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# Homology of Symplectic and Orthogonal Algebras

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In this paper we compute the homology of the infinite Lie algebra of orthogonal (resp. symplectic) matrices for an associative ring with involution over a characteristic zero field. It is shown (Theorem 5.5) to be the graded symmetric algebra over skew-dihedral homology. The surprising result is that the orthogonal and symplectic algebras have the same homology in the stable range.

Skew-dihedral homology is a variation of cyclic homology, which is studied in detail in [L]. The proof of the main theorem is based on computations in invariant theory (cf. [P1, P2, W]) for orthogonal and symplectic matrices. It follows the lines of proof of the analogue result for matrices given in [L–Q].

We also determine exactly the stable range (Theorem 7.1) and the first obstruction to stability (Theorem 7.3) in the symplectic case.

In the first part we give a comprehensive and detailed review of invariant theory for orthogonal and symplectic matrices. In particular we make explicit the relations between the hyperoctahedral group, the trace identities for matrices and the invariant space of the tensor algebra of matrices.

The homology result in the stable range was also announced in [F-T].

*Conventions.* Vector spaces will be abbreviated as spaces. If the group $G$ is acting on the space $V$, the space of invariants is $V^G = \{v \in V \mid \forall g \in G, g \cdot v = v\}$ and the space of coinvariants is $V_G = V/\{g \cdot v - v \mid g \in G, v \in V\}$. The signature of a permutation $\sigma$ is denoted $\varepsilon(\sigma) = \pm 1$.



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## I. Trace Identities

In this section we review the invariant theory of [W, P1] with substantial additions necessary for this paper. In particular we introduce the universal space $T_m$ for trace formulas. We use its universality on the one hand to understand its module structure over the hyperoctahedral group (Section 3) and on the other hand to compare it to some spaces of invariants (Section 4).

### 1. *∗-Algebras with Trace*

1.1. Definition. A $*$-algebra with trace is an algebra $R$ equipped with two linear maps:

(1) $\mathrm{Tr}: R \to R;\ r \mapsto \mathrm{Tr}(r)$,

(2) $*: R \to R;\ r \mapsto r^*$,

satisfying the following axioms:

(i) $\mathrm{Tr}(rs) = \mathrm{Tr}(sr)$,

(ii) $\mathrm{Tr}(r)\, s = s\, \mathrm{Tr}(r)$,

(iii) $\mathrm{Tr}(\mathrm{Tr}(r)\, s) = \mathrm{Tr}(r)\, \mathrm{Tr}(s)$,

(iv) $r^{**} = r$,

(v) $(rs)^* = s^* r^*$,

(vi) $\mathrm{Tr}(r^*) = \mathrm{Tr}(r) = \mathrm{Tr}(r)^*$.

In this paper, unless otherwise specified, an algebra will mean a $*$-algebra with trace.

1.2. Example. The main examples with which we will be concerned are the classical $*$-algebras associated to forms. Let $V$ be a finitely generated projective module over a commutative unitary ring $A$ and let $\langle -, - \rangle: V \times V \to A$ be a nondegenerate bilinear form which is $\varepsilon$-symmetric, i.e., $\langle u, v \rangle = \varepsilon \langle v, u \rangle$, $\varepsilon = +1$ or $-1$. The form induces an isomorphism $V \to V^* = \mathrm{Hom}_A(V, A)$ and hence a map $j: V \otimes_A V \to A$ with a structure of $*$-algebra with trace as follows:

(a) $u \otimes v \cdot w \otimes z = u \otimes \langle v, w \rangle z$,

(b) $\mathrm{Tr}(u \otimes v) = \langle v, u \rangle$,

(c) $(u \otimes v)^* = \varepsilon v \otimes u$.

### 2. *Universal ∗-Algebra with Trace*

2.1. *The Algebras F, T and U.* In the previous category we can construct as usual in universal algebra, free algebras. We start with a set of variables

$\{x_i\}_{i \in I}$ and add a new set of variables $\{x_i^*\}_{i \in I}$. We set $F = \{x_i, x_i^*\}_{i \in I}$ be the usual free associative algebra over the field $k$. In $F$ we can uniquely define an involution $*$ by the requirements:

$$* : x_i \mapsto x_i^*, \; x_i^* \mapsto x_i,$$

$$* \text{ is an antihomomorphism.}$$

In this way $F$ is a free algebra in the category of $*$-algebras. To add now the trace one considers the set $\mathbf{M}$ of all monomials in the $x_i$ and $x_i^*$. In $\mathbf{M}$ consider the equivalence relation generated by the two rules

(i) $M \sim M^*$ (involution),

(ii) $M_1 M_2 \sim M_2 M_1$ (cyclic equivalence).

If $M \in \mathbf{M}$ we denote by $\mathrm{Tr}(M)$ its equivalence class in $\mathbf{M}/\sim$ and construct the polynomial ring $T$ in the variables $\mathrm{Tr}(M)$.

Finally we construct $U = F \otimes_k T$. The ring $U$ is given a structure of a $*$-algebra with trace as follows.

(i) The map $M \mapsto \mathrm{Tr}(M)$ extends by linearity to a $k$-linear map $\mathrm{Tr}: F \to T$,

(ii) Set $(f \otimes t)^* = f^* \otimes t$,

(iii) Set $\mathrm{Tr}(f \otimes t) = 1 \otimes \mathrm{Tr}(f)\, t$.

One easily verifies that $U$ is the free algebra in the $x_i$'s in the category of $*$-algebras with trace. Remark that $T$ can be identified with a subalgebra of $U$ via $t \mapsto 1 \otimes t$.

2.2. *Linear Actions.* We introduce some representation theory in the picture as follows. The group $GL(2m, k)$ acts naturally on the $2m$-dimensional vector space $W$ with basis $\{x_1, ..., x_m, x_1^*, ..., x_m^*\}$ over the field $k$. Therefore there is a natural left action of the group $GL(2m, k)$ on $F$ induced by the linear action on $W$.

The map $*: W \to W$ can be decomposed in its $+1$ and $-1$ eigenspaces (if $\mathrm{char}\, k \neq 2$, which we always assume). The space $W^+$ has basis $(x_i + x_i^*)/2 = x_i^+$, $i = 1, ..., m$, and $W^-$ has basis $(x_i + x_i^*)/2 = x_i^-$, $i = 1, ..., m$. We have $x_i = x_i^+ + x_i^-$ and $x_i^* = x_i^+ - x_i^-$, and we refer to $x_i^+$ and $x_i^-$ as the symmetric and the antisymmetric parts of $x_i$.

