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# The diagonal of the Stasheff polytope

Jean-Louis Loday

Institut de Recherche Mathématique Avancée  
CNRS et Université Louis Pasteur  
7 rue R. Descartes  
67084 Strasbourg Cedex, France  
loday@math.u-strasbg.fr

*To Murray Gerstenhaber and Jim Stasheff*

**Summary.** We construct an  $A$ -infinity structure on the tensor product of two  $A$ -infinity algebras by using the simplicial decomposition of the Stasheff polytope. The key point is the construction of an operad  $AA$ -infinity based on the simplicial Stasheff polytope. The operad  $AA$ -infinity admits a coassociative diagonal and the operad  $A$ -infinity is a retract by deformation of it. We compare these constructions with analogous constructions due to Sanedidze-Umble and Markl-Shnider based on the Boardman-Vogt cubical decomposition of the Stasheff polytope.

**Key words:** Stasheff polytope, associahedron, operad, bar-cobar construction, cobar construction,  $A$ -infinity algebra,  $AA$ -infinity algebra, diagonal.

*2010 Mathematics Subject Classification.* 18D50, 55U35, 52B11

## Introduction

An associative algebra up to homotopy, or  $A_\infty$ -algebra, is a chain complex  $(A, d_A)$  equipped with an  $n$ -ary operation  $\mu_n$  for each  $n \geq 2$  verifying  $\mu \circ \mu = 0$ . See [15], or, for instance, [5]. Here we put

$$\mu := d_A + \mu_2 + \mu_3 + \cdots : T(A) \rightarrow T(A),$$

where  $\mu_n$  has been extended to the tensor coalgebra  $T(A)$  by coderivation. In particular  $\mu_2$  is not associative, but only associative up to homotopy in the following sense:

$$\mu_2 \circ (\mu_2 \otimes \text{id}) - \mu_2 \circ (\text{id} \otimes \mu_2) = d_A \circ \mu_3 + \mu_3 \circ d_{A^{\otimes 3}} .$$

Putting an  $A_\infty$ -algebra structure on the tensor product of two  $A_\infty$ -algebras is a long standing problem, cf. for instance [12, 2]. Recently a solution has

been constructed by Saneblidze and Umble, cf. [13, 14], by providing a diagonal  $A_\infty \rightarrow A_\infty \otimes A_\infty$  on the operad  $A_\infty$  which governs the  $A_\infty$ -algebras. Recall that, over a field, the operad  $A_\infty$  is the minimal model of the operad  $As$  governing the associative algebras. The differential graded module  $(A_\infty)_n$  of the  $n$ -ary operations is the chain complex of the Stasheff polytope. The method of Saneblidze and Umble consists in providing an explicit (i.e. combinatorial) diagonal of the Stasheff polytope considered as a cellular complex. In [11] Markl and Shnider give a construction of the Saneblidze-Umble diagonal by using the Boardman-Vogt model of  $As$ . This model is the bar-cobar construction on  $As$ , denoted  $\Omega BAs$ , in the operadic framework. It turns out that there exists a coassociative diagonal on  $\Omega BAs$ , which is constructed out of the diagonal of the cube. This diagonal, together with the quasi-isomorphisms  $q : A_\infty \rightarrow \Omega BAs$  and  $p : \Omega BAs \rightarrow A_\infty$  permit them to construct a diagonal on  $A_\infty$  by composition:

$$A_\infty \xrightarrow{q} \Omega BAs \rightarrow \Omega BAs \otimes \Omega BAs \xrightarrow{p \otimes p} A_\infty \otimes A_\infty .$$

The aim of this paper is to give an alternative solution to the diagonal problem by relying on the *simplicial decomposition of the Stasheff polytope* described in [8] and using the diagonal of the standard simplex. It leads to a new model  $AA_\infty$  of the operad  $As$ , whose dg module  $(AA_\infty)_n$  is the chain complex of a simplicial decomposition of the Stasheff polytope. Because of its simplicial nature, the operad  $AA_\infty$  has a coassociative diagonal (by means of the Alexander-Whitney map) and therefore we get a diagonal on  $A_\infty$  by composition:

$$A_\infty \xrightarrow{q'} AA_\infty \rightarrow AA_\infty \otimes AA_\infty \xrightarrow{p' \otimes p'} A_\infty \otimes A_\infty .$$

The map  $q' : A_\infty \rightarrow AA_\infty$  is induced by the simplicial decomposition of the associahedron. The map  $p' : AA_\infty \rightarrow A_\infty$  is slightly more involved to construct. It is induced by the deformation of the “main simplex” of the associahedron into the big cell of the associahedron. Here the main simplex is defined by the shortest path in the Tamari poset structure of the planar binary trees.

We compute the diagonal map on  $(A_\infty)_n$  up to  $n = 5$  and we find the same result as the Saneblidze-Umble diagonal. So it is reasonable to conjecture that they coincide.

In the last part we give a similar interpretation of the map  $p : \Omega BAs \rightarrow A_\infty$  constructed in [11] and giving rise to the Saneblidze-Umble diagonal. It is induced by the deformation of the “main cube” into the big cell.

**Acknowledgement** I thank Bruno Vallette for illuminating discussions on the algebras up to homotopy and Samson Saneblidze for sharing his drawings with me some years ago. Thanks to Emily Burgunder, Martin Markl, Samson

Saneblidze, Jim Stasheff and Ron Umble for their comments on the previous versions of this paper. I warmly thank the referee for his careful reading and his precious comments which helped me to improve this text.

This work is partially supported by the French agency ANR.

## 1 Stasheff polytope (associahedron)

We recall briefly the construction of the Stasheff polytope, also called associahedron, and its simplicial realization, which is the key tool of this paper. All chain complexes in this paper are made of free modules over a commutative ring  $\mathbb{K}$  (which can be  $\mathbb{Z}$  or a field).

