

PERMUTADS

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ABSTRACT. We unravel the algebraic structure which controls the various ways of computing the word $((xy)(zt))$ and its siblings. We show that it gives rise to a new type of operads, that we call permutads. It turns out that this notion is equivalent to the notion of “shuffle algebra” introduced by the second author. It is also very close to the notion of “shuffle operad” introduced by V. Dotsenko and A. Khoroshkin. It can be seen as a noncommutative version of the notion of nonsymmetric operads. We show that the role of the associahedron in the theory of operads is played by the permutohedron in the theory of permutads.

INTRODUCTION

In a product like $((xy)(zt))$ we usually do not care about the order of making the product: it can be either (xy) first and then (zt) , or the other way round. The purpose of this paper is to unravel the algebraic structure underlying the process of taking care about this order. For classical types of algebras the relevant tool is the notion of algebraic operad, which is based on rooted trees. We will show that in our case the relevant tool, that we call a permutad, is analogous but based on shuffles or, equivalently, on surjective maps between finite sets, in place of trees. It turns out that this is equivalent to the notion of “shuffle algebra” introduced by the second author in [12] and closely connected to the notion of “shuffle operad” introduced by Dotsenko and Khoroshkin in [2].

Any nonsymmetric operad can be seen as a permutad. This is the case of the operad As encoding associative algebras. Considered as a permutad we can construct its minimal model. We show that it is explicitly build out of the chain complex of the permutohedron. Recall that in the operadic framework one gets the associahedron (Stasheff polytope). As a result we have a concrete presentation of the notion of permutadic associative algebra up to homotopy. Another way of phrasing this result is to say that we have constructed an algebraic structure which entwines the permutohedrons while respecting their geometry.

In the operad setting the associative-like relation

$$x(yz) = q(xy)z,$$

where the parameter q is a scalar, is interesting only when $q = 0, 1$ or ∞ . Indeed, in the other cases the free algebra collapses. However, in the permutad setting, it makes sense for any value of q and, computing in this framework, leads to the appearance of the length function of the Coxeter group \mathbb{S}_n .

Date: May 26, 2011.

Key words and phrases. Tree, Hopf algebra, combinatorial, dendriform, pre-Lie, permutad.

This work has been partially supported by the Proyecto Fondecyt Regular 1100380, and the “MathAmSud” project OPECSHA 01-math-10.

Roughly speaking a permutad is a variation of the notion of nonsymmetric operad where the “parallel composition” axiom for the partial operations does not necessarily hold. We illustrate this fact by the following example. The nonsymmetric operad encoding the associative algebras with derivation is, in arity n , the polynomial algebra on n variables. In the permutad framework it is the algebra of *noncommutative* polynomials.

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Notation. Let $\underline{n} = \{1, \dots, n\}$ be a finite ordered set with n elements. By convention $\underline{0}$ is the empty set. The automorphisms group of \underline{n} is the symmetric group denoted by \mathbb{S}_n . Its unit element is denoted by 1_n (identity permutation). For the operadic terminology the reader may refer either to [10] or to [8].

Acknowledgement. We warmly thank Vladimir Dotsenko for his comment on shuffle algebras. Thanks also to Emily Burgunder for her comments on the first draft of this paper.

1. SURJECTIVE MAPS AND PERMUTADS

We introduce ad hoc terminology about surjective maps of finite sets. Then we give the “combinatorial definition” of a permutad.

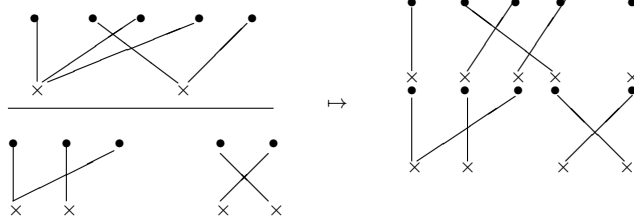
1.1. Surjective maps. Let n and k be positive integers such that $1 \leq k \leq n$. We denote by $Sur(\underline{n}, \underline{k})$ the set of surjective maps $t : \underline{n} \rightarrow \underline{k}$. We call *vertices* the elements in the target set \underline{k} . The set of vertices of the surjective map t is denoted by $\mathbf{vert} t$. The *arity* of $v \in \mathbf{vert} t$ is $|v| := \#t^{-1}(v) + 1$ (note the shift). When $k = 1$, $Sur(\underline{n}, \underline{k})$ contains only one element that we call a *corolla* (denoted by c_{n+1}), and when $k = n$ the set $Sur(\underline{n}, \underline{n})$ coincides with the symmetric group \mathbb{S}_n . We adopt the following notation: $Surj_n := \bigcup_k Sur(\underline{n}, \underline{k})$ (disjoint union). By convention $Surj_0 = \{\mathbf{1}\}$ and its unique element is considered as a surjective map $\mathbf{1} : \underline{0} \rightarrow \underline{0}$ (formally $\underline{0} = \emptyset$) with no vertex. In section 9 we list all the surjective maps for $n = 2, 3$.

1.2. Substitution. Let $t \in Sur(\underline{n}, \underline{k})$ and $t_i \in Sur(\underline{i_j}, \underline{m_j}), j = 1, \dots, k$, be surjective maps such that $i_j = \#t^{-1}(j)$. We put $m := \sum_j m_j$. By definition the *substitution* of $\{t_j\}$ in t is the surjective map $(t; t_1, \dots, t_k) \in Sur(\underline{n}, \underline{m})$ given by

$$(t; t_1, \dots, t_k)(a) := m_1 + \dots + m_{j-1} + t_j(b),$$

whenever $t(a) = j$ and a is the b th element in $t^{-1}(j)$.

Example with $k = 2$:



Observe that substitution is an associative operation.

The family of surjective maps endowed with the process of substitution could be described as a colored operad (cf. [13]), where the colors are integers, and also as a “multi-category” (cf. [4]), see 1.5.

1.3. A monad on arity-graded modules. Let \mathbb{K} be a commutative ring and let $\mathbb{N}^+\text{-Mod}$ be the category of positively graded \mathbb{K} -modules. An object M of $\mathbb{N}^+\text{-Mod}$ is a family $\{M_n\}, n \geq 1$, of \mathbb{K} -modules. In this paper we say “arity” in place of degree for these objects.

We define a monad in the category $\mathbb{N}^+\text{-Mod}$ of arity-graded modules as follows. First, for any M and any surjective map $t \in \text{Sur}(\underline{n}, \underline{k})$ we define a module

$$M_t := \bigotimes_{v \in \text{vert } t} M_{|v|},$$

where $|v|$ is the arity of the vertex v .

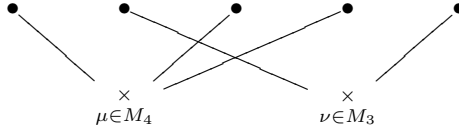
Second, for any M we define an arity-graded module $\mathbb{P}(M)$ as follows:

$$\mathbb{P}(M)_n := \bigoplus_{t \in \text{Sur}_{n-1}} M_t \text{ for } n \geq 2,$$

and $\mathbb{P}(M)_1 := \mathbb{K} \text{ id}$.

Explicitly the module $\mathbb{P}(M)_n$ is spanned by surjective maps with $n - 1$ inputs whose vertices are decorated by elements of M .

Example:



So, we have defined a functor

$$\mathbb{P} : \mathbb{N}^+\text{-Mod} \rightarrow \mathbb{N}^+\text{-Mod}.$$

There is a natural map $\iota(M) : M \rightarrow \mathbb{P}(M)$ which consists in associating to an element $\mu \in M_n$ the corolla c_n of arity n decorated by μ .