The subgroup of $GL(2m, k)$ which commutes with $*$ is clearly the product $GL(W^+) \times GL(W^-)$. It contains a diagonal $GL(m, k)$ which acts on the $x_i$'s and the $x_i^*$'s in the same way.

The equivalence relation of Section 2.1 is compatible with the action of $GL(W^+) \times GL(W^-)$ and hence we have a canonical action of this group on the free algebra $U = F \otimes_k T$ as a group of trace preserving isomorphisms denoted $f^g$, $f \in U$, $g \in GL(W^+) \times GL(W^-)$. Now we consider the group $D$ of diagonal matrices in $GL(m, k)$ and consider its basic character $d \mapsto \det(d) \in k^*$.

2.3. DEFINITION. An element $f \in U$ is said to be *multilinear* if for every $d \in D$ we have $f^d = \det(d) f$.

This formal definition means concretely that $f$ is a combination of monomials, each of which possesses for each index $i = 1, ..., m$ either the variable $x_i$ or the variable $x_i^*$ but only once. The multilinear elements of $U$ have necessarily degree $m$ and form a finite dimensional vector space denoted $U_m$.

EXAMPLE. $m = 2$. The multilinear monomials of $U$ are (up to equivalence) $x_1 x_2$, $x_2 x_1$, $x_1^* x_2$, $x_2^* x_1$, $x_1 x_2^*$, $x_2 x_1^*$, $x_1^* x_2^*$, $x_2^* x_1^*$, $\mathrm{Tr}(x_1) x_2$, $\mathrm{Tr}(x_1) x_2^*$, $\mathrm{Tr}(x_2) x_1^*$, $\mathrm{Tr}(x_2)\ x_1$, $\mathrm{Tr}(x_1 x_2)$, $\mathrm{Tr}(x_1^* x_2)$, $\mathrm{Tr}(x_1)$ $\mathrm{Tr}(x_2)$.

As $T$ is a subring of $U$ there is also defined the multilinear elements of $T$ which form a finite-dimensional vector space denoted $T_m$. For $m = 2$ it is generated by the last three elements of the above list.

2.4. *The Hyperoctahedral Group.* The hyperoctahedral group $H_m$ is the semi-direct product $(Z/2)^m \rtimes S_m$, where the symmetric group $S_m$ acts on $(Z/2)^m$ by permuting the factors. $H_m$ acts on $W$ (and therefore on $U$ and $T$) as follows: $S_m$ acts by the same permutation on the $x_i$'s and the $x_i^*$'s, the $i$th generator of $Z/2$ permutes $x_i$ and $x_i^*$ and leaves the other variables unchanged. So $H_m$ is the subgroup of the group of permutations $S_{2m}$ of $\{x_1, ..., x_m, x_1^*, ..., x_m^*\}$ which commute with the operator $*$.

2.5. PROPOSITION. *The map $j: U_m \to T_{m+1}$ given by $j(u) = \mathrm{Tr}(u x_{m+1})$ is $H_m$-equivariant and is an isomorphism.*

*Proof.* $H_m$ is supposed to fix the last coordinate $x_{m+1}$. We first prove that $U_m$ and $T_{m+1}$ are $H_m$-spaces. We know that the group $GL(m, k)$ (acting diagonally) is a group of trace isomorphism of the free algebra in $m$ variables. The subgroup $S_m$ of permutations normalizes the diagonal group $D$ and hence preserves the weight subspaces. In particular $S_m$ preserves $M_m$ and $T_m$ (hence $T_{m+1}$). Similarly $(Z/2)^m$ preserves $M_m$ and $T_m$. Finally it is immediate to check that $j$ is $H_m$-equivariant.

We have to show that $j$ is surjective and injective. Since $j$ maps monomials in monomials it is enough to work on monomials. Let us show first that $j$ is surjective. If $a = \mathrm{Tr}(M_1)\,\mathrm{Tr}(M_2) \cdots \mathrm{Tr}(M_h) \in T_{m+1}$ we can find a (unique) factor which we may assume to be $\mathrm{Tr}(M_h)$ which contains the variable $x_{m+1}$ (or $x_{m+1}^*$) linearly. Since $\mathrm{Tr}(M_h) = \mathrm{Tr}(M_h^*)$ we may assume without loss of generality that $M_h = Ax_{m+1}B$, hence $\mathrm{Tr}(M_h) = \mathrm{Tr}(BAx_{m+1})$ and $a = \mathrm{Tr}(\mathrm{Tr}(M_1) \cdots \mathrm{Tr}(M_{h-1})\, BAx_{m+1}) = j(\mathrm{Tr}(M_1) \cdots \mathrm{Tr}(M_{h-1})\, BA)$.

Now for the injectivity we follow essentially the same argument. If $u_1 = \mathrm{Tr}(M_1) \cdots \mathrm{Tr}(M_h)\, M_{h+1}$ and $u_2 = \mathrm{Tr}(N_1) \cdots \mathrm{Tr}(N_h)\, N_{h+1}$ are in $M_m$ and $j(u_1) = j(u_2)$ we must have, since the index $m+1$ appears in the elements $\mathrm{Tr}(M_{h+1}x_{m+1})$ and $\mathrm{Tr}(N_{h+1}x_{m+1})$, that $\mathrm{Tr}(M_{h+1}x_{m+1}) = \mathrm{Tr}(N_{h+1}x_{m+1})$ and $\mathrm{Tr}(M_1) \cdots \mathrm{Tr}(M_h) = \mathrm{Tr}(N_1) \cdots \mathrm{Tr}(N_h)$. Now when we apply cyclic equivalence to $M_{h+1}x_{m+1}$ we necessarily move $x_{m+1}$ away from the rightmost position and when we apply $*$ we exchange $x_{m+1}$ with $x^*_{m+1}$ so that $M_{h+1}x_{m+1}$ equivalent to $N_{h+1}x_{m+1}$ implies $M_{h+1} = N_{h+1}$ as desired.

## 3. *Determinantal Expressions*

3.1. We now consider the following two particular cases of $*$-algebras with trace of the form $\mathrm{End}(N)$, where $N$ is a $2m$-dimensional free module over a ring $A$. We suppose that $N$ is equipped with a basis $u_1, ..., u_{2m}$ and an $\varepsilon$-symmetric form $\langle -, - \rangle : N \times N \to A$ as in Section 1.2.