### 1.1 Planar binary trees

We denote by  $PBT_n$  the set of *planar binary trees* having  $n$  leaves:

$$PBT_1 := \{ \{\} \}, PBT_2 := \{ \begin{array}{c} \diagup \diagdown \\ | \end{array} \}, PBT_3 := \{ \begin{array}{c} \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \end{array} \},$$

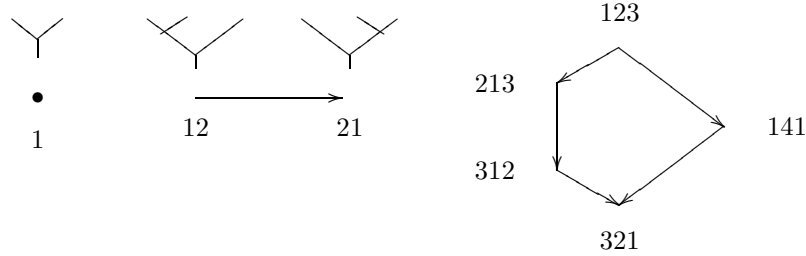
$$PBT_4 := \{ \begin{array}{c} \diagup \diagdown \\ | \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ | \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \diagup \diagdown \\ | \end{array} \}.$$

So  $t \in PBT_n$  has one root,  $n$  leaves,  $(n - 1)$  internal vertices,  $(n - 2)$  internal edges. Each vertex is binary (two inputs, one output). The number of elements in  $PBT_{n+1}$  is known to be the *Catalan number*  $c_n = \frac{(2n)!}{n!(n+1)!}$ . There is a partial order on  $PBT_n$ , called the *Tamari order*, defined as follows. On  $PBT_3$  it is given by

$$\begin{array}{c} \diagup \diagdown \\ | \end{array} \rightarrow \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \end{array}.$$

More generally, if  $t$  and  $s$  are two planar binary trees with the same number of leaves, there is a covering relation  $t \rightarrow s$  if and only if  $s$  can be obtained from  $t$  by replacing a local pattern like  $\begin{array}{c} \diagup \diagdown \\ | \end{array}$  by  $\begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \\ | \end{array}$ . In other words  $s$  is obtained from  $t$  by moving a leaf or an internal edge from left to right over a fork.

Examples:

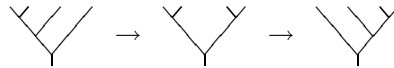


where the elements of  $PBT_4$  (listed above) are denoted 123, 213, 141, 312, 321, respectively. We recall from [7] how this way of indexing is obtained. First we label the leaves of a tree from left to right by  $0, 1, 2, \dots$ . Then we label the vertices by  $1, 2, \dots$  by saying that the label  $i$  vertex lies in between the leaves  $i - 1$  and  $i$  (drop a ball). To any binary tree  $t$  we associate a sequence of integers  $x_1 x_2 \dots x_{n-1}$  as follows:  $x_i = a_i b_i$  where  $a_i$  (resp.  $b_i$ ) is the number of leaves on the left (resp. right) side of the  $i$ th vertex.

### 1.2 Shortest path and long path

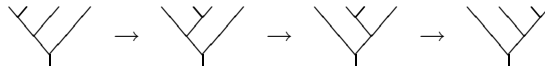
The Tamari poset admits an initial element: the left comb  $12 \dots (n - 1)$ , and a terminal element: the right comb  $(n - 1)(n - 2) \dots 1$ . There is a *shortest path* from the initial element to the terminal element. It is made of the trees which are the grafting of some left comb with a right comb. In  $PBT_n$  there are  $n - 1$  of them. This sequence of planar binary trees will play a significant role in the comparison of different cell realizations of the Stasheff polytope.

Example: the shortest path in  $PBT_4$ :

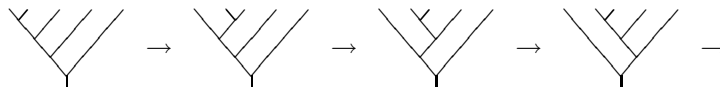


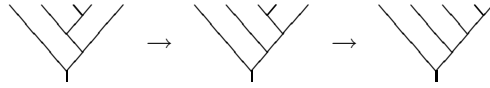
We also define “the long path” as follows. The *long path* from the left comb to the right comb is obtained by taking a covering relation at each step with the following rule: the vertex which is moved is the one with the smallest label (among the movable vertices, of course).

Examples:  $n = 2$



$n = 3$





Observe that there are (for  $n \geq 3$ ) other paths with the same length.

### 1.3 Planar trees

We now consider the planar trees for which an internal vertex has one root and  $k$  leaves, where  $k$  can be any integer greater than or equal to 2. We denote by  $PT_n$  the set of planar trees with  $n$  leaves:

$$PT_1 := \{\{\}\}, PT_2 := \{ \begin{array}{c} \diagup \\ \diagdown \end{array} \}, PT_3 := \{ \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \end{array}, \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \diagdown \end{array}, \begin{array}{c} \diagup \\ \diagdown \\ \diagup \end{array} \},$$

$$PT_4 := \{ \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \diagdown \end{array}, \dots, \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \diagdown \quad \diagup \\ \diagdown \end{array}, \dots, \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} \}.$$

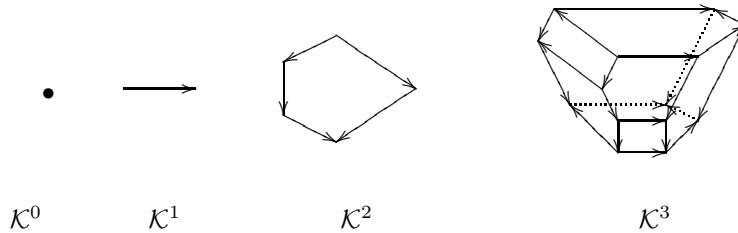
Each set  $PT_n$  is graded according to the number of internal vertices, i.e.  $PT_n = \bigcup_{p=1}^{p=n} PT_{n,p}$  where  $PT_{n,p}$  is the set of planar trees with  $n$  leaves and  $p$  internal vertices. For instance  $PT_{n,1}$  contains only one element which we call the  $n$ -*corolla* (the last element in the above sets). It is clear that  $PT_{n,n-1} = PBT_n$ .