1.4. Proposition. *The substitution of surjective maps defines a transformation of functors $\Gamma : \mathbb{P} \circ \mathbb{P} \rightarrow \mathbb{P}$ which is associative and unital. So $(\mathbb{P}, \Gamma, \iota)$ is a monad on arity-graded modules.*

Proof. From the definition of \mathbb{P} we get

$$\begin{aligned}
\mathbb{P}(\mathbb{P}(M))_n &= \bigoplus_{t \in \text{Surj}_{n-1}} \mathbb{P}(M)_t \\
&= \bigoplus_{t \in \text{Surj}_{n-1}} (\mathbb{P}(M)_{i_1} \otimes \cdots \otimes \mathbb{P}(M)_{i_k}) \\
&= \bigoplus_{t \in \text{Surj}_{n-1}} \left(\bigotimes_{j=1}^{j=k} \left(\bigoplus_{s \in \text{Surj}_{j-1}} M_s \right) \right).
\end{aligned}$$

Under the substitution of surjective maps we get an element of $\mathbb{P}(M)_n$, since at any vertex j of t we have an element of $\mathbb{P}(M)_{i_j} = \bigoplus_{s \in \text{Surj}_{j-1}} M_s$, that is a surjective map s and its decoration. We substitute this data at each vertex of t to get a new decorated surjective map, decorated by elements of M . Therefore we have obtained a linear map $\Gamma(M) : \mathbb{P}(\mathbb{P}(M))_n \rightarrow \mathbb{P}(M)_n$, which defines a morphism of arity-graded modules

$$\Gamma(M) : \mathbb{P}(\mathbb{P}(M)) \rightarrow \mathbb{P}(M).$$

Since it is functorial in M we get a transformation of functors $\Gamma : \mathbb{P} \circ \mathbb{P} \rightarrow \mathbb{P}$.

Since the substitution process is associative, Γ is associative. Substituting a vertex by a corolla does not change the surjective map. Substituting a surjective map to the vertex of a corolla gives the former surjective map. Hence Γ is also unital.

We have proved that $(\mathbb{P}, \Gamma, \iota)$ is a monad. \square

1.5. Colored operad. A *colored operad* is to an operad what a category is to a monoid. More precisely, there is a set of colors and for each operation of the operad there is a color assigned to each input and a color assigned to the output. In order for a composition like $\mu \circ (\nu_1, \dots, \nu_k)$ to hold the color of the output of ν_i has to be equal to the color of the i th input of μ . Of course the colors of the inputs of the composite are the colors of the ν_i 's and the color of the output is the color of the output of μ . See for instance [13].

The monad $(\mathbb{P}, \Gamma, \iota)$ defined above can be seen as a nonsymmetric colored operad, where the colors are the natural numbers.

1.6. Permutads. By definition a *permutad* \mathcal{P} is a unital algebra over the monad $(\mathbb{P}, \Gamma, \iota)$. So \mathcal{P} is an arity-graded module such that $\mathcal{P}_1 = \mathbb{K}\text{id}$ endowed with a morphism of arity-graded modules

$$\Gamma_{\mathcal{P}} : \mathbb{P}(\mathcal{P}) \rightarrow \mathcal{P}$$

compatible with the composition Γ and the unit ι . It means that the following diagrams are commutative:

$$\begin{array}{ccc}
 & \mathbb{P}(\mathbb{P}(\mathcal{P})) & \xrightarrow{\mathbb{P}(\Gamma_{\mathcal{P}})} & \mathbb{P}(\mathcal{P}) \\
 & \nearrow \mathbb{R} & & \downarrow \Gamma_{\mathcal{P}} \\
 (\mathbb{P} \circ \mathbb{P})(\mathcal{P}) & & & \mathbb{P}(\mathcal{P}) \\
 \downarrow \Gamma(\mathcal{P}) & & & \downarrow \Gamma_{\mathcal{P}} \\
 \mathbb{P}(\mathcal{P}) & \xrightarrow{\Gamma_{\mathcal{P}}} & & \mathcal{P}
 \end{array}$$

and

$$\begin{array}{ccc}
 \text{Id}(\mathcal{P}) & \xrightarrow{\iota(\mathcal{P})} & \mathbb{P}(\mathcal{P}) \\
 \searrow = & & \downarrow \Gamma_{\mathcal{P}} \\
 & & \mathcal{P}
 \end{array}$$

We, now, give a first example: the free permutad.

1.7. Proposition (Free permutad). *For any arity-graded module M such that $M_1 = 0$, the arity-graded module $\mathbb{P}(M)$ is a permutad which is the free permutad over M .*

Proof. The structure of permutad $\Gamma_{\mathbb{P}(M)}$ on $\mathbb{P}(M)$ is induced by Γ , that is

$$\Gamma_{\mathbb{P}(M)} : \mathbb{P}(\mathbb{P}(M)) = (\mathbb{P} \circ \mathbb{P})(M) \xrightarrow{\Gamma(M)} \mathbb{P}(M).$$

The proof that $\mathbb{P}(M)$ is free among permutads goes as follows. Let \mathcal{Q} be another permutad and let $f : M \rightarrow \mathcal{Q}$ be a morphism of arity-graded modules. The composite $\mathbb{P}(M) \xrightarrow{\mathbb{P}(f)} \mathbb{P}(\mathcal{Q}) \rightarrow \mathcal{Q}$, which uses the permutadic structure of \mathcal{Q} , extends the map f . It is straightforward to check that it is a map of permutads and that it is unique as a permutad morphism extending f . So $\mathbb{P}(M)$ is free over M . \square

1.8. Differential graded permutad. By replacing the category of modules over the ground ring \mathbb{K} by the category of differential graded modules (i.e. chain complexes), we obtain the definition of *differential graded permutads*, abbreviated into *dg permutads*. Explicitly, when (M, d) is a dg \mathbb{N}^+ -Mod we make $\mathbb{P}(M)$ into a dg \mathbb{N}^+ -Mod by defining the differential d by

$$d(t; \mu_1, \dots, \mu_k) := \sum_{i=1}^k (-1)^{\epsilon_i} (t; \mu_1, \dots, d\mu_i, \dots, \mu_k)$$

where $t : \underline{n} \rightarrow \underline{k}$ is a surjective map and $\epsilon_i = |\mu_1| + \dots + |\mu_{i-1}|$. Then the structure map $\Gamma_M : \mathbb{P}(M) \rightarrow M$ is required to be a dg \mathbb{N}^+ -Mod morphism.

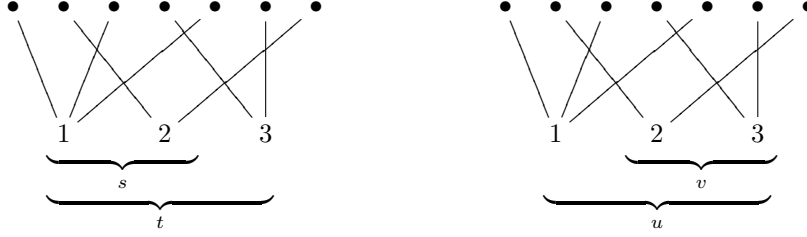
2. PARTIAL OPERATIONS AND SHUFFLE ALGEBRAS

The definition of a permutad that we have given is similar to the so-called ‘‘combinatorial’’ presentation of an operad (see Chapter 5 of [8]). Any surjection between finite sets can be obtained by successive substitutions of surjections with target set

of size 2. This property will enable us to give a definition of a permutad with a minimal data. It is similar to the so-called “partial” presentation of an operad.

As a consequence we will be able to identify the notion of permutad with the notion of shuffle algebra introduced by the second author in [12].

2.1. On the substitution of surjective maps with target set of size 2. Let us consider a surjective map $r : k \rightarrow \underline{3}$. We denote by ℓ, m, n the arity of the vertex 1, 2, 3 respectively. There are two ways to merge the three vertices to one vertex. Either we first merge 1 and 2, then we merge the resulting vertex with 3, or we first merge 2 and 3, then we merge 1 with the resulting vertex. The first process gives two surjective maps $s : \underline{\ell - 1 + m - 1} \rightarrow \underline{2}$ and $t : \underline{\ell - 1 + m - 1 + n - 1} \rightarrow \underline{2}$, and the second process gives $v : \underline{m - 1 + n - 1} \rightarrow \underline{2}$ and $u : \underline{\ell - 1 + m - 1 + n - 1} \rightarrow \underline{2}$.



From these two decompositions, r can be seen either as the substitution of s in t at 1, or as a substitution of v in u at 2.