*Case* $\varepsilon = +1$. Consider a symmetric $2m \times 2m$ matrix $Y = [y_{ij}]$, where the $y_{ij}$'s are indeterminates over $k$ (i.e., subject to the relations $y_{ij} = y_{ji}$). Let $A = {}_1A = k[y_{ij}]$ be the polynomial ring in these variables and $\langle -, - \rangle : N \times N \to {}_1A$ be such that $\langle u_i, u_j \rangle = y_{ij}$.

*Case* $\varepsilon = -1$. Same as before but now $Y = [y_{ij}]$ is skew-symmetric: $y_{ii} = 0$ and $y_{ij} = -y_{ji}$. Hence $A = {}_{-1}A$ is of dimension $m(2m-1)$ over $k$. The form on $N$ is then also skew-symmetric.

In both cases we have an action of the group $GL(2m, k)$ on $V$ given by the usual matrix multiplication and also an action of $GL(2m, k)$ on the polynomial ring $A$ extending by linearity the action $Y \mapsto \alpha Y^t \alpha$ of $\alpha \in GL(2m, k)$ on the generic matrix $Y$.

Considering the diagonal group $D$ of $GL(2m, k)$ we have again the notion of mutilinear elements denoted ${}_\varepsilon A_m$ or $A_m$ if there is no ambiguity.

3.2. LEMMA. *The space $A_m$ of multilinear elements of $A$ is spanned by the monomials of degree $m$*

$$y_{i_1 j_1} y_{i_2 j_2} \cdots y_{i_m j_m},$$

*where $(i_1, j_1, i_2, j_2, ..., i_m, j_m)$ is a permutation of $(1, 2, ..., 2m)$.*

This lemma shows that $A_m$ is generated by the products

$$\langle u_{i_1}, u_{j_2} \rangle \langle u_{i_2}, u_{j_2} \rangle \cdots \langle u_{i_m}, u_{j_m} \rangle,$$

where $(i_1, j_1, i_2, j_2, ..., i_m, j_m)$ is a permutation of $(1, 2, ..., 2m)$. If we consider now the group $S_{2m}$ which permutes the elements $u_1, ..., u_{2m}$, it induces an action on $y_{ij}$ by $\sigma(y_{ij}) = y_{\sigma(i)\sigma(j)}$ and hence an action on $A_m$. In fact $S_{2m}$ permutes the multilinear monomials. When $\varepsilon = +1$ the distinct monomials are a basis of $A_m$, while when $\varepsilon = -1$ we get only a basis up to sign (since $y_{ij} = -y_{ji}$). In the first case we have a permutation representation, while in the second case we have a sign-permutation representation.

We denote by $k[S_{2m}/H_m]^\varepsilon$ the $S_{2m}$-module whose underlying space is $k[S_{2m}/H_m]$ and where the action of $S_{2m}$ is multiplication on the left if $\varepsilon = +1$, and is multiplication on the left times the sign of the permutation if $\varepsilon = -1$.

3.3. PROPOSITION. *The subgroup of $S_{2m}$ fixing the monomial $M_0 = y_{1m+1} y_{2m+2} \cdots y_{m2m}$ up to sign is the hyperoctahedral group $H_m$. The representation of $S_{2m}$ on ${}_\varepsilon A_m$ is induced from the trivial representation of dimension one if $\varepsilon = +1$ and from the sign representation if $\varepsilon = -1$. In other words there is an isomorphism $\theta\colon k[S_{2m}/H_m]^\varepsilon \cong {}_\varepsilon A_m$.*

*Proof.* This is clear since every multilinear monomial is, up to sign, in the $S_{2m}$-orbit of $M_0$.

*Remark.* Explicitly we have $\theta(\sigma) = y_{\sigma(1)\sigma(m+1)} \cdots y_{\sigma(m)\sigma(2m)}$ if $\varepsilon = +1$ and $\theta(\sigma) = \mathrm{sgn}(\sigma)\, y_{\sigma(1)\sigma(m+1)} \cdots y_{\sigma(m)\sigma(2m)}$ if $\varepsilon = -1$.

By universality the assignment $x_i \mapsto u_i \otimes u_{m+i}$ (and therefore $x_i^* \to \varepsilon u_{m+i} \otimes u_i$) gives rise to a morphism of $*$-algebras with trace $\rho\colon U \to N \otimes N = \mathrm{End}(N)$ which restricts to a homomorphism (still denoted $\rho\colon T \to {}_\varepsilon A$). For instance, when $m = 2$, $\mathrm{Tr}(x_1 x_2)$ has image $\mathrm{Tr}(u_1 \otimes u_3 \cdot u_2 \otimes u_4) = \langle u_3, u_2 \rangle \mathrm{Tr}(u_1 \otimes u_4) = \langle u_3, u_2 \rangle \langle u_4, u_1 \rangle = y_{32}\, y_{41}$. Therefore it corresponds to the class of the permutation $\sigma = (12)$ in $k[S_4/H_2]$.

3.4. DEFINITION. The induced map $\rho\colon T_m \to {}_\varepsilon A_m$ on the multilinear elements is called the *coding map*.

The main result of this section is the following.

3.5. THEOREM. *The coding map $\rho\colon T_m \to {}_\varepsilon A_m$ is an isomorphism of $H_m$-modules.*

*Proof.* When we substitute $u_i \otimes u_{m+i}$ to $x_i$ in a monomial in $T_m$ we see by the rules of Section 1.1 that we obtain a monoid in the elements $(u_i, u_j)$ in which the $2m$ indices $1, 2, ..., 2m$ all appear exactly once. Conversely, given a monomial $a$ in $A_{2m}$ we can define uniquely a monomial in $T_m$ as follows.

1. Write $a=(u_{i_1}, u_{j_1})(u_{i_2}, u_{j_2})\cdots(u_{i_m}, u_{j_m})$.

2. Split $a$ as a product of monomials $a_i$ satisfying the following condition: $a_i$ is of the type

$$(u_{a_1}, u_{b_1})(u_{a_2}, u_{b_2})\cdots(u_{a_k}, u_{b_k}),$$

where $a_2 \equiv b_1+m$, $a_3 \equiv b_2+m$, ..., $a_1 \equiv b_k+m \bmod 2m$.

This can be clearly done, up to changing eventually the sign of $a$ (in case $\varepsilon=-1$), since we are assuming that all the indices 1, 2, ..., $2m$ appear exactly once. Hence given any factor $(u_s, u_t)$ there is an occurrence of $t+m$ (mod $2m$) as an index in another factor. If $s \equiv t+m$ we isolate $(u_s, u_t)$ as one of the monomials, otherwise by exchanging the two vectors in the scalar product in which $u_{t+m}$ appears (if necessary), we can collect $(u_s, u_t)(u_{t+m}, u_v)$ and proceed until we close the cycle.

