Lagrangian submanifolds

Examples and counter-examples

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Special thanks

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General question

To understand the topology of symplectic manifolds.

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- This is the local structure of the symplectic manifold near its Lagrangian submanifold (a version, due to Weinstein, of Darboux's theorem).

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Recent results by Viterbo, Biran-Cieliebak and Seidel $+$ work in progress.

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However, graded Lagrangian submanifolds deserve to be known.

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using Darboux

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if one insists on compactness

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- $\mathbf{P}^n(\mathbf{R}) \subset \mathbf{P}^n(\mathbf{C})$,
- special Lagrangian tori in Calabi-Yau 3-folds (Bryant),
- real forms of Lie groups, $SU(n)/(\mathbf{Z}/n) \subset \mathbf{P}^{n^2-1}(\mathbf{C})$, etc.

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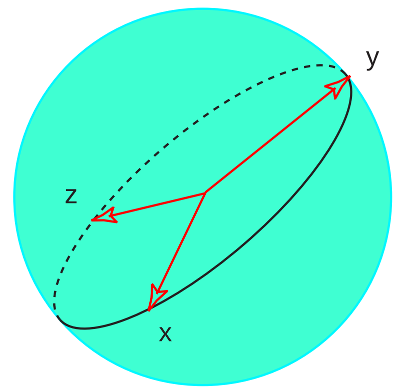
- $\xi \in \mathfrak{g}^\star$ a fixed point of the coadjoint action,
- Then, the G-orbits in $\mu^{-1}(\xi)$ are isotropic. In the example, $\mu^{-1}(0)$ is a single orbit, stabilizer is trivial, get a Lagrangian $SO(3)$ in $S^2 \times S^2 \times S^2$.

More on this example

This example is due to Chiang.

$$\{(x, y, z) \in S^2 \times S^2 \times S^2 \mid x + y + z = 0\}.$$



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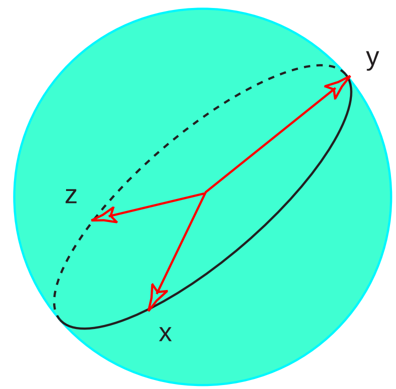

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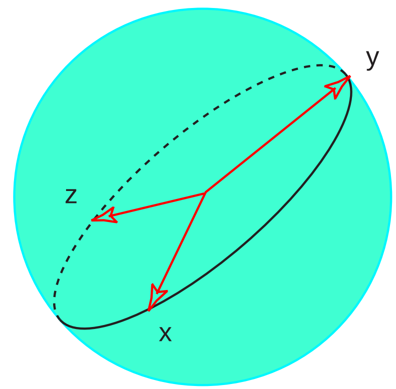

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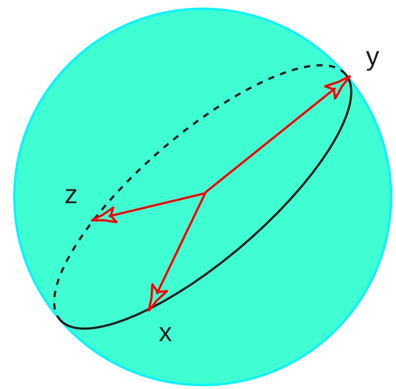

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Taking quotients, we get a Lagrangian embedding.

$$S^3/\Gamma = SO(3)/\mathfrak{S}_3 \hookrightarrow \mathrm{P}^3(\mathrm{C}).$$

The sphere

Seidel's idea, namely graded Lagrangians, allows to prove

Theorem. *Assume S^n is a Lagrangian submanifold of a compact monotone symplectic manifold W with $H_1(W;\mathbf{Z}) = 0$. Assume that W is endowed with a (non constant) periodic Hamiltonian. Let N be the generator of the subgroup*

$$\{\langle c_1(W), A\rangle \mid A \in H_2(W;\mathbf{Z})\} \subset \mathbf{Z}.$$

Then $n \equiv 0 \mod N$ and the sum of the weights of the linearized S^1-action at any point where the Hamiltonian is minimal is also congruent to n modulo N.