2.2. Partial presentation of a permutad. Let $(\mathcal{P}, \Gamma, \iota)$ be a permutad (so we suppose $\mathcal{P}_1 = \mathbb{K} \text{id}$). Any surjection t with target set of size 2 (i.e. $t : \underline{m - 1 + n - 1} \rightarrow \underline{2}$) determines a linear map that we denote by

$$\circ_t : \mathcal{P}_m \otimes \mathcal{P}_n \rightarrow \mathcal{P}_{m-1+n},$$

where $m = \#t^{-1}(1) + 1$. From the discussion of the previous paragraph it comes immediately that for any operations $\lambda \in \mathcal{P}_\ell, \mu \in \mathcal{P}_m, \nu \in \mathcal{P}_n$ one has

$$(\lambda \circ_s \mu) \circ_t \nu = \lambda \circ_u (\mu \circ_v \nu).$$

2.3. Theorem. *A permutad $(\mathcal{P}, \Gamma, \iota)$ is completely determined by the arity graded module $\{\mathcal{P}_n\}_{n \geq 1}$, with $\mathcal{P}_1 = \mathbb{K}$, and the partial operations*

$$\circ_t : \mathcal{P}_m \otimes \mathcal{P}_n \rightarrow \mathcal{P}_{m-1+n}, \quad t : \underline{m - 1 + n - 1} \rightarrow \underline{2}, \quad m - 1 = \#t^{-1}(1),$$

satisfying

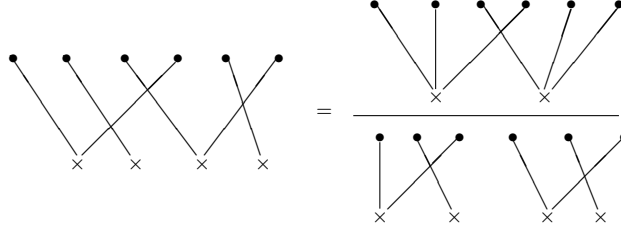
$$(\diamond) \quad (\lambda \circ_s \mu) \circ_t \nu = \lambda \circ_u (\mu \circ_v \nu),$$

for any surjective map r with target $\underline{3}$ such that $(t; s, c_{k+1}) = r = (u; c_{j+1}, v)$, for $\#r^{-1}(1) = j$ and $\#r^{-1}(3) = k$.

Proof. Let us start with a permutad $(\mathcal{P}, \Gamma, \iota)$. We define the partial operations by $\mu \circ_t \nu := \Gamma(t; \mu, \nu)$. From the associativity property of the permutad it follows that these partial operations satisfy the formula (\diamond) .

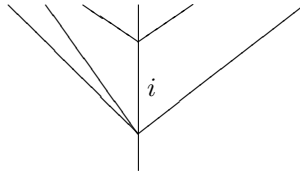
On the other hand, let us start with the arity graded module and the partial operations. Any surjective map s can be obtained by iteration of the substitution process from surjective maps with target $\underline{2}$. Hence we get $\Gamma(t; \mu_1, \dots, \mu_k)$.

Example:

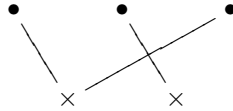


Its value does not depend on the choice of substitutions by virtue of formula (\diamond). So the structure map $\Gamma_k : \mathcal{P}_{i_1} \otimes \cdots \otimes \mathcal{P}_{i_k} \rightarrow \mathcal{P}_n$ is completely determined by the maps \circ_t . The associativity property is also a consequence of formula (\diamond). \square

2.4. The partial operations \circ_i . Let now $t : \underline{m-1+n-1} \rightarrow \underline{2}$ be such that the inverse image of the vertex 1 is made of consecutive elements $t^{-1}(1) = \{i, i+1, \dots, i+m-1\}$. So, once m and n have been chosen, t is completely determined by the integer i . We will sometimes denote it by \circ_i instead of \circ_t because it has properties similar to the partial operations in the operad framework (see [8] or [10]). More precisely $\circ_i : \mathcal{P}_m \otimes \mathcal{P}_n \rightarrow \mathcal{P}_{m-1+n}$ is similar to the operadic operation which corresponds to the tree



obtained by grafting a corolla on the i th leaf of another corolla. When $n = 2$ the surjective maps (that is the permutations) 1_2 and (12) correspond respectively to the operations \circ_1 and \circ_2 . For $n = 3$ there is only one surjective map $\underline{3} \rightarrow \underline{2}$ which is not of the type \circ_i , it is



Under the bijection between surjective maps and planar rooted trees (see for instance [11]), which describes the cells of the permutohedron, it corresponds to the dotted arrow in the hexagon of the proof of Proposition 9.7.

2.5. Proposition. *The \circ_i partial operations in a permutad \mathcal{P} satisfy the sequential composition relation:*

$$(\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu), \quad \text{for } 1 \leq i \leq l, 1 \leq j \leq m,$$

for any $\lambda \in \mathcal{P}_l, \mu \in \mathcal{P}_m, \nu \in \mathcal{P}_n$.

Proof. This is a particular case of the formula (\diamond) in Theorem 2.3. \square

Observe that, in a permutad, the partial operations \circ_i do not necessarily satisfy the parallel composition relation (see [8], Chapter 5).

2.6. Shuffles. By definition an (i_1, \dots, i_k) -shuffle in S_n , $n = i_1 + \dots + i_k$, is a permutation such that for any $j = 1, \dots, k$ the sequence of integers

$$i_1 + \dots + i_{j-1} + 1, i_1 + \dots + i_{j-1} + 2, \dots, i_1 + \dots + i_{j-1} + i_j$$

appear in this order in the sequence $[\sigma(1), \dots, \sigma(n)]$. For instance the $(1, 2)$ -shuffles are $[1, 2, 3], [2, 1, 3], [2, 3, 1]$. Following Stasheff, let us call *unshuffle* the inverse of a shuffle. So the $(1, 2)$ -unshuffles are $[1|2, 3], [2|1, 3], [3|1, 2]$.

2.7. Lemma. *There is a bijection between the set of shuffles $Sh(i_1, \dots, i_k) \subset S_n$ and the subset of $Sur(\underline{n}, \underline{k})$ made of surjective maps $t : \underline{n} \rightarrow \underline{k}$ such that $i_j = \#t^{-1}(j)$.*

Proof. Starting with a surjective map t we construct a sequence of integers $\sigma(1), \dots, \sigma(n)$ as follows: let $t^{-1}(j) = \{l_1^j < \dots < l_{i_j}^j\}$ be the inverse image of j by t , for $1 \leq j \leq k$. The sequence defines a permutation σ_t^{-1} whose image is:

$$(\sigma_t^{-1}(1), \dots, \sigma_t^{-1}(n)) := (l_1^1, \dots, l_{i_1}^1, l_1^2, \dots, l_{i_2}^2, \dots, l_1^k, \dots, l_{i_k}^k),$$

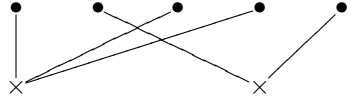
which is a (i_1, \dots, i_k) -unshuffle by construction. Taking the inverse σ_t gives a shuffle. It is immediate to check that we have a bijection as expected. \square

Example:

surjective map $\underline{5} \rightarrow \underline{3}$

$(3, 2)$ -unshuffle

$(3, 2)$ -shuffle



$[1, 3, 4|2, 5]$

$[1, 4, 2, 3, 5]$.

Given a pair of permutations $(\sigma, \tau) \in S_n \times S_m$ there is a natural way to construct the concatenation $\sigma \times \tau$ of them, which is an element of S_{n+m} . This product extends naturally to a product $\times : Sur(\underline{n}, \underline{k}) \times Sur(\underline{m}, \underline{h}) \rightarrow Sur(\underline{n+m}, \underline{k+h})$, by setting:

$$t \times w(j) := \begin{cases} t(j), & \text{for } 1 \leq j \leq n \\ w(j-n) + k, & \text{for } n+1 \leq j \leq n+m. \end{cases}$$

The product \times is associative.

A well-known result about shuffles (see for instance [1]), states that:

$$Sh(i_1 + i_2, i_3) \cdot (Sh(i_1, i_2) \times 1_{S_{i_3}}) = Sh(i_1, i_2, i_3) = Sh(i_1, i_2 + i_3) \cdot (1_{S_{i_1}} \times Sh(i_2, i_3)),$$

where 1_{S_n} denotes the identity of the group S_n and \cdot denotes the usual product in $S_{i_1+i_2+i_3}$.

The paragraph above shows that any (i_1, \dots, i_k) -shuffle σ may be written, in a unique way, as

$$\sigma = \sigma_1 \cdot (\sigma_2 \times 1_{S_{i_k}}) \cdot \dots \cdot (\sigma_{k-1} \times 1_{S_{i_3+\dots+i_k}}),$$

with $\sigma_j \in Sh(i_1 + \dots + i_{k-j}, i_{k-j+1})$.

Note that surjections with target size $\underline{2}$ correspond to shuffles of type (i_1, i_2) .