3. Each monomial of type 2 is of the form $\mathrm{Tr}(M)$, where $M$ is a monomial in the $x_i$'s or the $x_i^*$'s.

One easily sees that the only ambiguities implicit in this construction are the ones given by cyclic equivalence and $*$-equivalence, e.g., $m=2$, $x_1=u_1\otimes u_3$, $x_2=u_2\otimes u_4$, $T_2$ is spanned by $\mathrm{Tr}(x_1x_2)$, $\mathrm{Tr}(x_1^*x_2)$, $\mathrm{Tr}(x_1)\,\mathrm{Tr}(x_2)$, and $A_4$ is spanned by $(u_4, u_1)(u_3, u_2)$, $(u_2, u_1)(u_4, u_3)$, $(u_3, u_1)(u_4, u_2)$, which correspond to each other under $\rho$.

3.6. *Trace Identities.* We fix a vector space $V$ over $k$ of dimension $p$ and an $\varepsilon$-symmetric nondegenerate form on $V$ (so $p$ must be even if $\varepsilon=-1$) (cf. Section 1.1). For any choice of $m$ elements $\alpha_1, ..., \alpha_m \in \mathrm{End}(V)$ we have an *evaluation map* by replacing the variables $x_i$ in $U$ by the matrices $\alpha_i$.

3.7. Definition. A *trace identity* for $\mathrm{End}(V)$ is an element $f(x_1, ..., x_m)$ in $U$ which vanishes under all substitutions of the $x_i$'s by $\alpha_i$'s, $\alpha_i \in \mathrm{End}(V)$. A *multilinear trace identity* is a trace identity which is in $T_m$.

The kernel of the *evaluation map* $\mathrm{ev}: U \to (\mathrm{End}(V)^{\otimes m})^*$ (where $*$ means dual over $k$) is a two-sided ideal. It is well known that as soon as the base field is infinite this ideal of trace identities depends only on the dimension $p$ of $V$ and the value of $\varepsilon$. We will consider its restriction to $T$ and in particular to the space of multilinear elements $T_m$. The following theorem identifies the image of the multilinear trace identities by the coding map.

3.8. Theorem. [P1, P2]. (1) *For $\varepsilon=+1$ the space of multilinear trace identities of $p\times p$-matrices corresponds, under the coding map $\rho: T_m \to {}_1A_m$, to the intersection of ${}_1A_m$ with the ideal $I_{p+1}$ of ${}_1A$ generated by the $(p\times 1)\times(p+1)$ subdeterminants of the generic symmetric matrix $Y$.*

(2) *For $\varepsilon=-1$ the space of multilinear trace identities corresponds, under the coding map, to the intersection of ${}_{-1}A_m$ with the ideal $P_{p+2}$ of*

*$_{-1}A$ generated by the Pfaffians of the principal $(p+2)\times(p+2)$ minors of the generic skew-symmetric matrix $Y$.*

We have previously described $_{\varepsilon}A_m$ as an induced representation. In fact we can be much more precise in describing its structure. In general this can be done by standard Young diagrams and in characteristic zero it yields the following.

Recall that the irreducible representations of the symmetric group $S_t$ are indexed by partitions of $t$. If $\lambda: k_1 \geqslant \cdots \geqslant k_t$ is such a partition, we set $M_\lambda$ to be the corresponding module. We call the numbers $k_i$'s the rows of $\lambda$ and refer instead to the columns of $\lambda$ when considering the dual partition ($k_t$ columns). We set our conventions so that $M_t$ corresponds to the sign representation, while $M_{1\ldots1}$ is the trivial representation. With these notations one has

3.9. PROPOSITION. *As a representation of $S_{2m}$ we have*

(a) $_{I}A_m = \oplus M_\lambda$, *where $\lambda$ runs over all partitions with even columns;*

(b) $_{1}A_m \cap I_{p+1} = \oplus M_\lambda$, $\lambda = k_1 \geqslant \cdots \geqslant k_{2m}$ *as before and* $k_1 \geqslant p+1$;

(c) $_{-1}A_m = \oplus M_\lambda$, *$\lambda$ runs over all partitions with even rows;*

(d) $_{-1}A_m \cap P_{p+2} = \oplus M_\lambda$, $\lambda = k_1 \geqslant \cdots \geqslant k_{2m}$ *as in c) and* $k_1 \geqslant p+2$.

3.10. *Remark.* $_{-1}A_m \cong {}_{1}A_m \otimes \mathrm{sgn}$ as representations of $S_{2m}$ but no such analogy exists for cases (b) and (d).

## 4. *Lie Algebras Invariants*

In this section we use Proposition 3.3 and Theorems 3.5 and 3.8 to determine the structure of some spaces of coinvariants that will be needed in the sequel.

Let $V$ be a $p$-dimensional vector space over $k$ equipped with a non-degenerate $\varepsilon$-symmetric form. Let $gl_p = \mathrm{End}(V)$ be the Lie algebra of $(p\times p)$-matrices. The embedding $gl_p \subset gl_{p+1}$ is the embedding $\mathrm{End}(V) \subset \mathrm{End}(V\oplus k)$. The group of automorphisms $G(p)$ of $V$ is the orthogonal group $O(p)$ if $\varepsilon = +1$ or the symplectic group $Sp(p)$ if $\varepsilon = -1$.

The group $H_m$ acts on $gl_p^{\otimes m}$ by permuting the factors for elements of $S_m$ and by changing the element $\alpha_i$ of $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n$ into ${}^t\alpha_i$ (resp. ${}^T\alpha_i$) for the $i$th generator of $(\mathbf{Z}/2)^m$ (cf. Section 5).