The sphere—continuation

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so that $H^2(W; \mathrm{Z}) = \mathrm{Z}$ generated by a hyperplane section and that $N = n$.
The Lagrangian sphere S^n is the real part of W for the real structure given by complex conjugation

$$S[z_0, z_1, \ldots, z_{n+1}] = [\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_{n+1}]$$

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Theorem (Seidel). *If L is a Lagrangian submanifold of $\mathbf{P}^n(\mathbf{C})$, then $H^1(L; \mathbf{Z}/2n+2) \neq 0$.*

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In the Chiang example above, $n = 3$, $\pi_1(L) = \Gamma$ is an extension of $\mathfrak{S}_3$ such that $H_1(L; \mathbf{Z}) = \mathbf{Z}/4$ and $H^1(L; \mathbf{Z}/8) = \mathbf{Z}/4 \neq 0$.

Ideas of the proof

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$$HF(L, L) \cong \oplus_i H^i(L; \mathrm{Z}/2).$$

Notice that (under the assumptions made), this depends only on the manifold L.

Ideas of the proof—continuation

- Second ingredient: an addition to the structure of Floer cohomology (this is the Seidel grading), a $\mathbf{Z}/2N$-grading on this cohomology group. Notice that N depends only on the symplectic manifold W.

Ideas of the proof—continuation

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- Last ingredient: a subtle use (by Seidel) of a Hamiltonian circle action, which gives a periodicity (modulo $2N$) on this Floer cohomology.

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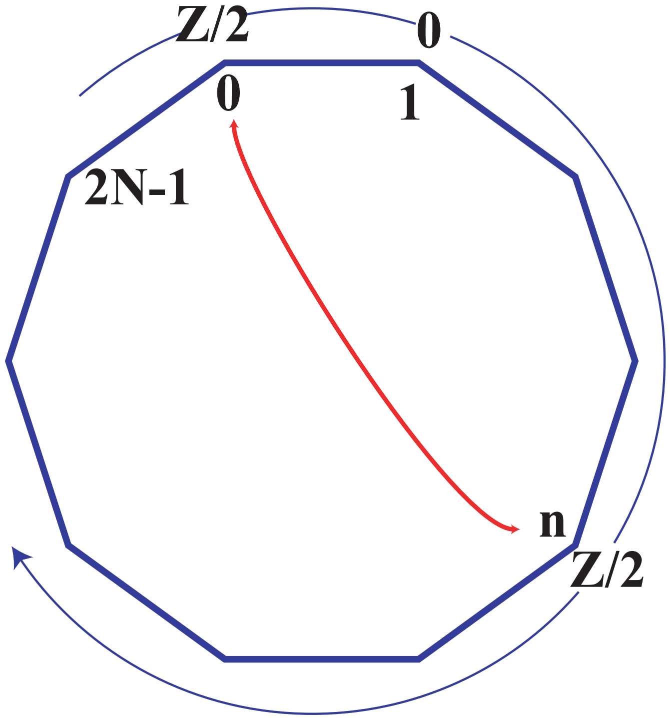

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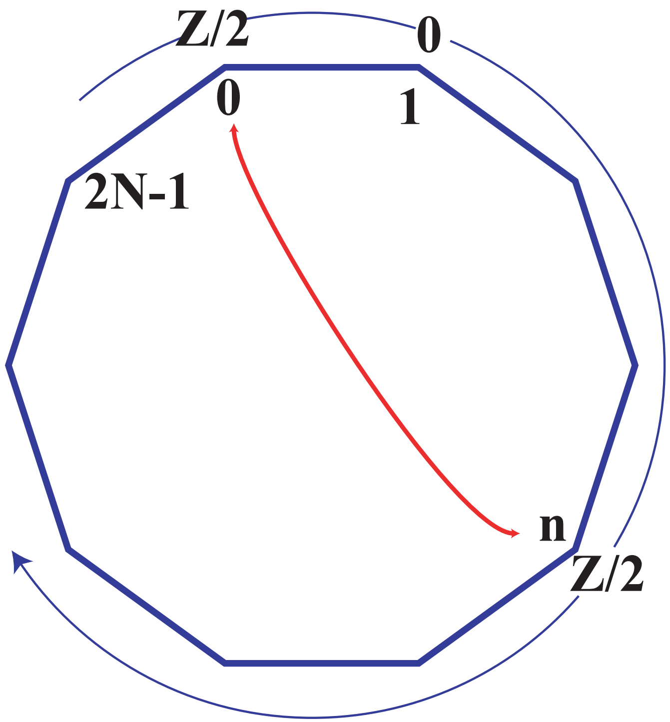