2.8. Lemma. *Let $t : \underline{n} \rightarrow \underline{2}$ be a surjective map and let $t_j \in Sur(\underline{i}_j, \underline{m}_j)$, for $j = 1, 2$, be such that $i_j = \#t^{-1}(j)$. In this case, we have the equality:*

$$\sigma_{(t; t_1, t_2)} = (t_1 \times t_2) \cdot \sigma_t,$$

where \cdot denotes the composition of maps.

Proof. Let $t^{-1}(1) = \{l_1 < \dots < l_{i_1}\}$ and $t^{-1}(2) = \{h_1 < \dots < h_{i_2}\}$. The surjection $(t; t_1, t_2)$ is described by the formula:

$$(t; t_1, t_2)(i) = \begin{cases} t_1(j), & \text{if } i = l_j, \text{ for } 1 \leq j \leq i_1, \\ t_2(j) + m_1, & \text{if } i = h_j, \text{ for } 1 \leq j \leq i_2. \end{cases}$$

On the other hand, we have that

$$\sigma_t(j) = \begin{cases} i, & \text{if } j = l_i, \\ i_1 + k, & \text{if } j = h_k. \end{cases}$$

So,

$$(t_1 \times t_2) \cdot \sigma_t(j) = \begin{cases} t_1(i), & \text{if } j = h_1, \\ t_2(k) + m_1, & \text{if } j = h_k, \end{cases}$$

which ends the proof. \square

2.9. Proposition. *With the same notation that in Lemma 2.8, we have that,*

$$\sigma_{(t; t_1, t_2)} = (\sigma_{t_1} \times \sigma_{t_2}) \cdot \sigma_t.$$

Proof. First, it is easy to check that $\sigma_{t_1 \times t_2} = \sigma_{t_1} \times \sigma_{t_2}$.

So, by Lemma 2.8, to end the proof it suffices to verify that for any (i_1, i_2) -shuffle α , we have the equality

$$\sigma_{(t_1 \times t_2) \cdot \alpha} = \sigma_{t_1 \times t_2} \cdot \alpha.$$

We know that the permutation $\sigma_{(t_1 \times t_2) \cdot \alpha}$ is determined by the family of subsets $((t_1 \times t_2) \cdot \alpha)^{-1}(i)$, for $1 \leq i \leq m_1 + m_2$. For $1 \leq i \leq m_1 + m_2$, it is straightforward to see that:

- (1) For $1 \leq i \leq m_1$, the subset $((t_1 \times t_2) \cdot \alpha)^{-1}(i)$ coincides with $\alpha^{-1}(t_1^{-1}(i))$.
- (2) For $m_1 + 1 \leq i \leq m_1 + m_2$, the subset $((t_1 \times t_2) \cdot \alpha)^{-1}(i)$ is equal to $\alpha^{-1}(t_2^{-1}(i - m_1) + i_1)$.

The paragraph above implies that

$$\sigma_{(t_1 \times t_2) \cdot \alpha}^{-1} = \alpha^{-1} \cdot \sigma_{t_1 \times t_2}^{-1} = \alpha^{-1} \cdot (\sigma_{t_1}^{-1} \times \sigma_{t_2}^{-1}),$$

and the proof is over. \square

2.10. Shuffle algebra [12]. A *shuffle algebra* is a graded \mathbb{K} -module $A = \bigoplus_{n \geq 0} A_n$ such that $A_0 = \mathbb{K}1$ endowed with binary operations

$$\bullet_\gamma : A_n \otimes A_m \rightarrow A_{n+m}, \text{ for } \gamma \in Sh(n, m),$$

verifying:

$$(\ddagger) \quad x \bullet_\gamma (y \bullet_\delta z) = (x \bullet_\sigma y) \bullet_\lambda z,$$

whenever $(1_n \times \delta) \cdot \gamma = (\sigma \times 1_r) \cdot \lambda$ in $Sh(n, m, r)$. It is also supposed that 1 is a unit on both sides. Since any k -shuffle $\sigma \in Sh(i_1, \dots, i_k)$ can be written as a composition of 2-shuffles, the above relation implies that for any such σ there is a well-defined map

$$\bullet_\sigma : A_{i_1} \otimes \dots \otimes A_{i_k} \rightarrow A_{i_1 + \dots + i_k}.$$

The relationship with permutads is given by the following result.

2.11. Proposition. *There is an equivalence between permutads \mathcal{P} and shuffle algebras A . It is given by $\mathcal{P}_{n+1} = A_n$, $\circ_t = \bullet_{\sigma_t}$, where t is a surjective map with target $\underline{2}$.*

Proof. Lemma 2.7 gives a bijection between surjective maps $\underline{n} \rightarrow \underline{k}$ and k -shuffles, which restricts to a bijection between surjective maps with target $\underline{2}$ and 2-shuffles.

Note that the identity $1_n \in S_n$ corresponds via this bijection to the corolla $c_{n+1} \in \text{Sur}(\underline{n}, \underline{1})$, that is $\sigma_{c_{n+1}} = 1_n$. So, Proposition 2.9 implies that for any surjective map r of target $\underline{3}$, such that $r = (t; s, c_{k+1}) = (u; c_{j+1}, v)$, we have that $\sigma_r = (\sigma_s \times 1_k) \cdot \sigma_t = (1_j \times \sigma_v) \cdot \sigma_u$.

From this relation, the relation (\ddagger) between shuffles used in the definition of a shuffle algebra corresponds, via the bijection, to the relation (\diamond) between surjective maps proved in Theorem 2.3. \square

Several examples of shuffle algebras, and therefore of permutads have been given in [12].

3. PERMUTADS AND NONSYMMETRIC OPERADS

We show that any nonsymmetric operad gives rise to a permutad.

3.1. Nonsymmetric operad. Let us recall that a nonsymmetric operad (we often write ns operad for short) can be defined as we defined a permutad but with planar trees in place of surjections. Here we consider only the trees which have at least two inputs at each vertex. The monad of planar trees is denoted by PT and a ns operad is an algebra over this monad (cf. for instance the combinatorial definition of a ns operad in [8] section 5.8.5).

3.2. From ns operads to permutads. The bijection between permutations and planar leveled trees described in the Appendix (see 9.4) induces a map $\varphi : \mathbb{S}_n \rightarrow PBT_{n+1}$ between permutations and planar binary trees. We still denote by φ its extension to all the surjections, that is to all the cells of the permutohedron \mathbb{P}^{n-1} , with values in the set of planar trees:

$$\varphi : \text{Sur}j_n \rightarrow PT_{n+1}.$$

This map is obviously compatible with substitution, and therefore we get a morphism of monads

$$\varphi : \mathbb{P} \rightarrow PT.$$

As a consequence we get the following result.

3.3. Proposition. *There is a functor from the category of ns operads to the category of permutads, induced by φ .*

Observe that the partial operation \circ_i in the ns operad framework is exactly the partial operation with same notation in the permutad framework.

4. BINARY QUADRATIC PERMUTADS

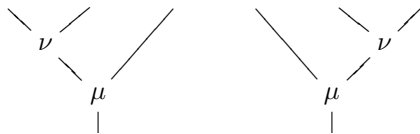
For binary permutads it will prove helpful to replace the permutations by the leveled planar binary trees in order to handle explicitly the operations. We refer to the Appendix for details on this bijection.

4.1. **Definition.** A *binary permutad* is a permutad which is generated by binary operations. In other words, it is the quotient of a free permutad $\mathbb{P}(M)$, where the \mathbb{N}^+ -module M is concentrated in arity 2: $M = (0, E, 0, \dots)$.

By 1.3 a typical element of $\mathbb{P}(M)$ is a surjection such that the arity of each vertex is 2 (hence a permutation), whose vertices are decorated by elements of E . Under the isomorphism between bijections and leveled binary trees, cf. 9.4, one can see such an element as a leveled binary tree where the vertices are decorated by elements of E .

A *quadratic permutad* is a permutad whose relations involve only elements made of surjections on $\underline{2}$.

For binary quadratic permutads, the modules involved in the definition (generators and relations) are the same as in the ns operad framework. Indeed the surjections $(1_2; \mu, \nu)$ and $((12); \mu, \nu)$ correspond respectively to the operations $\mu \circ_1 \nu$ and $\mu \circ_2 \nu$ that is to the trees



In other words the difference between permutads and ns operads begins only at arity 4. So, the data to define a binary quadratic permutad is the same as the data to define a binary quadratic ns operad. It may happen that, for a given set of generators and relations, the permutad is exactly the ns operad with same presentation (via the functor φ , see 3.3). As we will see it is not always the case (free object for instance).