4.1. PROPOSITION. *There is a map $(gl_p^{\otimes m})_{G(p)} \to k[S_{2m}/H_m]^{\varepsilon}$ which is $H_m$-equivariant and which is an isomorphism for $p \geqslant m$ when $G = O$ $(\varepsilon = +1)$ and for $p \geqslant 2m$ when $G = Sp$ $(\varepsilon = -1)$.*

*Proof.* The restriction of the evaluation map (see Section 3.6) to $T_m$ lands in $\mathrm{Hom}(gl_p^{\otimes m}, k) = (gl_p^{\otimes m})^*$. By the first theorem of invariant theory

[W, P1] its span is the invariant space $((gl_p^{\otimes m})^*)^{G(p)}$. In the following diagram of $H_m$-equivariant maps

$$k[S_{2m}/H_m]^\varepsilon \xrightarrow{\theta} {}_\varepsilon A_m \xleftarrow{\rho} T_m \xrightarrow{ev} ((gl_p^{\otimes m})^*)^{G(p)} = ((gl_p^{\otimes m})_{G(p)})^*$$

the map $\theta$ is an isomorphism by Section 3.3, $\rho$ is an isomorphism by Theorem 3.5, and $ev$ is determined by Theorem 3.8.

Let $\varepsilon = +1$. By Theorem 3.8 the ideal of trace identities corresponds, under the coding map, to the intersection of ${}_\varepsilon A$ with the ideal $I_{p+1}$ generated by the $(p+1)\times(p+1)$ subdeterminants of $Y$. As we are interested only in the multilinear elements, that is, the space generated by the monomials

$$y_{i_1j_1}y_{i_2j_2}\cdots y_{i_mj_m},$$

where $i_1, j_1, ..., i_m, j_m$ is a permutation of $1, 2, ..., 2m$, we see that the first time this intersection is nonzero is for $p+1=m$. Therefore this space is 0 for $p \geqslant m$.

Let $\varepsilon = -1$. In this case we need to consider the Pfaffians of the principal $(p+2)\times(p+2)$ minors of $Y$. There is no such Pfaffian in ${}_{-1}A_m$ if $p \geqslant 2m$.

Therefore if $p$ is large enough $ev$ is an isomorphism and so is the composite $ev \circ \rho^{-1} \circ \theta$. Hence the isomorphism of the proposition is the dual of $ev \circ \rho^{-1} \circ \theta$ once $k[S_{2m}/H_m]^\varepsilon$ is identified with its dual, and this can be done canonically as this space has a preferred basis.

4.2. *The Pfaffian and Its Augmentation Map.* The Pfaffian Pf($Y$) of the generic matrix $Y = [y_{ij}]_{1 \leqslant i,j \leqslant 2m}$ is an element of ${}_{-1}A_m$. We describe its image in $k[S_{2m}/H_m]$ under $\theta$. Choose in each class $(\sigma) \in S_{2m}/H_m$ a permutation $\sigma \in S_{2m}$ such that $\sigma(i) < \sigma(m+i)$ and denote by $S$ the set of these permutations. The Pfaffian is given by the following formula (see, for instance, N. Bourbaki, "Algèbre," Chap. 9, Sect. 5, n°2)

$$\theta\left(\sum_{\sigma \in S} \operatorname{sgn}(\sigma)(\sigma)\right) = (-1)^{m+1}\,\mathrm{Pf}(Y).$$

The Pfaffian generates a one-dimensional vector space, so dually it gives an augmentation map $\pi: k[S_{2m}/H_m]^{-1} \to k$ given by $\pi(\sigma) = (-1)^{m+1}\varepsilon(\sigma)$ for $\sigma \in S$.

4.3. PROPOSITION. *Let* $\varepsilon = -1$. *The map* $(gl_{2m-2}^{\otimes m})_{Sp(2m-2)} \to (gl_{2m}^{\otimes m})_{Sp(2m)}$ *is injective and, under the identification* $(gl_{2m}^{\otimes m})_{Sp(2m)} = k[S_{2m}/H_m]$, *its cokernel is the map* $\pi$.

*Proof.* Following the proof of Proposition 4.1 we see that it suffices to identify the kernel of $ev$.

Let $\varepsilon = -1$ and $p+2=2m$. By Theorem 3.8 the ideal of trace identities corresponds, under the coding map, to the intersection ${}_{-1}A_m \cap P_{2m}$. This space is of dimension one and spanned by the principal Pfaffian $\mathrm{Pf}(Y)$. Hence, by Section 4.2, the space $(gl_{2m-2}^{\otimes m})_{Sp(2m-2)}$ can be identified with the kernel of $\pi$.

4.4. *Remark.* The orthogonal case is more complicated because in the first nontrivial case $(p+1=m)$ we have ${}_1A_m \cap I_m = M_{mm}$ (cf. Proposition 3.9). This representation is of dimension $(2m)!/(m+1)!\, m!$.

## II. Homology of Lie Algebras

### 5. *Lie Algebra Homology and Dihedral Homology*

Let $k$ be a field of characteristic zero and $A$ an associative $k$-algebra with 1. We do not assume that $A$ is commutative. The Lie algebra of $n \times n$ matrices with coefficients in $A$ is denoted $gl_n(A)$. The *embedding* $gl_n(A) \to gl_{n+1}(A)$ is obtained by inserting a last column and a last row of zeroes. The limit $\lim gl_n(A)$ under the inclusions is denoted $gl(A)$. We now suppose that $A$ is equipped with an (anti)-involution on $gl_n(A)$, $a \mapsto \bar{a}$ (i.e., $(\overline{a\ b}) = \bar{b}\,\bar{a}$) which is the identity on $k$. It induces as usual an involution on $gl_n(A)$, $\alpha \mapsto {}^t\alpha$, defined by $({}^t\alpha)_{ij} = \bar{\alpha}_{ji}$ which is compatible with the inclusions.

5.1. *Orthogonal Matrices.* By definition we have

$$o_n(A) = \{\alpha \in gl_n(A) \mid {}^t\alpha = -\alpha\}.$$

This is a sub-Lie algebra of $gl_n(A)$. We put $o(A) = \lim o_n(A)$. If we think of $gl_n(A)$ as split by the involution ${}^t(-)$, that is, $gl_n(A) = gl_n(A)^+ \oplus gl_n(A)^-$, then $o(A) = gl(A)^-$.

5.2. *Symplectic Matrices.* In order to make the inclusions compatible with symplectic matrices we choose $J_n = j \oplus \cdots \oplus j$, where

$$j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is the matrix of the symplectic form on $A^{2n}$. Therefore we have

$$sp_{2n}(A) = \{\alpha \in gl_{2n}(A) \mid {}^T\alpha = -\alpha\}$$

where ${}^T\alpha = -J_n\,{}^t\alpha J_n$, and there is a canonical inclusion $sp_{2n}(A) \to sp_{2n+2}(A)$. As usual we put $sp(A) = \lim sp_{2n}(A)$. If we think of $gl(A)$ as split by the involution $\alpha \mapsto {}^T\alpha$, then $gl(A) = gl(A)^+ \oplus gl(A)^-$ and $sp(A) = gl(A)^-$.