We order the vertices of a planar tree by using the same procedure as for the planar binary trees.

### 1.4 The Stasheff polytope, alias associahedron

The *associahedron* is a cellular complex  $\mathcal{K}^n$  of dimension  $n$ , first constructed by Jim Stasheff [15], which can be realized as a convex polytope whose extremal vertices are in one-to-one correspondence with the planar binary trees in  $PBT_{n+2}$ . We showed in [7] that it is the convex hull of the points  $M(t) = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ , where the computation of the  $x_i$ 's has been recalled in 1.1. The edges of the polytope are indexed by the covering relations of the Tamari poset.

Examples:

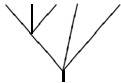


Its  $k$ -cells are in one-to-one correspondence with the planar trees in  $PT_{n+2, n+1-k}$ . For instance the 0-cells are indexed by the planar binary trees, and the top cell is indexed by the corolla.

It will prove helpful to adopt the notation  $\mathcal{K}^t$  to denote the cell in  $\mathcal{K}^n$  indexed by  $t \in PT_{n+2}$ . For instance, if  $t$  is the corolla, then  $\mathcal{K}^t = \mathcal{K}^n$ . As a space  $\mathcal{K}^t$  is the product of  $p$  associahedrons (or associahedra, as you like), where  $p$  is the number of internal vertices of  $t$ :

$$\mathcal{K}^t = \mathcal{K}^{i_1} \times \dots \times \mathcal{K}^{i_p}$$

where  $i_j + 2$  is the number of inputs of the  $j$ th internal vertex of  $t$ . For instance,

if  $t =$   , then  $\mathcal{K}^t = \mathcal{K}^1 \times \mathcal{K}^1$ .

The shortest path and the long path defined combinatorially in 1.1 give rise to concrete paths on the associahedron.

To the cellular complex  $\mathcal{K}^n$  we associate its chain complex  $C_*(\mathcal{K}^n)$ . The module of  $k$ -chains admits the set of trees  $PT_{n+2, n+1-k}$  as a basis:

$$C_k(\mathcal{K}^n) = \mathbb{K}[PT_{n+2, n+1-k}].$$

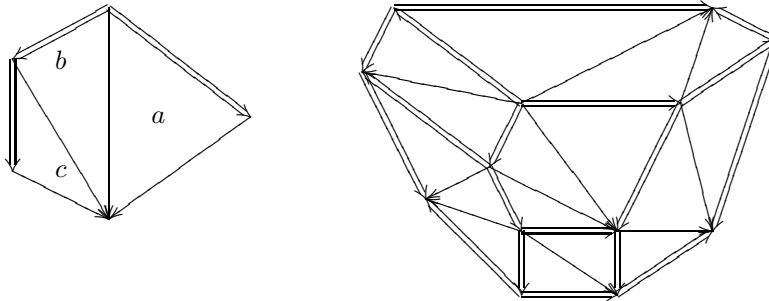
In particular  $C_0(\mathcal{K}^n) = \mathbb{K}[PBT_{n+2}]$  and  $C_n(\mathcal{K}^n) = \mathbb{K} t_{n+2}$  where  $t_{n+2}$  is the corolla.

### 1.5 The simplicial associahedron

In [8] we constructed a simplicial set  $\mathcal{K}_{\text{simp}}^n$  whose geometric realization gives a simplicial decomposition of the associahedron. In other words the associahedron  $\mathcal{K}^n$  is viewed as a union of  $n$ -simplices (there are  $(n + 1)^{n-1}$  of them). This simplicial decomposition is constructed inductively as follows. We fatten the simplicial set  $\mathcal{K}_{\text{simp}}^{n-1}$  into a new simplicial set  $\text{fat}\mathcal{K}_{\text{simp}}^{n-1}$ , cf. [8]. Then  $\mathcal{K}_{\text{simp}}^n$  is defined as the cone over  $\text{fat}\mathcal{K}_{\text{simp}}^{n-1}$  (as in the original construction of Stasheff [15]).

For  $n = 1$ , we have  $\mathcal{K}_{\text{simp}}^1 = \mathcal{K}^1 = [0, 1]$  (the interval).

Examples:  $\mathcal{K}_{\text{simp}}^2$  and  $\text{fat}\mathcal{K}_{\text{simp}}^3$



Since, in the process of fattening, the new cells are products of smaller dimensional associahedrons we get the following main property.

