Riemannian manifold L. This is a manifold V of dimension $2n-2$ and it is well known (see again [**3**]) that this is indeed a symplectic manifold.

The symplectic cutting is thus the union of the symplectic manifold V and the complement of the zero section in the open disc bundle $D(T^\star L)$, with its original symplectic structure. It is then possible to compactify this symplectic cutting by adding to it the zero section that we had removed. The symplectic manifold is compact (if L was compact) and contains L as a Lagrangian submanifold.

Proposition 4.1. *Let L be a compact n-dimensional Riemannian manifold the geodesic flow of which is periodic. There exists a polarized compact symplectic manifold W, the Lagrangian skeleton of which is L and the polarization of which is the space of geodesics in L.*

Proof. The only thing which is left to prove is the existence of a Morse-Bott function the critical levels of which are L and V. The function

$$f([x,y,z]) = \|y\|^2 ,$$

that is, the square of the original Hamiltonian, has all the desired properties. □

4.2. Lagrangian spheres in V. By construction, the complement of V in W is the disc bundle of $T^\star L$. On the other side, the complement of V is fibered over L. Call

$$\pi : W - L \longrightarrow V$$

the projection, namely the map which, as the reduction at the level 1, maps a point and a tangent vector to the geodesic through this point tangent to this vector.

The symplectic manifold W contains a lot of isotropic spheres (of dimension $n-1$), namely all the fibers

$$S_r(T^\star_x L) = \{y \in T^\star_x L \mid \|y\| = r\}$$

for $r \in]0,1[$. We have already used a special case of the next proposition in § 2.1 (this was Lemma 2.4).

Proposition 4.2. *The composition*

$$S_r(T^\star_x L) \subset W - L \xrightarrow{\ \pi\ } L$$

is a Lagrangian embedding.

So that the symplectic manifold V contains many Lagrangian spheres.

Proof. The map under consideration is clearly injective. Let us check that this is an immersion. The kernel of the tangent mapping to

$$S_r(T^\star L) \subset W - L \xrightarrow{\ \pi\ } V$$

is generated by the fundamental vector field of the circle action at this level (the level r) of H. We just have to check that this vector field is tangent to the fibers of the cotangent bundle. This is indeed the case, because its image under

$$T_{(x,y)}(T^\star L) \xrightarrow{T_{(x,y)}\pi} T^\star_x L$$

is $y/\|y\|$, a nonzero vector. □

The Lagrangian sphere which is the image of $S_r(T^\star_x L)$ is the set of all the geodesics through x. Notice that two such spheres, images of fibers at x and x' always intersect: their intersection consists of the geodesics through x and x'. In general this is a unique point. In view of Proposition 4.3, this is to be compared with the results proven in [**5**] about the intersections of Lagrangian spheres in quadrics.

4.3. The examples. Let us show now that the examples obtained from periodic geodesic flows as in § 4 are also the examples which appear in [**2**] as examples of symplectic manifolds satisfying Proposition 4.1.

Proposition 4.3. *The space of geodesics is isomorphic, as symplectic manifold, to*

- *the quadric Q^{n-1} when L is the round sphere S^n or the real projective space $\mathbf{RP}^n$;*
- *the codimension-2 submanifold M^{4n-2} when L is the complex projective space $\mathbf{CP}^n$.*

Proof. A geodesic of the round sphere is an oriented great circle, that is, the intersection with the sphere of an oriented plane of the Euclidean space $\mathbf{R}^{n+1}$. The space of geodesics of S^n is thus the Grassmannian $\widetilde{G}_2(\mathbf{R}^{n+1})$ and so is the space of geodesics of $\mathbf{RP}^n$. Hence, both are the quadric Q^{n-1}.

A geodesic in $\mathbf{CP}^n$ is a geodesic in a unique complex projective line $\mathbf{CP}^1 \subset \mathbf{CP}^n$. Hence the manifold of oriented geodesics is an S^2-bundle over $G_2(\mathbf{C}^{n+1})$. As we have seen it in the $\mathbf{CP}^1 = S^2$

case, to give an oriented geodesic in $\mathbf{CP}^1$ is equivalent to giving a point in $\mathbf{CP}^1$. Hence the space of oriented geodesics in $\mathbf{CP}^n$ is

$$\begin{aligned} V &= \big\{(P,\ell) \in G_2(\mathbf{C}^{n+1} \times \mathbf{CP}^n \mid \ell \subset P\big\} \\ &= \{(\ell,d) \in \mathbf{CP}^n \times \mathbf{CP}^n \mid \ell \perp d\} \\ &= M^{4n-2} \subset \mathbf{CP}^n \times \mathbf{CP}^n \end{aligned}$$

as expected. □

Let us now describe the symplectic manifolds obtained by symplectic cutting in this situation.