Our aim is to compute the homology of the Lie algebras $o(A)$ and $sp(A)$ with coefficients in the trivial module $k$ (cf. [K]). For sake of brevity $g(A)$ (resp. $g_n(A)$) will denote $o(A)$ (resp. $o_n(A)$) or $sp(A)$ (resp. $sp_n(A)$), depending on the choice of $\varepsilon$. These homology groups are obtained by considering the complex $(\Lambda^* g(A), d)$, where $\Lambda^* g(A)$ is the exterior algebra and $d$ is the differential

$$d(\alpha_1 \wedge \cdots \wedge \alpha_n) = \sum_{1 \leqslant i,j \leqslant n} (-1)^{i+j+1} [\alpha_i, \alpha_j] \wedge \alpha_1 \wedge \cdots \wedge \hat{\alpha}_i \wedge \cdots \wedge \hat{\alpha}_j \wedge \cdots \wedge \alpha_n.$$

As we are dealing with the stable case (that is, $g(A)$ instead of $g_n(A)$) the homology $H_*(g(A), k)$ is a commutative Hopf algebra, the algebra structure being given by the direct sum of matrices. Therefore, by Hopf's theorem, it is the graded symmetric algebra over its primitive part. In order to compute this primitive part we need to introduce *dihedral homology.*

5.3. *Dihedral Homology.* Consider the tensor product $A^{\otimes n+1}$ of $n$ copies of $A$ over $k$. We write $(a_0, a_1, ..., a_n)$ instead of $(a_0 \otimes a_1 \otimes \cdots \otimes a_n)$ for a generator of $A^{\otimes n+1}$. The dihedral group $D_{n+1} = \{x, y / x^{n+1} = y^2 = 1, yxy = x^{-1}\}$ is acting on $A^{\otimes n+1}$ by

$$x(a_0, a_1, ..., a_n) = (-1)^n (a_n, a_0, ..., a_{n-1})$$

and

$$y(a_0, a_1, ..., a_n) = (-1)^{n(n+1)/2} (\bar{a}_0, \bar{a}_n, \bar{a}_{n-1}, ..., \bar{a}_1).$$

We will sometimes use the fact that $D_n$ is also generated by $x$ and $z = x^{-1}y$. The module of coinvariants

$$(A^{\otimes n+1})_{D_{n+1}}$$

is denoted $\mathbf{D}_n(A)$ (it is the quotient of $A^{\otimes n+1}$ which makes the action of $x$ and $y$ trivial). If we modify the action of $y$ by $(-1)$ we denote by ${}_{-1}\mathbf{D}_n(A)$ the resulting module. The Hochschild boundary $b\colon A^{\otimes n+1} \to A^{\otimes n}$ is given by

$$b(a_0, ..., a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, ..., a_i a_{i+1}, ..., a_n) + (-1)^n (a_n a_0, a_1, ..., a_{n-1}).$$

It is compatible with passing to the quotient by the action of the dihedral group (cf. [L]). As a consequence it defines a complex denoted $(\mathbf{D}_*(A), b)$ (resp. $({}_{-1}\mathbf{D}_*(A), b)$).

5.4. Definition [L]. The dihedral (resp. skew-dihedral) homology of $A$ is

$$HD_n(A) = H_n(\mathbf{D}_*(A), b) \text{ (resp. } {}_{-1}HD_n(A) = H_n({}_{-1}\mathbf{D}_*(A), b)).$$

5.5. THEOREM. *Let $k$ be a characteristic zero field and $A$ an associative $k$-algebra with unity. Then there are isomorphisms*

$$\operatorname{Prim} H_*(o(A), k) = {}_{-1}HD_{*-1}(A),$$
$$\operatorname{Prim} H_*(sp(A), k) = {}_{-1}HD_{*-1}(A).$$

As a corollary we see that $o(A)$ and $sp(A)$ have the same homology.

Another way of expressing the theorem is to say that $H_*(o(A), k)$ is canonically isomorphic to the graded symmetric tensor algebra over ${}_{-1}HD_{*-1}(A)$ (and similarly for $sp(A)$).

## 6. *Proof of the Stable Theorem*

The proof is along the same lines as that of Theorem 6.2 in [L-Q], which asserts that $\operatorname{Prim} H_*(gl(A), k) = HC_{*-1}(A)$, where $HC_*(A)$ is cyclic homology. However, the result of invariant theory that we are going to use is different. We recall it from Proposition 4.1: the map $(ev \circ \rho^{-1} \circ \theta)^* : (gl_p^{\otimes n})_{G(p)} \to k[S_{2n}/H_n]^\varepsilon$ is an isomorphism of $H_n$-modules for $p \geqslant n$ in the orthogonal case $(G = O)$ and $p \geqslant 2n$ in the symplectic case$(G = Sp)$.

Furthermore we remark that the following diagram is commutative,

$$\begin{array}{ccc} (gl_p^{\otimes n})_G & \longrightarrow & k[S_{2n}/H_n]^\varepsilon \\ \downarrow & & \downarrow \\ (gl_p^{\otimes n})_{GL} & \longrightarrow & k[S_n] \end{array}, \tag{6.1}$$

where the right-hand-side vertical map is dual to the composite $S_n \subset S_{2n} \to S_{2n}/H_n$.

6.2. *First Step.* By hypothesis $g(A) = (gl(A))_{\mathbf{Z}/2}$, where the generator $\eta$ of $\mathbf{Z}/2$ acts by $\alpha \mapsto -{}^t\alpha$ (resp. $\alpha \mapsto -{}^T\alpha$). Therefore $g(A)^{\otimes n} = ((gl(A)^{\otimes n})_{(\mathbf{Z}/2)^n}$. Hence the exterior product can be rewritten

$$\begin{aligned} A^* g(A) &= (g(A)^{\otimes n})_{S_n} = ((gl(A)^{\otimes n})_{(\mathbf{Z}/2)^n})_{S_n} \\ &= (gl(A)^{\otimes n})_{H_n} = (gl^{\otimes n} \otimes A^{\otimes n})_{H_n}. \end{aligned}$$

In this setting $\eta_i \in (\mathbf{Z}/2)^n \subset H_n$ acts on $\alpha \in gl^{\otimes n}$ by $\eta_i(\alpha) = (\alpha_1 \otimes \cdots \otimes {}^t\alpha_i \otimes \cdots \otimes \alpha_n)$ (or $(\alpha_1 \otimes \cdots \otimes {}^T\alpha_i \otimes \cdots \otimes \alpha_n)$) and on $A^{\otimes n}$ by $\eta_i(a) = (a_1 \otimes \cdots \otimes -\bar{a}_i \otimes \cdots \otimes a_n)$. A permutation $\sigma \in S_n$ acts on $gl^{\otimes n}$ by permuting the variables and on $A^{\otimes n}$ by permuting the variables *and* multiplication by the signature.