One can find many examples of binary quadratic ns operads in the Encyclopedia [14]. As shown below, in the *q-permAs* case the underlying arity modules can be very different in the operad case and in the permutad case (for instance if $q = -1$ and $n \geq 4$, then $\dim (-1)\text{-permAs}_n = 1$ and $\dim (-1)\text{-As}_n = 0$).

4.2. **Parametrized associative permutad.** Let $q \in \mathbb{K}$ be a parameter. We define the *parametrized associative permutad q-permAs* as the permutad generated by one element in arity 2, denoted by μ , and satisfying the relation

$$\Gamma((12); \mu, \mu) = q \Gamma(1_2; \mu, \mu),$$

where 1_2 is the identity and $(12) = [21]$ is the cycle. Equivalently this relation can be written $\mu \circ_2 \mu = q \mu \circ_1 \mu$. We will show that $(q\text{-permAs})_n$ is one-dimensional for any $n \geq 1$. The permutadic composition gives the following result.





4.3. **Proposition.** *For any $n \geq 1$ the module $(q\text{-permAs})_n$ is one-dimensional spanned by $\Gamma(1_n; \mu, \dots, \mu)$. For any permutation $\sigma \in \mathbb{S}_n$, considered as a surjective map from \underline{n} to \underline{n} , we have the following equality*

$$\Gamma(\sigma; \mu, \dots, \mu) = q^{\ell(\sigma)} \Gamma(1_n; \mu, \dots, \mu),$$

where $\ell(\sigma)$ is the length of σ in the Coxeter group \mathbb{S}_n .

Proof. We consider the Coxeter presentation of the symmetric group \mathbb{S}_n with generators s_1, \dots, s_{n-1} (transpositions). The length of σ , denoted by $\ell(\sigma)$, is the number of Coxeter generators in a minimal writing of σ . Consider the poset of permutations

equipped with the weak Bruhat order. So, $\sigma \rightarrow \tau$ is a covering relation iff τ is obtained from σ by the left multiplication by a Coxeter generator and $\ell(\tau) = \ell(\sigma) + 1$. Under the bijection between permutations and leveled binary trees, cf. 9.4, there are two different covering relations:

- those which are obtained through a local application of  \rightarrow  ,
- those which are obtained by some change of levels, like  \rightarrow  .

The property that we use is the following, proved in the appendix, cf. Proposition 9.7:

the subposet of the poset \mathbb{S}_n made of the covering relations of first kind is a connected graph.

The relation which defines the permutad $q\text{-permAs}$ is precisely

$$\Gamma\left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \text{---} \end{array}; \mu, \mu\right) = q\Gamma\left(\begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \text{---} \end{array}; \mu, \mu\right).$$

Since any σ can be related to 1_n by a sequence of covering relations of the first kind, we can apply repeatedly the relation and we get the expected formula. \square

We remark that in the permutadic case we do not encounter the obstruction $q^3 = q^2$ met in the operadic case. In the operadic case the module $(q\text{-permAs})_n$ is one-dimensional, when $n \geq 4$, only for $q = 0, 1$ or ∞ (compare with [9]).

4.4. Corollary. *The associative permutad permAs presented by a binary operation and the associativity relation is one dimensional in each arity. Hence it is the permutad associated to the ns operad As .*

Proof. It is a particular case of Proposition 4.3 since $\text{permAs} = 1\text{-permAs}$. Therefore it is the permutad obtained from the ns operad As by the functor φ of Proposition 3.3. \square

5. ALGEBRA OVER A PERMUTAD

We define the notion of an algebra over a permutad. We show that the notion of permutad is the relevant tool to study products for which the two ways of computing $((xy)(zt))$ differ.

5.1. Definition of an algebra over a permutad. Recall that any \mathbb{N}^+ -module M determines an endofunctor of $\mathbb{K}\text{-Mod}$ by

$$M(A) := \bigoplus_{n \geq 1} M_n \otimes A^{\otimes n}.$$

By definition, an *algebra over the permutad \mathcal{P}* is a \mathbb{K} -module A endowed with linear maps

$$\mathbb{P}(\mathcal{P})_n \otimes A^{\otimes n} \rightarrow A,$$

whence a linear map $\mathbb{P}(\mathcal{P})(A) \rightarrow A$. In particular the map $\iota : \mathcal{P} \rightarrow \mathbb{P}(\mathcal{P})$ gives rise to

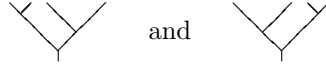
$$\mathcal{P}(A) \rightarrow A, (\mu; a_1, \dots, a_n) \mapsto \mu(a_1, \dots, a_n).$$

As a special case of the property of \mathcal{P} with respect to \mathbb{P} the following equality holds:

$$(\mu \circ_i \nu)(a_1, \dots, a_{m-1+n}) = \mu(a_1, \dots, a_{i-1}, \nu(a_i, \dots, a_{i+m-1}), a_{i+m}, \dots, a_{m-1+n})$$

for any $\mu \in \mathcal{P}_m, \nu \in \mathcal{P}_n, 1 \leq i \leq m$, and any elements a_j of A .

5.2. Computation in algebras over a permutad. If the permutad \mathcal{P} has generating operations M , then it is a quotient of the free permutad $\mathbb{P}(M)$. An element in $\mathbb{P}(M)$ is a ‘‘computational pattern’’ in the sense that, given a sequence of elements in a \mathcal{P} -algebra A , we can compute an element of A out of this pattern. So, the parenthesizings of the operad framework, i.e. planar trees, are replaced here by leveled planar trees. For instance, supposing that M is determined by one binary operation, the two computational patterns



give, a priori, two distinct values denoted by:

$$((xy)_1(zt)_2) \quad \text{and} \quad ((xy)_2(zt)_1)$$

respectively. Here is an explicit example.

5.3. Examples of computation. In the case of an algebra A over the permutad $q\text{-permAs}$ we have

$$((ab)_2(cd)_1) = q((ab)_1(cd)_2)$$

for any $a, b, c, d \in A$. In the particular case of the free algebra on one generator x , the elements x^n , defined inductively as $x^n = x^{n-1}x$, span this algebra. We compute

$$((xx)_1(xx)_2) = qx^4 \quad \text{and} \quad ((xx)_2(xx)_1) = q^2x^4.$$

6. ASSOCIATIVE PERMUTAD AND THE PERMUTOHEDRON

In the operad framework the operad As , which encodes the associative algebras, admits a minimal model which is described explicitly in terms of the Stasheff polytope (associahedron). It means that this minimal model is a differential graded operad whose space of n -ary operations is the chain complex of the $(n-2)$ -dimensional *associahedron* (considered as a cell complex). When we consider As as a permutad, denoted by $permAs$, then one can also construct its minimal model. We show that, in arity n , this differential graded permutad is given by the chain complex of the $(n-2)$ -dimensional *permutohedron*.

6.1. Associative permutad. We consider an arity-graded module which is 0 except in arity 2 for which it is one-dimensional, spanned by μ . The free permutad on μ , denoted $permMag(\mu)$ is such that $permMag(\mu)_n \cong S_{n-1}$ since the non-zero decorated surjective maps are bijections, cf. Proposition 1.7. For instance, in arity 3, we get

$$\mu \circ_1 \mu = \Gamma(1_2; \mu, \mu) = \begin{array}{c} \bullet \\ \downarrow \\ \times \\ \mu \end{array} \quad \begin{array}{c} \bullet \\ \downarrow \\ \times \\ \mu \end{array} = [1 \ 2]$$

$$\mu \circ_2 \mu = \Gamma((12); \mu, \mu) = \begin{array}{ccc} \bullet & & \bullet \\ & \searrow & \swarrow \\ \times & & \times \\ \mu & & \mu \end{array} = [2 \ 1]$$

Let us put the relation $\Gamma(1_2; \mu, \mu) = \Gamma((12); \mu, \mu)$ (which is the associativity relation $\mu \circ_1 \mu = \mu \circ_2 \mu$) and denote the associated permutad by *permAs* (this is the permutad 1-*permAs* introduced in 4.2).