**Proposition 1.6** *The simplicial decomposition of a face  $\mathcal{K}^{i_1} \times \dots \times \mathcal{K}^{i_k}$  of  $\mathcal{K}^n$  is the product of the simplicializations of each component  $\mathcal{K}^{i_j}$ .*

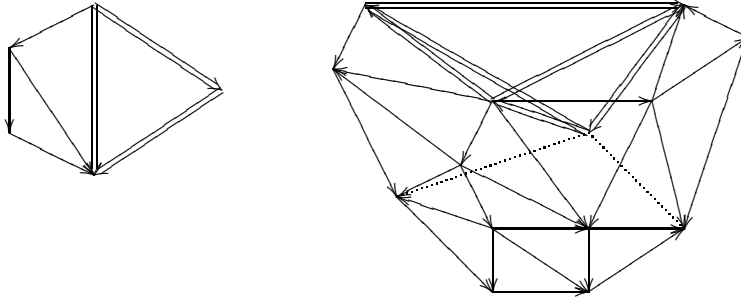
*Proof.* It is immediate from the inductive procedure which constructs  $\mathcal{K}^n$  out of  $\mathcal{K}^{n-1}$ . □

Considered as a cellular complex, still denoted  $\mathcal{K}_{\text{simp}}^n$ , the simplicialized associahedron gives rise to a chain complex denoted  $C_*(\mathcal{K}_{\text{simp}}^n)$ . This chain complex is the normalized chain complex of the simplicial set. It is the quotient of the chain complex associated to the simplicial set, divided out by the degenerate simplices (cf. for instance [9] Chapter VIII). A basis of  $C_0(\mathcal{K}_{\text{simp}}^n)$  is given by  $PBT_{n+2}$  and a basis of  $C_n(\mathcal{K}_{\text{simp}}^n)$  is given by the  $(n + 1)^{n-1}$  top simplices (in bijection with the parking functions, cf. [8]). It is zero higher up.

In the sequel “a simplex of  $\mathcal{K}_{\text{simp}}^n$ ” always mean a nondegenerate simplex of  $\mathcal{K}_{\text{simp}}^n$ .

Among the top simplices there is a particular one which we call the *main simplex*. Its vertices are indexed by the planar binary trees which are part of the shortest path constructed in 1.1 (observe that the shortest path has  $n + 1$  vertices).

Examples (the main simplex is highlighted):



## 2 The operad $AA_\infty$

We construct the operad  $AA_\infty$  and we construct a diagonal on it. A morphism from the operad  $A_\infty$  governing the associative algebras up to homotopy to the operad  $AA_\infty$  is deduced from the simplicial structure of the associahedron.

### 2.1 Differential graded non-symmetric operad [10]

By definition a *differential graded non-symmetric operad*, dgns operad for short, is a family of chain complexes  $\mathcal{P}_n = (\mathcal{P}_n, d)$  equipped with chain complex morphisms

$$\gamma_{i_1 \dots i_n} : \mathcal{P}_n \otimes \mathcal{P}_{i_1} \otimes \dots \otimes \mathcal{P}_{i_n} \rightarrow \mathcal{P}_{i_1 + \dots + i_n},$$

which satisfy the following associativity property. Let  $\mathcal{P}$  be the endofunctor of the category of chain complexes over  $\mathbb{K}$  defined by  $\mathcal{P}(V) := \bigoplus_n \mathcal{P}_n \otimes V^{\otimes n}$ . The maps  $\gamma_{i_1 \dots i_n}$  give rise to a transformation of functors  $\gamma : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$ . This transformation of functors  $\gamma$  is supposed to be associative. Moreover we suppose that  $\mathcal{P}_0 = 0, \mathcal{P}_1 = \mathbb{K}$  (trivial chain complex concentrated in degree 0). The transformation of functors  $\text{Id} \rightarrow \mathcal{P}$  determined by  $\mathcal{P}_1$  is supposed to be a unit for  $\gamma$ . So we can denote by  $\text{id}$  the generator of  $\mathcal{P}_1$ . Since  $\mathcal{P}_n$  is a graded module,  $\mathcal{P}$  is bigraded. The integer  $n$  is called the ‘‘arity’’ in order to differentiate it from the degree of the chain complex.

## 2.2 The fundamental example $A_\infty$

The operad  $A_\infty$  is a dgns operad constructed as follows:

$$A_{\infty, n} := C_*(\mathcal{K}^{n-2}) \text{ (chain complex of the cellular space } \mathcal{K}^{n-2}\text{)}.$$

Let us denote by  $As^i$  the family of one dimensional modules  $(As^i_n)_{n \geq 1}$  generated by the corollas (unique top cells). It is easy to check that there is a natural identification of graded (by arity) modules  $A_\infty = \mathcal{T}(As^i)$ , where  $\mathcal{T}(As^i)$  is the free ns operad over  $As^i$ . This identification is given by grafting on the leaves as follows. Given trees  $t, t_1, \dots, t_n$  where  $t$  has  $n$  leaves, the tree  $\gamma(t; t_1, \dots, t_n)$  is obtained by identifying the  $i$ th leaf of  $t$  with the root of  $t_i$ . For instance:

$$\gamma\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{Y} \end{array}; \begin{array}{c} \diagup \quad \diagdown \\ \text{Y} \end{array}, \begin{array}{c} \diagup \quad \diagdown \\ \text{Y} \end{array}\right) = \begin{array}{c} \diagup \quad \diagdown \\ \text{Y} \end{array}.$$

Moreover, under this identification, the composition map  $\gamma$  is a chain map, therefore  $A_\infty$  is a dgns operad.

This construction is a particular example of the so-called ‘‘cobar construction’’  $\Omega$ , i.e.  $A_\infty = \Omega As^i$  where  $As^i$  is considered as the cooperad governing the coassociative coalgebras (cf. [10]).

For any chain complex  $A$  there is a well-defined dgns operad  $\text{End}(A)$  given by  $\text{End}(A)_n = \text{Hom}(A^{\otimes n}, A)$ . An  $A_\infty$ -algebra is nothing but a morphism of operads  $A_\infty \rightarrow \text{End}(A)$ . The image of the corolla under this isomorphism is the  $n$ -ary operation  $\mu_n$  alluded to in the introduction.

## 2.3 Hadamard product of operads, operadic diagonal

Given two operads  $\mathcal{P}$  and  $\mathcal{Q}$ , their Hadamard product, also called tensor product, is the operad  $\mathcal{P} \otimes \mathcal{Q}$  defined as  $(\mathcal{P} \otimes \mathcal{Q})_n := \mathcal{P}_n \otimes \mathcal{Q}_n$ . The composition map is simply the tensor product of the two composition maps.