Proposition 4.4. *The symplectic manifold obtained as the symplectic cutting of the cotangent bundle is isomorphic, as a symplectic manifold, to*

- *the quadric Q^n if $L = S^n$ is the round sphere;*
- *the complex projective space $\mathbf{CP}^n$ if $L = \mathbf{RP}^n$ is the real projective space;*
- *the product $\mathbf{CP}^n \times \mathbf{CP}^n$ endowed with the symplectic form $\omega \oplus -\omega$ if $L = \mathbf{CP}^n$ is the complex projective space.*

Proof. Let us call W the manifold we suspect to be the symplectic cutting, L the Riemannian manifold we started from and V the space of oriented geodesics. We need to check

- that L is indeed a Lagrangian submanifold in W,
- similarly, that V is a symplectic submanifold,
- that there exists a periodic Hamiltonian

$$H : W - L \longrightarrow \mathbf{R}$$

  such that
  - the manifold of fixed points is V,
  - the regular levels are isomorphic with $S(T^\star L)$ with the geodesic flow of L.

In the three cases, the first two properties are clearly satisfied. For the construction of the Hamiltonian action, let us consider first the case of the n-sphere.

Geodesics on S^n, definition of an S^1-action. The simplest thing to do is probably to consider our suspect, the quadric Q^n, as being the space of geodesics on the $(n+1)$-sphere. Let us add to $\mathbf{R}^{n+1}$ a summand generated by a vector e_0 (the corresponding coordinates denoted x_0,y_0). Hence the manifold Q^n of geodesics in S^{n+1} is a symplectic manifold that contains the two submanifolds

- L, the set of geodesics through e_0, which is the unit sphere $S^n \subset T_{e_0}S^{n+1}$,
- V, the set of geodesics contained in $\mathbf{R}^{n+1} = e_0^\perp$, which is the symplectic submanifold depicted in Proposition 4.3.

The complement of our S^n in Q^n consists of the geodesics in S^{n+1} that do not pass through e_0. Our S^1-action will simply rotate such a geodesic around e_0, in the 3-space spanned by e_0 and the geodesic itself, in the sense defined by the orientation of the geodesic itself on the equatorial circle (see Figure 2). Clearly, this action does not extend to Q^n, since there is no coherent way to rotate the great circles through e_0. To write down the Hamiltonian of this action (thus proving that it is indeed

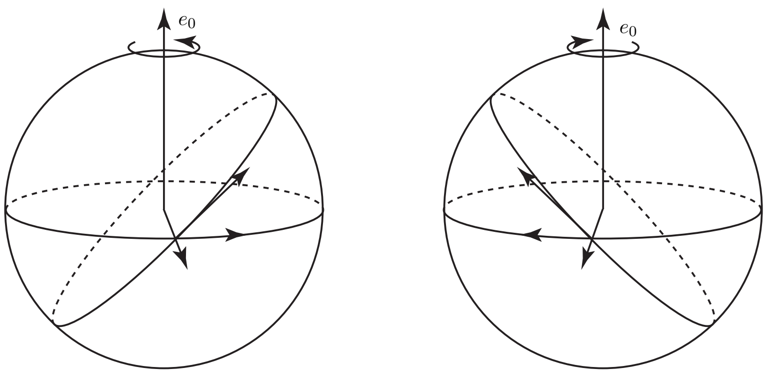

FIGURE 2

a Hamiltonian action), il is enough, as everything takes place in dimension 3, to check the $n = 2$ case. Since the identification of the space of oriented geodesics of S^2 with S^2 is by the map

$$\begin{aligned} V_2(\mathbf{R}^3) &\longrightarrow S^2 \\ (x, y) &\longmapsto x \times y, \end{aligned}$$

we are describing the circle action around e_0 on the complement of the equatorial circle in S^2 by direct rotations in one hemisphere and indirect ones in the other one. On the first hemisphere, this is a Hamiltonian action with Hamiltonian $z \mapsto z_0$ (the component on e_0) while in the second one, this is $z \mapsto -z_0$, hence our Hamiltonian is $z \mapsto |z_0|$, namely $|(x \times y)_0| = \sqrt{1 - x_0^2 - y_0^2}$. The Hamiltonian we are looking for is thus

$$\begin{aligned} H : Q^n - S^n &\longrightarrow \mathbf{R} \\ [x + iy] &\longrightarrow \sqrt{1 - x_0^2 - y_0^2} \end{aligned}$$

(using the notation $[x + iy]$ as in § 1.2). Before we proceed to the two other cases, let us make a few remarks on this example.