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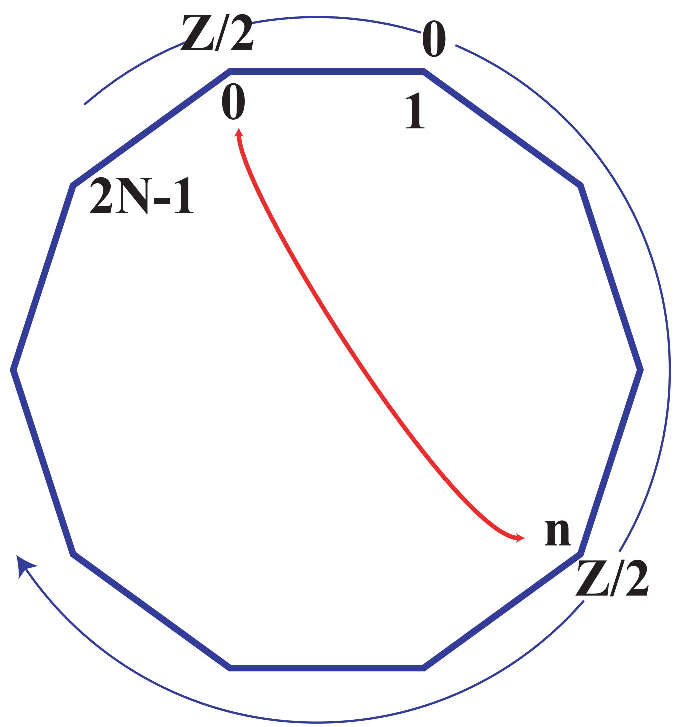

This can be periodic only if $n = N$ and the period is n... as in the example of the Lagrangian sphere in the quadric

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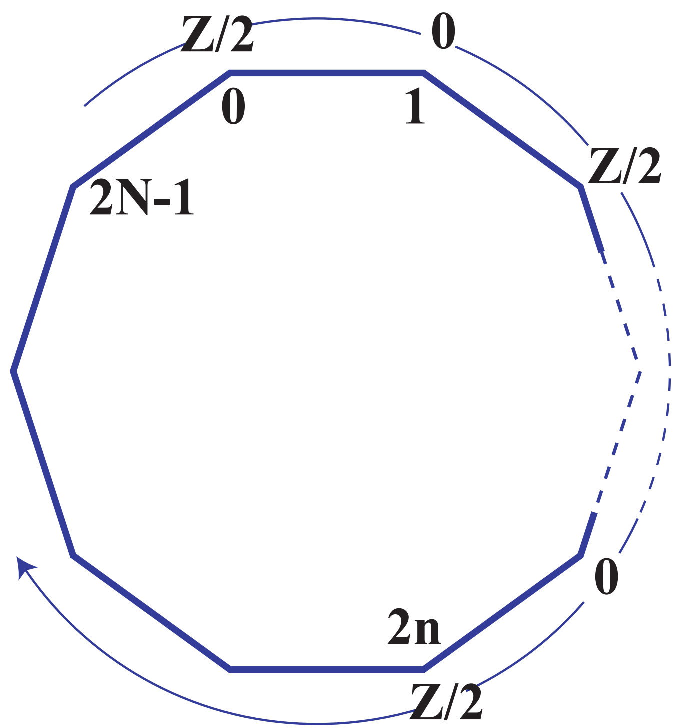

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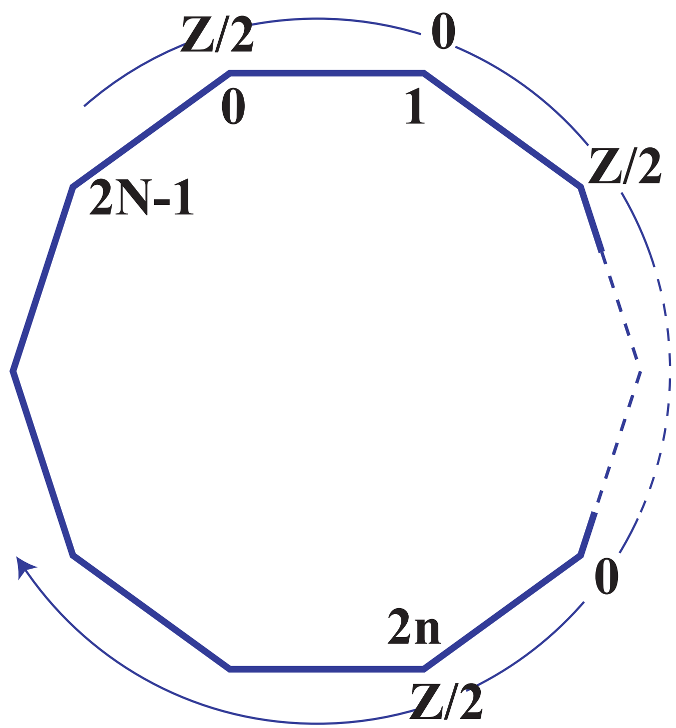