A diagonal on a nonsymmetric operad  $\mathcal{P}$  is a morphism of operads  $\Delta : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P}$ , which is compatible with the unit. Explicitly it is given by

chain complex morphisms  $\Delta : \mathcal{P}_n \rightarrow \mathcal{P}_n \otimes \mathcal{P}_n$  which commute with the composition in  $\mathcal{P}$ . In other words, the following diagram, where  $m := i_1 + \dots + i_n$ , is commutative:

$$\begin{array}{ccccc}
 \mathcal{P}_n \otimes \mathcal{P}_{i_1} \otimes \dots & \xrightarrow{\gamma} & \mathcal{P}_m & \xrightarrow{\Delta} & \mathcal{P}_m \otimes \mathcal{P}_m \\
 \downarrow \Delta \otimes \Delta \otimes \dots & & & & \uparrow \gamma \otimes \gamma \\
 (\mathcal{P}_n \otimes \mathcal{P}_n) \otimes (\mathcal{P}_{i_1} \otimes \mathcal{P}_{i_1}) \otimes \dots & \xrightarrow{\cong} & (\mathcal{P}_n \otimes \mathcal{P}_{i_1} \otimes \dots) \otimes (\mathcal{P}_n \otimes \mathcal{P}_{i_1} \otimes \dots) & & 
 \end{array}$$

We do not ask for  $\Delta$  to be coassociative.

### 2.4 Tensor product of $A_\infty$ -algebras

It is a long-standing problem to decide if, given two  $A_\infty$ -algebras  $A$  and  $B$ , there is a natural  $A_\infty$ -structure on their tensor product  $A \otimes B$  which extends the natural dg nonassociative algebra structure, cf. [12, 2]. It amounts to construct a diagonal on  $A_\infty$ , i.e. an operad morphism  $\Delta : A_\infty \rightarrow A_\infty \otimes A_\infty$ , since, by composition, we get an  $A_\infty$ -structure on  $A \otimes B$ :

$$A_\infty \rightarrow A_\infty \otimes A_\infty \rightarrow \text{End}(A) \otimes \text{End}(B) \rightarrow \text{End}(A \otimes B) .$$

Let us recall that the classical associative structure on the tensor product of two associative algebras can be interpreted operadically as follows. There is a diagonal on the operad  $As$  given by

$$As_n \rightarrow As_n \otimes As_n, \quad m_n \mapsto m_n \otimes m_n ,$$

where  $m_n$  is the standard  $n$ -ary operation in the associative framework. Since we want the diagonal  $\Delta$  on  $A_\infty$  to be compatible with the diagonal on  $As$  ( $\mu_2 \mapsto m_2$ ), there is no choice in arity 2, and we have  $\Delta(\mu_2) = \mu_2 \otimes \mu_2$ . Observe that these two elements are in degree 0. In arity 3, since  $\mu_3$  is of degree 1 and  $\mu_3 \otimes \mu_3$  of degree 2, this last element cannot be the answer. In fact there is already a choice (parameter  $a$ ) for a solution:

$$\begin{aligned}
 \Delta(\text{trivalent tree}) &= a \left( \text{tree}_1 \otimes \text{tree}_2 + \text{tree}_3 \otimes \text{tree}_4 \right) \\
 &+ (1-a) \left( \text{tree}_5 \otimes \text{tree}_6 + \text{tree}_7 \otimes \text{tree}_8 \right) .
 \end{aligned}$$

By some tour de force Samson Sanedlidze and Ron Umble constructed such a diagonal on  $A_\infty$  in [13]. Their construction was re-interpreted in [11] by Markl and Shnider through the Boardman-Vogt construction (see section 4 below for a brief account of their work). We will use the simplicialization of the associahedron described in [8] to give a solution to the diagonal problem.

## 2.5 Construction of the operad $AA_\infty$

We define the dgns operad  $AA_\infty$  as follows. The chain complex  $AA_{\infty,n}$  is the chain complex of the simplicialization of the associahedron considered as a cellular complex (cf. 1.5):

$$AA_{\infty,n} := C_*(\mathcal{K}_{\text{simp}}^{n-2}).$$

In low dimension we take  $AA_{\infty,0} = 0$ ,  $AA_{\infty,1} = \mathbb{K} \text{id}$ . So a basis of  $AA_{\infty,n}$  is made of the (nondegenerate) simplices of  $\mathcal{K}_{\text{simp}}^{n-2}$ . Let us now construct the composition map

$$\gamma = \gamma^{AA_\infty} : AA_{\infty,n} \otimes AA_{\infty,i_1} \otimes \cdots \otimes AA_{\infty,i_n} \rightarrow AA_{\infty,i_1+\cdots+i_n}.$$

We denote by  $\Delta^k$  the standard  $k$ -simplex. Let  $\iota : \Delta^k \hookrightarrow \mathcal{K}_{\text{simp}}^{n-2}$  be a cell, i.e. a linear generator of  $C_k(\mathcal{K}_{\text{simp}}^{n-2})$ . Given such cells

$$\iota_0 \in AA_{\infty,n}, \iota_1 \in AA_{\infty,i_1}, \dots, \iota_n \in AA_{\infty,i_n}$$

we construct their image  $\gamma(\iota_0; \iota_1, \dots, \iota_n) \in AA_{\infty,m}$ , where  $m := i_1 + \cdots + i_n$  as follows. We denote by  $k_i$  the dimension of the cell  $\iota_i$ .