6.2. A quasi-free dg permutad. We describe a quasi-free dg permutad ΩpermAs^i as follows. The reason for the fancy notation will be explained later. Let V be the arity-graded module which is one-dimensional in each arity $n \geq 2$ and 0 in arity 1. We denote the linear generator in arity n by m_n , so $V_n = \mathbb{K} m_n$. We declare that the homological degree of m_n is $n - 2$. As a permutad ΩpermAs^i is the free permutad on the graded \mathbb{N}^+ -module V (two gradings: homological and arity). It comes immediately from Proposition 1.7 that $(\Omega \text{permAs}^i)_n$ can be identified to the free module on the surjections (i.e. the set Surj_n), hence the cells of the permutohedron of dimension $n - 2$, cf. 9.3. Under this identification the operation m_n is identified with the big cell c_n of the $(n - 2)$ -dimensional permutohedron. We put on it the boundary map of the permutohedron, cf. 9.3.

6.3. Theorem. *The permutadic structure of ΩpermAs^i is compatible with the boundary map, hence ΩpermAs^i is a dg quasi-free permutad such that*

$$(\Omega \text{permAs}^i)_n \cong C_\bullet(\mathbb{P}^{n-2}).$$

Proof. We introduced the notion of dg permutad in 1.8. We need to prove that the structure map

$$\mathbb{P}(\mathbb{P}(V))_n \rightarrow \mathbb{P}(V)_n$$

is compatible with the boundary map. It is sufficient to check that the substitution at any vertex is compatible, that is, to check that for $t : \underline{m-1+n-1} \rightarrow \underline{2}$ the map

$$\circ_t : \mathbb{P}(V)_m \otimes \mathbb{P}(V)_n \rightarrow \mathbb{P}(V)_{m-1+n}$$

commutes with the differential. Since any cell of \mathbb{P}^k is a product of permutohedrons of lower dimensions, it suffices to check this property for $c_n \in \mathbb{P}(V)_n$ and $c_m \in \mathbb{P}(V)_m$. The element $c_m \circ_t c_n$ is in fact a cell of the permutohedron which is the product of two permutohedrons $\mathbb{P}^{m-2} \times \mathbb{P}^{n-2}$. Its boundary is

$$\partial(\mathbb{P}^{m-2} \times \mathbb{P}^{n-2}) = \partial(\mathbb{P}^{m-2}) \times \mathbb{P}^{n-2} \cup \mathbb{P}^{m-2} \times \partial(\mathbb{P}^{n-2}).$$

Therefore we have, at the chain complex level,

$$d(c_m \circ_t c_n) = dc_m \circ_t c_n \pm c_m \circ_t dc_n$$

as expected. □

6.4. Proposition. *The dg permutad ΩpermAs^i is a quasi-free model of the permutad *permAs*.*

Proof. The augmentation map $\Omega \text{permAs}^i \rightarrow \text{permAs}$ sends all the 0-cells to μ_n and the other higher dimensional cells to 0. It is obviously a map of dg permutads (trivial differential graded structure for *permAs*). It is a quasi-isomorphism, since the permutohedron is contractible. So we have constructed a quasi-free model of the permutad *permAs*. □

6.5. Comments on Koszul duality theory. One can write down a Koszul duality theory for permutads along the lines of what was down for operads in [8]. In particular $permAs$ is a Koszul permutad and one can construct its minimal model out of the co-permutad $permAs^i$. The dg permutad $\Omega permAs^i$ is in fact the cobar construction over the copermutad $permAs^i$.

For any invertible element $q \in \mathbb{K}$ the permutad $q\text{-}permAs$ can be shown to be Koszul, contrarily to the operadic case for which it is Koszul only for $q = 0, 1, \infty$.

6.6. $permAs$ -algebras up to homotopy. From the explicit description of the minimal model of the permutad $permAs$ we get the following definition of a $permAs_\infty$ -algebra, analogous to the definition of an A_∞ -algebra.

A $permAs_\infty$ -algebra is a chain complex (A, d) over \mathbb{K} equipped with linear maps of degree $n - 2$: $m_t : A^{\otimes n} \rightarrow A$ for any cell t of the permutohedron \mathbb{P}^{n-2} . These maps are supposed to satisfy the following properties:

$$\partial(m_t) = \sum_s \pm m_s,$$

where the sum is over the cells s of codimension 1 in the boundary of the cell t .

EXAMPLES. Let us adopt the shuffle notation for the cells of the permutohedron. So the top cell of \mathbb{P}^{n-2} is $\{1\ 2 \dots n-1\}$ and the map $m_{\{1\ 2 \dots n-1\}}$ is playing the role of the operadic operation m_n . In low dimensions the formulas are the following:

$$\begin{aligned} \partial(m_{\{1\}}) &= 0 \\ \partial(m_{\{12\}}) &= m_{\{1\ | 2\}} - m_{\{2\ | 1\}} \\ \partial(m_{\{123\}}) &= m_{\{12\ | 3\}} + m_{\{2\ | 13\}} + m_{\{23\ | 1\}} - m_{\{1\ | 23\}} - m_{\{13\ | 2\}} - m_{\{3\ | 12\}}. \end{aligned}$$

The interest of this structure lies in the following ‘‘Homotopy Transfer Theorem’’: if a chain complex (W, d) is an algebra over the permutad $permAs$, then any deformation retract (V, d) of (W, d) is naturally equipped with a structure of $permAs_\infty$ -algebra. This is part of the Koszul duality theory for permutads alluded to above.

7. THE PERMUTAD OF ASSOCIATIVE ALGEBRAS WITH DERIVATION

We describe explicitly the permutad of associative algebras equipped with a derivation. We show that, in arity n , it is the algebra of noncommutative polynomials in n variables. Recall that, in the operad framework, it is the commutative polynomials which occur, cf. [7].

7.1. Permutads with 1-ary operations. In the previous sections we supposed, for simplicity, that a permutad had only one 1-ary operation, namely id . But there is no obstruction to extend this notion. When working with the leveled planar trees, for instance, it suffices to admit the vertices with one input (for instance the ladders). We need this generalization for the following example.

7.2. The permutad $permAsDer$. We denote by $permAsDer$ the permutad which is generated by a unary operation D and a binary operation μ , which satisfy the

following relations:

$$\left\{ \begin{array}{l} \mu \circ_1 \mu = \mu \circ_2 \mu, \\ D \circ_1 \mu = \mu \circ_1 D + \mu \circ_2 D, \\ (\alpha \circ_i D) \circ_j \mu = (\alpha \circ_j \mu) \circ_i D, \\ (\alpha \circ_i \mu) \circ_{j+1} D = (\alpha \circ_j D) \circ_i \mu, \end{array} \right.$$

for any operation α and $i < j$.

Observe that the first relation is the associativity of μ , the second relation is saying that D is a derivation, the third and fourth relations say that the operations D and μ commute for parallel composition.

7.3. Theorem. *As a vector space permAsDer_n is isomorphic to the space of non-commutative polynomials in n variables:*

$$\text{permAsDer}_n = \mathbb{K}\langle x_1, \dots, x_n \rangle.$$

Let $t : \underline{n} \rightarrow \underline{2}$ be the surjective map given by the unshuffle $\{i_1, \dots, i_m \mid j_1, \dots, j_n\}$.

The composition map \circ_t is given by

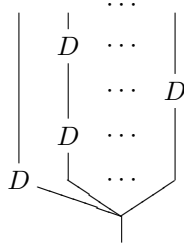
$$(P \circ_t Q)(x_1, \dots, x_{n+m-1}) = P(x_{j_1}, \dots, x_{j_{i-1}}, x_{i_1} + \dots + x_{i_m}, x_{j_i}, \dots, x_{j_n}) Q(x_{i_1}, \dots, x_{i_m}).$$

Under this identification the operations id, D, μ correspond to $1_1, x_1 \in \mathbb{K}\langle x_1 \rangle$ and to $1_2 \in \mathbb{K}\langle x_1, x_2 \rangle$ respectively. More generally the operation

$$(\cdot ((\mu \circ_{j_k} D) \circ_{j_{k-1}} D) \cdots \circ_{j_1} D)$$

corresponds to the noncommutative monomial $x_{j_k} x_{j_{k-1}} \cdots x_{j_1}$.