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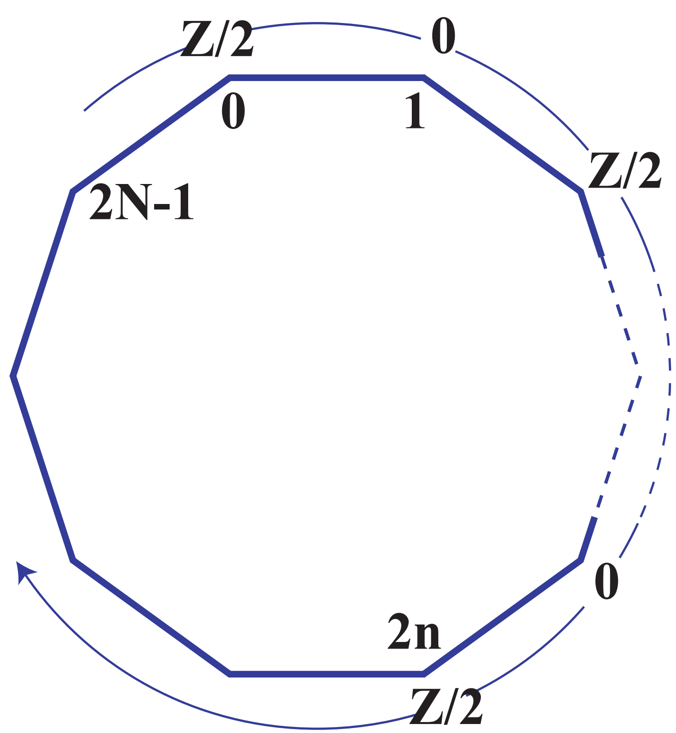

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is Lagrangian, as soon as $\mathbf{P}^n(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$ is endowed with the product symplectic form $\omega \oplus \omega$

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Note that this is deduced from the Polterovich example above.

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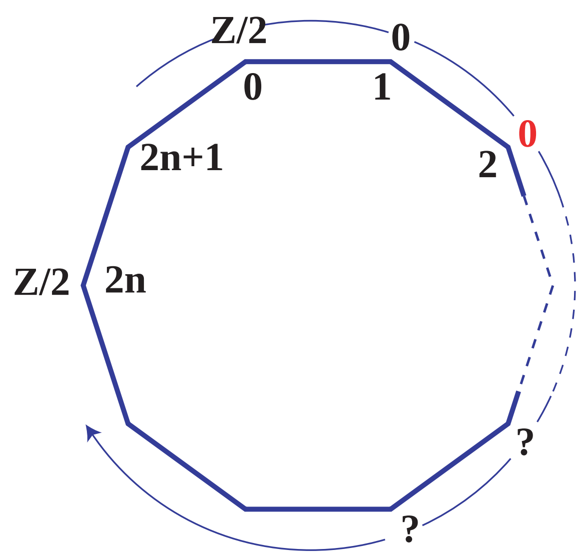

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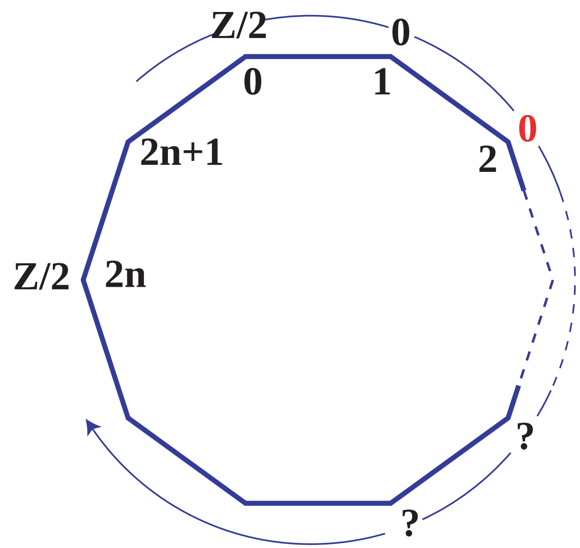

This can certainly not be periodic of period 4.

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a theorem which was proved by rather different methods. Let us go back to the grading.