Remarks 4.5

- The fact that the action does not extend can be seen on H as the fact that this function is not smooth on the n-sphere.
- The complement of Q^{n-1} in Q^n is endowed with a projection onto S^n. A geodesic which is not included in the hyperplane $e_0^\perp$ meets this hyperplane at two points, we associate to it the geodesic through e_0 and these two points, endowed with the orientation defined by that of the original geodesic. Moreover, the fiber of a geodesic through e_0 is the set of all the geodesics through a given point u in $e_0^\perp$ which are not included in $e_0^\perp$. This can be identified with a disc D^n, the upper hemisphere in S^{n+1} intersected with the hyperplane $u^\perp$.
- From the opposite side, the complement of L is endowed with a projection onto V. Any oriented geodesic that does not contain e_0 is mapped to an oriented geodesic in $e_0^\perp$. The fiber of a geodesic γ in $e_0^\perp$ consists of all the geodesics in the subspace generated by the plane of γ and e_0 that do not contain e_0 itself. This is a sphere S^2 minus a circle S^1, from which the selection of an orientation selects a component, namely the fiber is a disc D^2.
- These two projections reflect the fact that there is indeed, as expected, a function with two critical values on Q^n. If an oriented geodesic γ on S^{n+1} is defined by a point $x \in S^{n+1}$ and a unit tangent vector $y \in T_x S^{n+1}$, put $f(\gamma) = 1 - x_0^2 - y_0^2$. This depends only on γ and has minimum on S^n (where it takes the value 0) and maximum on Q^{n-1} (where it takes the value 1). This is of course the (smooth) square of our Hamiltonian H.

The case of $\mathbf{RP}^n$. The case of $\mathbf{RP}^n$ is a corollary. We know that the space of geodesics of $\mathbf{RP}^n$ is the same as that of the double covering S^n, namely the quadric Q^{n-1}. The cotangent bundle to $\mathbf{RP}^n$ is the quotient of that of S^n under the antipodal action. We deduce the expected result from the existence of the $2:1$ covering map $Q^n \to \mathbf{CP}^n$, branched along Q^{n-1} and that extends the covering map $S^n \to \mathbf{RP}^n$:

$$\begin{aligned} Q^n \subset \mathbf{CP}^{n+1} - \{[1, 0, \ldots, 0]\} &\longrightarrow \mathbf{CP}^n \\ [z_0, z_1, \ldots, z_{n+1}] &\longmapsto [z_1, \ldots, z_{n+1}]. \end{aligned}$$

The case of $\mathbf{CP}^n$. This is also a consequence of the case of S^n (of the case of S^2, in fact). We need to construct a Hamiltonian action on the complement of the diagonal in $\mathbf{CP}^n \times \mathbf{CP}^n$, namely on the set of pairs of distinct lines in $\mathbf{C}^{n+1}$. Two distinct complex lines span a complex plane and everything will take place in this plane... so that we only need to consider the case $n = 1$, that is, the case of geodesics in $\mathbf{CP}^1$ (namely, S^2) and we want to check that the symplectic cut in this case is $\mathbf{CP}^1 \times \mathbf{CP}^1$ (with the symplectic structure $\omega \oplus -\omega$). But in the S^2 case we have seen that this is a complex quadric of dimension 2, indeed a $\mathbf{CP}^1 \times \mathbf{CP}^1$. Let us identify $\mathbf{CP}^1 \times \mathbf{CP}^1$ with $Q^2 \subset \mathbf{CP}^3$ by the map

$$([x, y], [a, b]) \longmapsto [x\overline{a} + y\overline{b}, i(x\overline{a} - y\overline{b}), i(x\overline{b} + y\overline{a}), (x\overline{b} - y\overline{a})].$$