6.3. *Second Step.* By Proposition 6.4 of [L-Q] the complex $(A^* g_p(A), d)$ is quasi-isomorphic to the complex of coinvariants

$((\Lambda^* g_p(A))_{G(p)}, d)$. Therefore stably $(\Lambda^* g(A), d)$ is quasi-isomorphic to $((\Lambda^* g(A))_G, d)$. These last modules can be written as

$$(\Lambda^n g(A))_G = ((gl^{\otimes n} \otimes A^{\otimes n})_{H_n})_G = ((gl^{\otimes n})_G \otimes A^{\otimes n})_{H_n},$$

and so we are in position to apply our invariant theory result recalled above. This gives

$$(\Lambda^n g(A))_G = (k[S_{2n}/H_n] \otimes A^{\otimes n})_{H_n}.$$

The actions of $H_n$ were described above.

6.4. *Third Step.* As we are interested in the primitive part, by the same argument as in [L-Q, p. 585], it is sufficient to compute the primitive part of the complex of coinvariants.

Let $U_n$ be the conjugation class of $\tau = (1\ 2 \cdots n)$ in $S_n$. Let $V_n$ be the subset of $S_{2n}/H_n$ made of the elements of the form $(ug)$, $u \in (\mathbf{Z}/2)^n$, $g \in U_n \subset S_n$. The primitive part of degree $n$ of $(\Lambda^* g(A))_G$ is

$$(k[V_n] \otimes A^{\otimes n})_{H_n}$$

because, first, these elements are primitive and, second, the envelopping algebra is $(\Lambda^* g(A))_G$.

6.5. LEMMA. *There is an $H_n$-equivariant bijection $H_n/D_n \cong V_n$.*

*Proof.* Consider the action of $H_n$ on $S_{2n}/H_n$ by conjugation (or equivalently by multiplication). The orbit of $\tau \in S_n \subset S_{2n}$ is exactly $V_n$ and the subgroup of $H_n$ fixing $\tau$ is generated by $\tau$ and $\eta\omega$, where $\eta = \prod_i \eta_i$ and $\omega = (1\ n)(2\ n-1)(3\ n-2) \cdots$. This is because $\eta\omega\tau(\eta\omega)^{-1} = \eta\omega\tau\omega^{-1}\eta^{-1} = \eta\tau^{-1}\eta^{-1} = \tau\Delta(\tau^{-1})$. This group is isomorphic to $D_n$ by the application $x \mapsto \tau$ and $z \mapsto \eta\omega$.

*End of the Proof of Theorem* 5.5. Putting everything together we get

$$\begin{aligned} \mathrm{Prim}((\Lambda^n g(A)_G) &= (k[V_n] \otimes A^{\otimes n})_{H_n} = (k[H_n/D_n] \otimes A^{\otimes n})_{H_n} \\ &= (k \otimes A^{\otimes n})_{D_n} = (A^{\otimes n})_{D_n}. \end{aligned}$$

The action of $D_n$ on $A^{\otimes n}$ is via $H_n$, which is explicitly

$$x(a_1, ..., a_n) = (-1)^{n-1} (a_n, a_1, ..., a_{n-1})$$

$$z(a_1, ..., a_n) = (-1)^{n(n+1)/2} (\bar{a}_n, \bar{a}_{n-1}, ..., \bar{a}_1).$$

As $y=xz$ we have $y(a_1, ..., a_n)=(-1)^{(n+1)(n+2)/2}(\bar{a}_1, \bar{a}_n, ..., \bar{a}_2)$. Therefore the primitive part is ${}_{-1}\mathbf{D}_n(A)$ because the action of $y$ on $(n+1)$ variables is

$$y(a_0, a_1, ..., a_n)=-(-1)^{n(n+1)/2}(\bar{a}_0, \bar{a}_n, ..., \bar{a}_1).$$

It is immediate to check that the boundary induces the Hochschild boundary (like in the $gl$ case) and the proof of Theorem 5.5 is completed.

*Remark.* The primitive part of the homology of $gl(A)$ could also be computed using Theorem 3.8 i.e., quotienting by the action of $G$ instead of $GL$. It is the same proof as above but without taking the coinvariants by $(\mathbf{Z}/2)^n$.

## 7. *Stability and Obstruction to Stability*

It is clear from Section 6 that there is some stability for the homology of $o(A)$ and of $sp(A)$. We will precise this phenomenon and compute the first obstruction to stability in the symplectic case.

7.1. THEOREM. *Let $k$ be a characteristic zero field and $A$ a unitary associative $k$-algebra with involution. The stabilization homomorphism $H_p(o_{n-1}(A), k) \to H_p(o_n(A), k)$ is an isomorphism for $p<n-1$ and a surjection for $p=n-1$. The stabilization homomorphism $H_p(sp_{2n-2}(A), k) \to H_p(sp_{2n}(A), k)$ is an isomorphism for $p \leqslant n-1$.*

*For $p=n$ and $\varepsilon=-1$ the cokernel is a quotient of $\Lambda^n A^-$, more precisely there is an exact sequence*

$$H_n(sp_{2n-2}(A), k) \to H_n(sp_{2n}(A), k) \to \Lambda^n_{A^+}(A^-/[A^+, A^-]) \to 0.$$

*Proof.* As in Section 6 we use the complex of coinvariants $(\Lambda^* g_n(A))_{G(n)}$ to compute the homology of $g_n(A)$. We are interested in the cokernel $L_*$ of $(\Lambda^* g_{n-1}(A))_{G(n-1)} \to (\Lambda^* g_n(A))_{G(n)}$ whose homology is the relative homology $H_*(g_n(A), g_{n-1}(A); k)$. From Section 4 it follows that $(gl_n^{\otimes p})_{G(n)}$ can be identified with a subspace of $k[S_{2p}/H_p]$ that we denote $I_{n,p}$ in the orthogonal case and $I_{(n/2),p}$ in the symplectic case. With this convention there are isomorphisms $I_{n,p}=I_{p,p}=k[S_{2p}/H_p]$ as soon as $n \geqslant p$. As $(\Lambda^p o_n(A))_{O(n)}$ (resp. $(\Lambda^p sp_{2n}(A))_{Sp(2n)}$ is isomorphic to

$$(I_{n,p} \otimes A^{\otimes p})_{H_p}$$

it follows that $L_p=0$ for $p \leqslant n-1$. As a consequence the stabilization homomorphism is an isomorphism for $p<n-1$ and a surjection for $p=n-1$.