Seidel's grading

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This is C^n, hence

$$\Lambda_n = U(n)/O(n).$$

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$$U^m(n) = \{(A, z) \in U(n) \times S^1 \mid \det(A)^2 = z^m\}$$

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We have a $\mathbf{Z}/2N$-covering map with a lift

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A Hamiltonian S^1-action may change the lift

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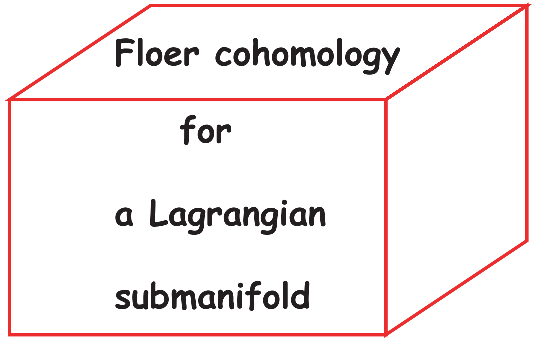

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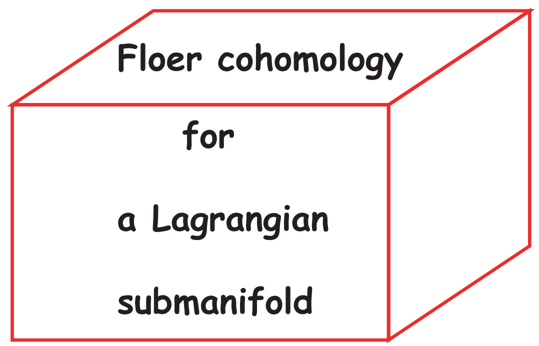

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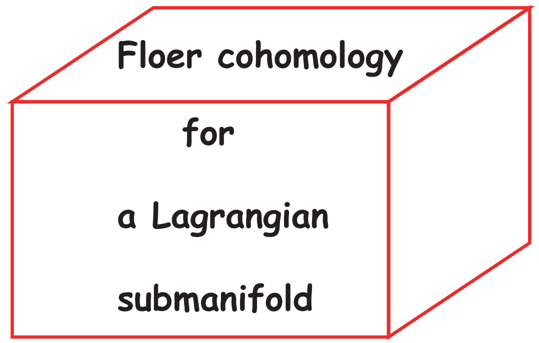

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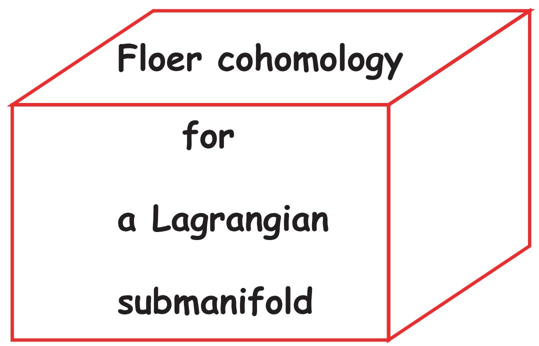

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A Hamiltonian action which changes the lift by the action of $k \in \mathbf{Z}/2N$ defines an isomorphism

$$HF^i(L, L) \longrightarrow HF^{i+k}(L, L).$$

Work in progress

What should we add to a cotangent bundle $T^\star L$ in order to construct a symplectic manifold?

Look at the Lagrangian barriers of Biran from the other side.

FIM

The end

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A symplectic manifold W is endowed with almost complex structures compatible with the symplectic form, all of which define the same Chern classes, in particular, the first Chern class $c_1(W) \in H^2(W; Z)$ is well defined.

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For instance, for $W = \mathrm{P}^n(\mathrm{C})$, $c_1 = (n+1)$ times the generator and $N = n + 1$.

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A symplectic manifold W is monotone if
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For instance, $\mathrm{P}^n(\mathrm{C})$ is a monotone symplectic manifold. $\mathrm{P}^n(\mathrm{C}) \times \mathrm{P}^n(\mathrm{C})$, endowed with the symplectic form $\omega \oplus \omega$ is monotone as well.

Definitions 3 for the statement

A Hamiltonian S^1-action on a symplectic manifold W with $H_1(W; \mathbb{Z}) = 0$ is simply a symplectic action. There exists a function

$$H : W \longrightarrow \mathbb{R}$$

the critical points of which are the fixed points of the S^1-action.

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For instance, $u \cdot [z_0, \ldots, z_n] = [uz_0, \ldots, z_n]$ is a Hamiltonian S^1-action on $\mathbb{P}^n(\mathbb{C})$, associated with the function

$$H([z_0, \ldots, z_n]) = \frac{1}{2} \frac{|z_0|^2}{\sum |z_i|^2}$$

Definitions 4 for the statement

On the tangent space at a fixed point, the S^1-action linearizes, in complex coordinates, a Hamiltonian

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- the maximum of H, point $[1, 0, \ldots, 0]$, weights $(-1, -1, \ldots, -1)$.