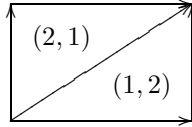
Let  $t_n$  be the  $n$ -corolla in  $PT_n$  and let  $s := \gamma(t_n; t_{i_1}, \dots, t_{i_n}) \in PT_m$  be the grafting of the trees  $t_{i_1}, \dots, t_{i_n}$  on the leaves of  $t_n$ . As noted before this is the composition in the operad  $A_\infty$ . The tree  $s$  indexes a cell  $\mathcal{K}^s$  of the space  $\mathcal{K}^{m-2}$ , which is combinatorially homeomorphic to  $\mathcal{K}^{n-2} \times \mathcal{K}^{i_1-2} \times \cdots \times \mathcal{K}^{i_n-2}$ . In other words it determines a map

$$s_* : \mathcal{K}^{n-2} \times \mathcal{K}^{i_1-2} \times \cdots \times \mathcal{K}^{i_n-2} = \mathcal{K}^s \hookrightarrow \mathcal{K}^{m-2}.$$

The product of the inclusions  $\iota_j, j = 0, \dots, n$ , defines a map

$$\iota_0 \times \iota_1 \times \cdots \times \iota_n : \Delta^{k_0} \times \Delta^{k_1} \times \cdots \times \Delta^{k_n} \hookrightarrow \mathcal{K}^{n-2} \times \mathcal{K}^{i_1-2} \times \cdots \times \mathcal{K}^{i_n-2}.$$

Let us recall that a product of standard simplices can be decomposed into the union of standard simplices. These pieces are indexed by the multi-shuffles  $\alpha$ . Example:  $\Delta^1 \times \Delta^1 = \Delta^2 \cup \Delta^2$ :



So, for any multi-shuffle  $\alpha$  there is a map

$$f_\alpha : \Delta^l \rightarrow \Delta^{k_0} \times \Delta^{k_1} \times \cdots \times \Delta^{k_n},$$

where  $l = k_0 + \cdots + k_n$ . By composition of maps we get

$$s_* \circ (\iota_0 \times \cdots \times \iota_n) \circ f_\alpha : \Delta^l \rightarrow \mathcal{K}^{m-2}$$

which is a linear generator of  $C_l(\mathcal{K}_{\text{simp}}^{m-2})$  by construction of the triangulation of the associahedron, cf. [7]. By definition  $\gamma(\iota_0; \iota_1, \dots, \iota_n)$  is the algebraic sum of the cells  $s_* \circ (\iota_0 \times \cdots \times \iota_n) \circ f_\alpha$  over the multi-shuffles.

**Proposition 2.6** *The graded chain complex  $AA_\infty$  and  $\gamma$  constructed above define a dgns operad, denoted  $AA_\infty$ . The operad  $AA_\infty$  is a model of the operad  $As$ .*

*Proof.* We need to prove associativity for  $\gamma$ . It is an immediate consequence of the associativity for the composition of trees (operadic structure of  $A_\infty$ ) and the associativity property for the decomposition of the product of simplices into simplices.

Since the associahedron is contractible, taking the homology gives a graded linear map  $C_*(\mathcal{K}_{\text{simp}}^{n-2}) \rightarrow \mathbb{K} m_n$ , where  $m_n$  is in degree 0. This map sends any planar binary tree having  $n$  leaves to  $m_n$ , and obviously induces an isomorphism on homology. These maps assemble into a dgns operad morphism  $AA_\infty \rightarrow As$ . Since it is a quasi-isomorphism,  $AA_\infty$  is a resolution of  $As$ , that is a model of  $As$  in the category of dgns operads.  $\square$

**Proposition 2.7** *The operad  $AA_\infty$  admits a coassociative diagonal.*

*Proof.* This diagonal  $\Delta : AA_\infty \rightarrow AA_\infty \otimes AA_\infty$  is determined by its value in arity  $n$  for all  $n$ , that is a chain complex morphism

$$C_*(\mathcal{K}_{\text{simp}}^{n-2}) \rightarrow C_*(\mathcal{K}_{\text{simp}}^{n-2}) \otimes C_*(\mathcal{K}_{\text{simp}}^{n-2}).$$

This morphism is defined as the composite

$$C_*(\mathcal{K}_{\text{simp}}^{n-2}) \xrightarrow{\Delta_*} C_*(\mathcal{K}_{\text{simp}}^{n-2} \times \mathcal{K}_{\text{simp}}^{n-2}) \xrightarrow{AW} C_*(\mathcal{K}_{\text{simp}}^{n-2}) \otimes C_*(\mathcal{K}_{\text{simp}}^{n-2}),$$

where  $\Delta_*$  is induced by the diagonal on the simplicial set, and where  $AW$  is the Alexander-Whitney map. Observe that under the identification

$$C_*(\mathcal{K}_{\text{simp}}^{n-2}) \otimes C_*(\mathcal{K}_{\text{simp}}^{n-2}) = C_*(\mathcal{K}_{\text{simp}}^{n-2} \times \mathcal{K}_{\text{simp}}^{n-2})$$

the composite morphism maps a  $k$ -simplex into a  $2k$ -simplex. Let us recall from [9], Chapter VIII, the construction of the  $AW$  map. Denote by  $d_0, \dots, d_k$  the face operators of the simplicial set. If  $x$  is a simplex of dimension  $k$ , then we define  $d_{\max}(x) := d_k(x)$ . So, for instance  $(d_{\max})^2(x) = d_{k-1}d_k(x)$ . By definition the  $AW$  map on  $C_k$  is given by

$$(x, y) \mapsto \sum_{i=0}^k ((d_{\max})^{k-i}(x), (d_0)^i(y)).$$

We need to check that the diagonal is compatible with the operad structure, that is, the diagram of 2.3 for  $\mathcal{P} = AA_\infty$  is commutative. First we remark that the diagonal of a product of standard simplices satisfies the following commutativity property:

$$\begin{array}{ccc} C_*(\Delta^k \times \Delta^l) & \xrightarrow{\Delta} & C_*(\Delta^k \times \Delta^l) \otimes C_*(\Delta^k \times \Delta^l) \\ \downarrow \Delta \otimes \Delta & & \downarrow = \\ C_*(\Delta^k \times \Delta^k) \otimes C_*(\Delta^l \times \Delta^l) & \xrightarrow{\cong} & C_*(\Delta^k \times \Delta^l \times \Delta^k \times \Delta^l) \end{array}$$

where the isomorphism  $\cong$  involves the switching isomorphism  $V \otimes V' \cong V' \otimes V$ . A similar property holds for a finite product of simplices. Starting with a linear generator

$$(\iota; \iota_1, \dots, \iota_n) \in AA_{\infty, n} \otimes AA_{\infty, i_1} \otimes \cdots \otimes AA_{\infty, i_n}$$

we see that  $\Delta(\gamma(\iota; \iota_1, \dots, \iota_n))$  is made of diagonals of products of simplices. Applying the preceding result we can rewrite this element as the composite of diagonals of simplices. Hence we get

$$\Delta(\gamma(\iota; \iota_1, \dots, \iota_n)) = \gamma(\Delta(\iota); \Delta(\iota_1), \dots, \Delta(\iota_n))$$

as expected.

The coassociativity property follows from the coassociativity property of the Alexander-Whitney map.  $\square$

## 2.8 Comparing $A_\infty$ to $AA_\infty$

Since  $\mathcal{K}_{\text{simp}}^n$  is a decomposition of  $\mathcal{K}^n$ , there is a chain complex map

$$q' : C_*(\mathcal{K}^n) \rightarrow C_*(\mathcal{K}_{\text{simp}}^n),$$

where a cell of  $\mathcal{K}^n$  is sent to the algebraic sum of the simplices it is made of.

**Proposition 2.9** *The map  $q' : A_\infty \rightarrow AA_\infty$  induced by the maps  $q' : C_*(\mathcal{K}^n) \rightarrow C_*(\mathcal{K}_{\text{simp}}^n)$  is a quasi-isomorphism of dgns operads.*

*Proof.* It is sufficient to prove that the maps  $q'$  on the chain complexes are compatible with the operadic composition:

$$q'(\gamma^{As}(t; t_1, \dots, t_n)) = \gamma^{AA_\infty}(q'(t); q'(t_1), \dots, q'(t_n)).$$

This equality follows from the definition of  $\gamma^{AA_\infty}$  given in 2.5 and Proposition 1.6.  $\square$

Moreover we have commutative diagrams:

$$\begin{array}{ccc} C_*(\mathcal{K}^{n-2}) & \xrightarrow{q'} & C_*(\mathcal{K}_{\text{simp}}^{n-2}) \\ & \searrow H_* & \swarrow H_* \\ & & \mathbb{K}\mu_n \end{array} \quad \begin{array}{ccc} A_\infty & \xrightarrow{q'} & AA_\infty \\ & \searrow H_* & \swarrow H_* \\ & & As \end{array}$$

## 3 From $AA_\infty$ to $A_\infty$

The aim of this section is to construct a quasi-inverse to  $q'$ , that is a quasi-isomorphism of dgns operads  $p' : AA_\infty \rightarrow A_\infty$ . We first construct chain maps  $p' : C_*(\mathcal{K}_{\text{simp}}^n) \rightarrow C_*(\mathcal{K}^n)$  by using a deformation of the main simplex to the top cell of the associahedron. This is obtained by an inflating process that we first describe on the cube and on the product of simplices.

### 3.1 Deformation of the cube

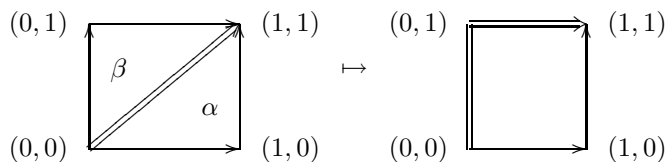
The cube  $I^n$  is a polytope whose vertices are indexed by  $(x_1, \dots, x_n)$ , where  $x_i = 0$  or 1. The long path in  $I^n$  is, by definition, the path

$$(0, \dots, 0, 0) \rightarrow (0, \dots, 0, 1) \rightarrow (0, \dots, 1, 1) \rightarrow \dots \rightarrow (1, \dots, 1, 1).$$

The cube is a cell complex which can be decomposed into  $n!$  top simplices, i.e. viewed as the realization of a simplicial set  $I^n_{\text{simp}}$ . The simplex which corresponds to the identity permutation is called the main simplex of the cube. Let us describe the deformation from the main simplex to the cube, which gives rise to a chain map

$$p' : C_*(I^n_{\text{simp}}) \rightarrow C_*(I^n).$$

We work by induction on  $n$ . In  $I^2_{\text{simp}}$  the main simplex, denoted  $\alpha$ , is deformed to the square by pushing the diagonal to the long path:

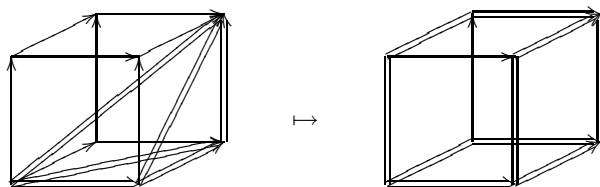


So  $p'$  is given by the identity on the boundary and by

$$\begin{aligned} ((0, 0), (1, 1)) &\mapsto ((0, 0), (0, 1)) + ((0, 1), (1, 1)), \\ \alpha &\mapsto I^2, \\ \beta &\mapsto 0, \end{aligned}$$

on the interior simplices. So, under this inflating process, the main simplex  $\alpha$  is mapped to the whole square and the other simplex  $\beta$  is flattened. More generally, the main simplex of  $I^n_{\text{simp}}$  is deformed into the top cell of  $I^n$  by sending the diagonal to the long path. The other edges of the main simplex are deformed according to the lower dimensional deformation.