Graphically the operation $x_{j_k} x_{j_{k-1}} \cdots x_{j_1}$ is pictured as a planar decorated tree with levels as follows (example: $x_1 x_2 x_n x_2$):



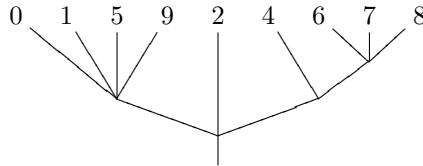
Proof. Up to a minor change of notation and terminology the following result has been proved in [7] for the case $\circ_t = \circ_i$, that is when the inverse image of 1 for t is made of consecutive elements. The proof for any t is similar. \square

8. PERMUTADS AND SHUFFLE OPERADS

Following the work of E. Hoffbeck [3], V. Dotsenko and A. Khoroshkin introduced in [2] the notion of shuffle operad. It is based on the combinatorial objects with substitution called shuffle trees. It turns out that the surjective maps can be considered as particular shuffle trees, whence the relationship between shuffle operads and permutads.

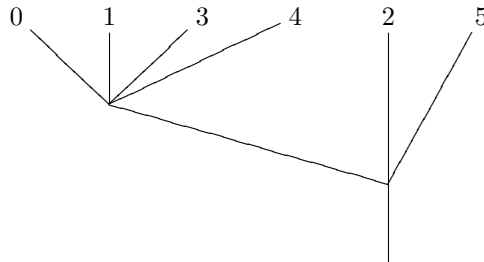
8.1. Shuffle trees and shuffle operads. A *shuffle tree* is a planar tree equipped with a labeling of the leaves by integers $\{0, 1, 2, \dots, n\}$ satisfying some condition stated below. First, we label each edge of the tree as follows. The leaves are already labelled. Any other edge is the output of some vertex v of the tree. We label this edge by $\min(v)$ which is the minimum of the labels of the inputs of v . Second, the condition for a labeled tree to be called a shuffle tree is that, for each vertex, the labels of the inputs, read from left to right, are increasing.

EXAMPLE:

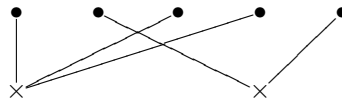


The substitution of a shuffle tree at a vertex of a shuffle tree still gives a shuffle tree. Hence we can define a monad on arity-graded modules (see [2], or [8], Chapter 8 for details). An algebra over this monad is by definition a *shuffle operad*.

8.2. Left combs and permutads. A *left comb* is a planar tree such that, at each vertex, all the inputs but possibly the most left one, is a leaf. A *shuffle left comb* is a shuffle tree whose underlying tree is a left comb. There is a bijection between shuffle left combs and surjective maps as follows. Order the vertices of the left comb downwards from 1 to k . The surjective map $f : \underline{n} \rightarrow \underline{k}$ is such that $f^{-1}(j)$ is the set of labels of the leaves pertaining to the vertex number j . Here is an example:



gives the surjective map:



8.3. Proposition. *The bijection between surjections and shuffle left combs is compatible with substitution. As a consequence any shuffle operad gives rise to a permutad. For instance any symmetric operad gives rise to a permutad.*

Proof. The first statement is proved by direct inspection. Since a shuffle operad is an algebra over the monad of shuffle trees, it suffices to restrict oneself to the shuffle left combs to get a permutad.

It is shown in [2] that any symmetric operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 1}$ gives a shuffle operad $\mathcal{P}^{sh} = \{\mathcal{P}_n^{sh}\}_{n \geq 1}$ with $\mathcal{P}_n^{sh} = \mathcal{P}(n)$. Hence a fortiori any symmetric operad gives rise to a permutad. \square

8.4. On the “partial” presentation of a permutad. Replacing surjections by left comb shuffle trees we observe that any left comb shuffle tree can be obtained by successive substitutions of left comb shuffle trees which have only 2 vertices. In fact these trees are the only left comb shuffle trees with 2 levels and also the only ones which give partial operations (compare with section 8.2 of [8]).

8.5. The permutad associated to the symmetric operad Ass . We consider the symmetric operad Ass encoding the associative algebras. We know that $\mathcal{P}(n) = \mathbb{K}[\mathbb{S}_n]$. Hence the shuffle operad Ass^{sh} associated to it is such that $Ass_n^{sh} = \mathbb{K}[\mathbb{S}_n]$. Let us denote by $permAss^{sh}$ this shuffle operad viewed as a permutad. It has two linear generators in arity 2 that we denote by 1 (the identity in the group \mathbb{S}_2) and τ (the flip in \mathbb{S}_2) respectively. These two operations generate 8 operations in arity 3. Since $Ass_3^{sh} = \mathbb{K}[\mathbb{S}_3]$ is of dimension 6, it means that there are two quadratic relations. An easy computation shows that they are:

$$\begin{array}{ccc}
 \begin{array}{c} \bullet \\ | \\ \times \\ 1 \end{array} & \begin{array}{c} \bullet \\ | \\ \times \\ \tau \end{array} & = & \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \times & \times \\ \tau & 1 \end{array} \\
 \\
 \begin{array}{c} \bullet \\ | \\ \times \\ \tau \end{array} & \begin{array}{c} \bullet \\ | \\ \times \\ 1 \end{array} & = & \begin{array}{c} \bullet & \bullet \\ / & \diagdown \\ \times & \times \\ 1 & \tau \end{array}
 \end{array}$$

This permutad looks analogous to the ns operad $Dend$ encoding dendriform algebras. Indeed, for each of them the dimension of the space of operations is the same as the dimension of the free object on one generator, shifted by one.

9. APPENDIX: THE PERMUTOHEDRON

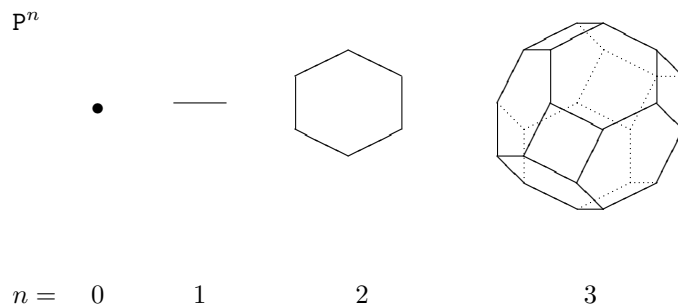
We define and construct the permutohedron together with several ways of labelling its cells: either by surjections or by leveled planar rooted trees. Then, we prove a lemma on some subposet of the weak Bruhat poset of permutations.

9.1. The permutohedron as a cell complex. For any permutation $\sigma \in \mathbb{S}_n$ we associate a point $M(\sigma) \in \mathbb{R}^n$ with coordinates

$$M(\sigma) = (\sigma(1), \dots, \sigma(n)).$$

Since $\sum_i \sigma(i) = \frac{n(n+1)}{2}$, the points $M(\sigma)$ lie in the hyperplane $\sum_i x_i = \frac{n(n+1)}{2}$ of \mathbb{R}^n . The convex hull of the points $M(\sigma)$, $\sigma \in \mathbb{S}_n$, forms a convex polytope \mathbb{P}^{n-1} of

dimension $n - 1$, whose vertices are precisely the $M(\sigma)$'s.



The polytope P^n is called the *permutahedron*. It is a cell complex. The cells of the permutahedron P^{n-1} are in one to one correspondence with the surjective maps $\underline{n} \rightarrow \underline{k}$, cf. for instance [11]. For $k = n$ we obtain the bijection between the set of vertices and the permutations since any surjective map between finite sets is bijective. For $k = 1$ there is only one map which corresponds to the big cell. More generally a surjective map $t = \underline{n} \rightarrow \underline{k}$ corresponds to a $n - k$ dimensional cell P^{n-1} . Let $i_j = \#t^{-1}(j)$. The subcell corresponding to t is of the form $P^{i_1-1} \times \dots \times P^{i_k-1}$.

9.2. Examples. The permutahedron in dimension 1 and 2 together with their permutadic operations (surjective maps between finite sets):



In the following picture the surjections with three, resp. two, resp. one, outputs encode the vertices, resp. edges, resp. 2-cell of P^2 :

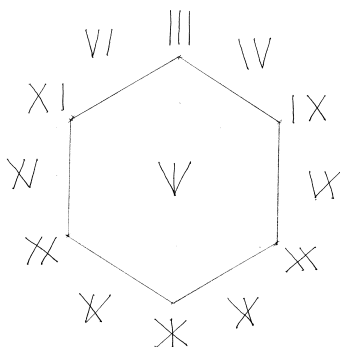


FIGURE 1. The permutad clock

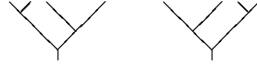
In figure 2 below we indicate the bijections labelling the vertices.