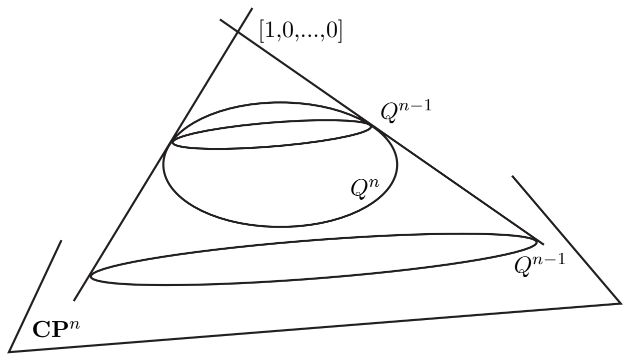

FIGURE 3

Thanks to the complex conjugation, the symplectic form induced on $\mathbf{CP}^1 \times \mathbf{CP}^1$ is indeed $\omega \oplus -\omega$. The diagonal $\mathbf{CP}^1 \subset \mathbf{CP}^1 \times \mathbf{CP}^1$ is mapped to the points of the form

$$\begin{aligned}[z_0, z_1, z_2, z_3] &= [|x|^2 + |y|^2, i(|x|^2 - |y|^2), i(x\overline{y} + \overline{x}y), x\overline{y} - y\overline{x}] \\ &= [1, iu_1, iu_2, iu_3] \text{ with } (u_1, u_2, u_3) \in S^2 \subset \mathbf{R}^3.\end{aligned}$$

The "antidiagonal"

$$M^2 = \left\{([x, y], [a, b]) \mid x\overline{a} + y\overline{b} = 0\right\}$$

is mapped onto the conic Q^1 of equations

$$z_0 = 0 \text{ and } z_1^2 + z_2^2 + z_3^2 = 0.$$

After homogenization of the formula $H([x + iy]) = \sqrt{1 - x_0^2 - y_0^2}$, the Hamiltonian we are looking for is

$$\begin{aligned}H([x, y], [a, b]) &= \sqrt{1 - \frac{\left|x\overline{a} + y\overline{b}\right|^2}{(|x|^2 + |y|^2)(|a|^2 + |b|^2)}} \\ &= \frac{|ay - bx|}{\sqrt{(|x|^2 + |y|^2)(|a|^2 + |b|^2)}} \text{ on } \mathbf{CP}^1 \times \mathbf{CP}^1, \text{ and} \\ H([z], [w]) &= \sqrt{1 - \frac{|\langle z, w\rangle|^2}{\|z\|^2 \|w\|^2}} \text{ on } \mathbf{CP}^n \times \mathbf{CP}^n.\end{aligned}$$

This ends the proof of Proposition 4.4. For the sake of completeness, as in the case of S^n, let us add a few remarks.

Remarks 4.6. There are two projections

$$\mathbf{CP}^n \times \mathbf{CP}^n - \mathbf{CP}^n \xrightarrow{\ p\ } M^{4n-2}$$

$$\text{and } \mathbf{CP}^n \times \mathbf{CP}^n - M^{4n-2} \xrightarrow{\ q\ } \mathbf{CP}^n.$$

The first one is defined by $p(\ell, d) = (\ell, d')$ where d' is the unique line orthogonal to ℓ in the plane spanned by ℓ and d (note that ℓ and d are distinct). The fiber of a pair (ℓ, d') of two orthogonal lines is the set of lines $d \neq \ell$ in the plane spanned by ℓ and d'; such a line is the graph of a linear map $d' \to \ell$, so that $p^{-1}(\ell, d') \cong \operatorname{Hom}(\ell, d) \cong \mathbf{C}$.

The map q is defined by $q(\ell, d) = \ell$. Here $d \not\subset \ell^\perp$, so that the fiber of ℓ consists of all the lines in $\mathbf{C}^{n+1} - \ell^\perp$. As above, $q^{-1}(\ell) \cong \operatorname{Hom}(\ell, \ell^\perp) \cong \mathbf{C}^n$.

The last remark is of course that the square of our Hamiltonian, namely the function

$$f([z], [w]) = 1 - \frac{|\langle z, w\rangle|^2}{\|z\|^2 \|w\|^2}$$

is a Morse-Bott function with critical submanifolds M and $\mathbf{CP}^n$ as expected.

□

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