In order to compute the relative homology group in the critical case and when $\varepsilon = -1$ we recall that by Proposition 4.4 the cokernel of $I_{n-1,n} \to I_{n,n}$ is one-dimensional. Consider the following diagram:

$$\begin{array}{ccccc} (I_{n,n+1} \otimes A^{\otimes n+1})_{H_{n+1}} & \xrightarrow{d} & (I_{n,n} \otimes A^{\otimes n})_{H_n} & \xrightarrow{d} & (I_{n,n-1} \otimes A^{\otimes n-1})_{H_{n-1}} \\ \downarrow & & \downarrow & & \downarrow \\ L_{n+1} & \longrightarrow & L_n = (k \otimes A^{\otimes n})_{H_n} & \longrightarrow & 0 \end{array} \qquad (7.2)$$

We note that there is a canonical isomorphism

$$(k \otimes A^{\otimes n})_{H_n} \cong \Lambda^n A^-.$$

The projection map $\pi_*$ has a splitting $s_*$ induced by the map $s$, which sends $1 \in k$ to the class of the trivial permutation $(\ ) \in I_{n,n}$. The element corresponding to $s_*(1 \otimes (a_1, ..., a_n)) = (\ ) \otimes (a_1, ..., a_n)$ in $(gl_{2n}(A)^{\otimes n})_{Sp(2n)}$ is the class of

$$E_{11}^{a_1} \otimes \cdots \otimes E_{nn}^{a_n}$$

whose image by $d$ is 0. Therefore $d \circ s_* = 0$.

The first consequence of this relation is that the connecting map $H_n(sp_{2n}(A), sp_{2n-2}(A); k) \to H_{n-1}(sp_{2n-2}(A); k)$ is trivial, which implies the isomorphism of the theorem for $p = n-1$. It also implies that $H_n(L_*)$ is the cokernel of the stabilization map for $p = n$.

Second, by changing $n$ to $n+1$, it shows that the image of

$$(I_{n,n+1} \otimes A^{\otimes n+1})_{H_{n+1}} \text{ in } (I_{n,n} \otimes A^{\otimes n})_{H_n} \text{ by } d$$

is the same as the image Im of

$$(I_{n+1,n+1} \otimes A^{\otimes n+1})_{H_{n+1}}.$$

From this we deduce by inspection of diagram (7.2) that $H_n(L_*)$ is the quotient of $\Lambda^n A^-$ by Im.

In order to determine the subspace Im of $\Lambda^n A^-$ we compute the image of the primitive part of

$$(I_{n+1,n+1} \otimes A^{\otimes n+1})_{H_{n+1}} \text{ by } \pi \circ d.$$

From the proof of Theorem 5.5 we know that this is the image of $\pi' \circ b$ with

$$b: (A^{\otimes n+1})_{D_n} \to (A^{\otimes n})_{D_n} \qquad \text{and} \qquad \pi': (A^{\otimes n})_{D_n} \to (A^{\otimes n})_{H_n}.$$

An element $a = (a_0, ..., a_n)$ of $A^{\otimes n+1}$ is called homogeneous if for all $i$ the element $a_i$ is in $A^+$ or in $A^-$. On homogeneous elements $\pi' \circ b$ is trivial unless all $a_j$ are in $A^-$ but for one index, let say $i$, for which $a_i \in A^+$. On this element we have

$$\begin{aligned}\pi' \circ b(a_0^-, \ldots, a_{i-1}^-, a_i^+, a_{i+1}^-, \ldots, a_n^-) &= (-1)^{i-1} a_0^- \wedge \cdots \wedge a_{i-1}^- a_i^+ \wedge a_{i+1}^- \wedge \cdots \wedge a_n^- \\ &\quad + (-1)^i a_0^- \wedge \cdots \wedge a_{i-1}^- \wedge a_i^+ a_{i+1}^- \wedge \cdots \wedge a_n^- .\end{aligned}$$

Hence $\Lambda^n A^-/\mathrm{Im}(\pi' \circ b) = \Lambda^n_{A^+}(A^-/[A^+, A^-])$, where $A^-/[A^+, A^-]$ is viewed as an $A^+$-module.

The relations induced by the nonprimitive part of $(gl_{2n+2}(A)^{\otimes n+1})_{Sp(2n+2)}$ are of the form

$$\bar{b}(a) \wedge a' \wedge a'' \wedge \cdots \pm a \wedge \bar{b}(a') \wedge a'' \wedge \cdots \pm \cdots = 0,$$

and they are consequences of the relations coming from the primitive part. It is now immediate that $H_n(L_*) = \Lambda^n A^{-1}/\mathrm{Im} = \Lambda^n_{A^+}(A^-/[A^+, A^-])$, which proves the theorem.

7.4. Corollary. *If $A$ has trivial involution (and is therefore commutative) then, in the symplectic case, the stability homomorphism (cf. Theorem 7.1) is an isomorphism for $p \leqslant n$ and a surjection for $p = n+1$.*

## References


[F-T] B. L. Feigin and B. L. Tsygan, Additive in *K*-theory, *in* "Arithmetic and Geometry," pp. 67–209, Lecture Notes in Mathematics, Vol. 1289, Springer-Verlag, New York/Berlin, 1987.

[K] J.-L. Koszul, Homologie et cohomologie des algèbres de Lie, *Bull. Soc. Math. France* **78** (1950), 65–127.

[L] J-L. Loday, Homologies diédrale et quaternionique, *Adv. in Math.* **66** (1987), 119–148.

[L-Q] J-L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, *Comm. Math. Helv.* **59** (1984), 565–591.

[P1] C. Procesi, The invariant theory of $n \times n$-matrices, *Adv. in Math.* **19** (1976), 306–381.

[P2] C. Procesi, Trace identities and standard diagrams, *in* "Ring Theory" (Proceedings, Antwerp Conference, 1978), pp. 191–218, Lecture Notes in Pure and Applied Mathematics, Dekker, New York, 1979.

[W] H. Weyl, "The Classical Groups," Princeton Univ. Press, Princeton, NJ, 1946.