9.3. The chain complex of the permutohedron. Since the permutohedron is a cell complex, we can take its associated chain complex $C_\bullet(\mathbb{P}^n)$. In degree k , the space $C_k(\mathbb{P}^n)$ is spanned by the k -dimensional cells, that is by the surjective maps $\underline{n+1} \rightarrow \underline{n+1-k}$. The boundary map on the big cell c_n is given by the formula

$$d(c_n) = \sum_t \text{sgn}(t)t$$

where the sum is over all the surjective maps t with target $\underline{2}$. The sign $\text{sgn}(t)$ involved in this formula is $\text{sgn}(t) := \text{sgn}(\sigma_t)(-1)^{\#t^{-1}(1)}$ where σ_t is the shuffle associated to the surjective map t (cf. Lemma 2.7). See 9.2 for examples.

9.4. Vertices of the permutohedron and leveled planar binary trees. We consider the set of leveled planar binary rooted trees with $n+1$ leaves. By “leveled” we mean that any vertex is assigned a level, and there is only one vertex per level. Of course the levels are compatible with the structure of the tree. For instance the following trees are acceptable and different:



There is a bijection between leveled planar binary trees and permutations obtained as follows. Starting with a tree we choose to enumerate the vertices of the tree by $1, \dots, n$ from left to right and enumerate the levels similarly from top to bottom. Since there is only one vertex per level we get a bijection. For instance the permutations corresponding to the aforementioned trees are:

$$[1 \ 3 \ 2] \qquad [2 \ 3 \ 1]$$

Forgetting the levels of the trees gives a surjective map

$$\varphi : \mathbb{S}_n \rightarrow PBT_{n+1}.$$

This map appears naturally when comparing the free dendriform algebra on n generators with the free associative algebra, cf. [5].

9.5. Remark. One can extend the preceding bijection to a bijection between all the leveled planar trees and all the surjective maps (i.e. the cells of the permutohedron). But we do not need this interpretation here.

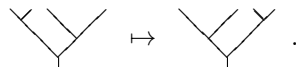
9.6. On a property of the weak Bruhat order. The 1-skeleton of the permutohedron can be oriented such that each edge

$$\begin{array}{ccc} \sigma & & \omega \\ \bullet & \longrightarrow & \bullet \end{array}$$

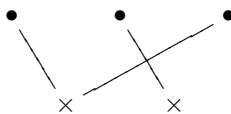
corresponds to the left multiplication by some Coxeter generator $\omega = \sigma s_i$ such that $\ell(\omega) = \ell(\sigma) + 1$. This partial order is called the *weak Bruhat order*. So the 1-skeleton is the geometric realization of a poset with minimal element $[1 \ 2 \ \dots \ n] = 1_n$ and maximal element $[n \ n-1 \ \dots \ 1] = s_1 s_2 \dots s_{n-1} \dots s_2 s_1$.

Under the bijection between permutations and leveled binary trees, there are two different *types of covering relations*:

(1) those obtained through a local application of 

(2) those obtained by some change of levels, like 

In P^2 there is only one edge which corresponds to a covering relation of type (2). It corresponds to the surjection:



(dotted arrow in Figures 2 and 3). The following result is used in the proof of Proposition 4.3 which determines the associative permutad.

9.7. Proposition. *The subposet of the weak Bruhat poset S_n made of the covering relations of first kind is a connected graph.*

We first prove a Lemma.

9.8. Lemma. *If one can exchange i and $i + 1$ for any fixed integer $\ell \leq i$, then one can exchange ℓ and k for any $\ell \leq i < k$.*

Proof. It suffices to show that one can exchange i and $i + 2$. It follows from the following sequence of moves:

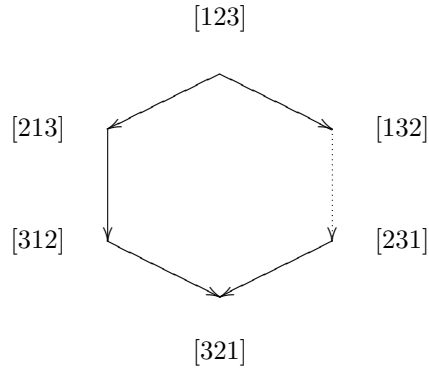
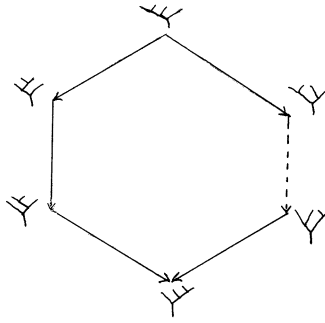
$$\begin{aligned} & [\dots i \dots i + 2 \dots i + 1 \dots] \\ & [\dots i + 1 \dots i + 2 \dots i \dots] \\ & [\dots i + 2 \dots i + 1 \dots i \dots] \\ & [\dots i + 2 \dots i \dots i + 1 \dots] \end{aligned}$$

□

Proof. [of Proposition 9.7] A covering relation, from a permutation to another one, is obtained by left multiplication by some Coxeter generator s_i . The effect consists in exchanging the elements i and $i + 1$ in the image of the permutation. If it is of the first kind, we call it an *admissible move*. If the elements which lie in the interval between i and $i + 1$ contains only elements which are less than i , then applying s_i is admissible. For instance, for $n = 3$, the only covering relation which is not admissible is s_1 applied to $[132]$, see Figure 2 and Figure 3.

We will show that, starting with some permutation σ , there is always a path of admissible moves leading to the permutation with i and $i + 1$ exchanged. We work by induction on the number of elements between i and $i + 1$ which are larger than $i + 1$.

a) If there is none, then applying s_i is admissible.

FIGURE 2. P^2 and bijectionsFIGURE 3. P^2 and trees

b) If there is one, for instance $i + 2$, then the following path is a solution (see the permutohedron above):

$$\begin{aligned}
 & [\dots i - 2 \dots i \dots i - 1 \dots] \\
 & [\dots i - 2 \dots i - 1 \dots i \dots] \\
 & [\dots i - 1 \dots i - 2 \dots i \dots] \\
 & [\dots i \dots i - 2 \dots i - 1 \dots] \\
 & [\dots i \dots i - 1 \dots i - 2 \dots] \\
 & [\dots i - 1 \dots i \dots i - 2 \dots]
 \end{aligned}$$

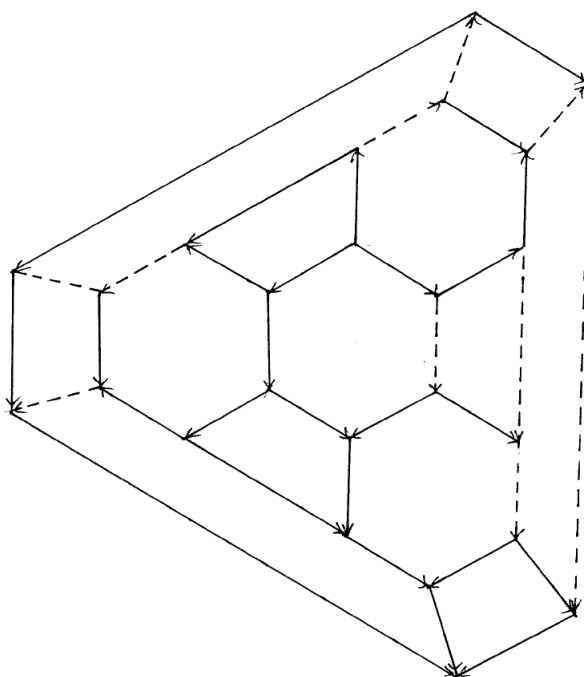
c) We now achieve a double recursion by using Lemma 9.8. We suppose that the r elements $i + 2, \dots, i + 1 + r$ lie in between i and $i + 1$. By induction we suppose that the assertion is valid for $r - 2$ elements. Using this induction we build the

following path (here the integers k_1, \dots, k_r are greater than $i + 1$):

$$\begin{aligned}
 & [i \dots k_1 \dots k_2 \dots k_r \dots i + 1] \\
 & [i \dots i + 1 \dots k_2 \dots k_r \dots k_1] \\
 & [i + 1 \dots i \dots k_2 \dots k_r \dots k_1] \\
 & [i + 1 \dots k_1 \dots k_2 \dots k_3 \dots k_r \dots i]
 \end{aligned}$$

This ends the proof of connectivity by admissible moves. □

Here is the graph corresponding to P^3 :



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