Here is the example of the 3-dimensional cube:



### 3.2 Deformation of a product of simplices

Similarly we define a deformation of the product of simplices  $\Delta^r \times \Delta^s$  by inflating the main simplex as follows. Let us denote by  $\{\underline{0}, \dots, \underline{r}\}$  the vertices of  $\Delta^r$ . The *main simplex* of  $\Delta^r \times \Delta^s$  is chosen as being the simplex  $\Delta^{r+s}$  with vertices

$$(\underline{0}, \underline{0}), (\underline{1}, \underline{0}), \dots, (\underline{r}, \underline{0}), (\underline{r}, \underline{1}), \dots, (\underline{r}, \underline{s}), \dots$$

We deform the main simplex into the whole product by induction on  $s$ . So it suffices to give the image of the edge  $((\underline{0}, \underline{0}), (\underline{r}, \underline{s}))$ . We send it to the “long path” defined as

$$(\underline{0}, \underline{0}), (\underline{0}, \underline{1}), \dots, (\underline{0}, \underline{s}), (\underline{1}, \underline{s}), \dots, (\underline{r}, \underline{s}), \dots$$

Under this deformation the main simplex becomes the whole product and all the other simplices are flattened. This deformation defines a chain complex morphism

$$C_*((\Delta^r \times \Delta^s)_{\text{simp}}) \rightarrow C_*(\Delta^r \times \Delta^s)$$

where, on the right side,  $\Delta^r \times \Delta^s$  is considered as a cell complex with only one  $r + s$  cell.

Observe that some simplices may happen to be deformed into cells of various dimensions. For instance in  $\Delta^2 \times \Delta^1$  the triangle with vertices  $(\underline{0}, \underline{0}), (\underline{1}, \underline{0}), (\underline{2}, \underline{1})$  is deformed into the union of a square (with vertices  $(\underline{0}, \underline{0}), (\underline{1}, \underline{0}), (\underline{1}, \underline{1}), (\underline{2}, \underline{1})$ ) and an edge (with vertices  $(\underline{1}, \underline{1}), (\underline{2}, \underline{1})$ ). Its image under the chain morphism is the square.

### 3.3 Deformation of the associahedron

We construct a topological deformation of the simplicial associahedron by pushing the main simplex to the whole associahedron. All the other simplices are going to be flattened. This topological deformation will induce the chain map  $p'$  we are looking for. This inflating process is analogous to what we did for the cube and the product of simplices above. We work by induction on the dimension.

For  $n = 1$ , there is no deformation since  $\mathcal{K}_{\text{simp}}^1 = \mathcal{K}^1$ . For  $n = 2$  the deformation is the identity on the boundary and the only edge of the main simplex which is not on the boundary is “pushed” to the long path.

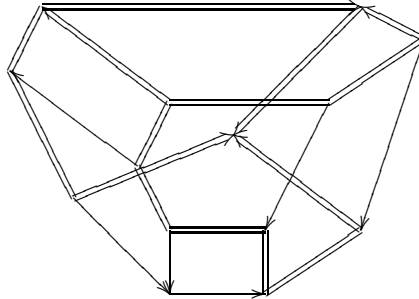


In the meantime, the other interior edge is pushed to the union of two boundary edges and the two other top simplices are flattened.

For higher  $n$  we use the inductive construction of  $\mathcal{K}_{\text{simp}}^n$  out of  $\text{fat}\mathcal{K}_{\text{simp}}^{n-1}$ . We suppose that the deformation is known for any  $i < n$  and we construct it on  $\text{fat}\mathcal{K}_{\text{simp}}^{n-1}$ . The simplicial set  $\text{fat}\mathcal{K}_{\text{simp}}^{n-1}$  is the union of the simplicial sets of the form  $\mathcal{K}_{\text{simp}}^t = \mathcal{K}_{\text{simp}}^i \times \mathcal{K}_{\text{simp}}^j$  indexed by some trees  $t$  with one and only one internal edge. The main simplex of this product is the main simplex  $\Delta^{i+j}$  of  $\Delta^i \times \Delta^j$  where  $\Delta^i$ , resp.  $\Delta^j$ , is the main simplex of  $\mathcal{K}_{\text{simp}}^i$ , resp.  $\mathcal{K}_{\text{simp}}^j$ . The deformation is obtained by, first, deforming the main simplex  $\Delta^{i+j}$  into  $\Delta^i \times \Delta^j$  as described in 3.2 and then use the inductive hypothesis (deformation from the main simplex to the associahedron).

The deformation of the interior cells is obtained by pushing the main simplex of  $\mathcal{K}_{\text{simp}}^n$  to the top cell. It is determined by the image of the edges of the main simplex. By induction, it suffices to construct the image of the edge which goes from the vertex indexed by the left comb (initial element) to the vertex indexed by the right comb (terminal element). We choose to deform it to the long path of the associahedron as constructed in 1.2. Since any simplex of  $\mathcal{K}_{\text{simp}}^n$  is either on the boundary, or is a cone (for the last vertex) over a simplex in the boundary, we are done. In particular, the edge going from a 0-simplex labelled by the tree  $t$  to the right comb is deformed into a path made of 1-cells of the associahedron, constructed with the same rule as in the construction of the long path.

The deformed tetrahedron:



### 3.4 The map $p' : C_*(\mathcal{K}_{\text{simp}}^n) \rightarrow C_*(\mathcal{K}^n)$

We define the map  $p'$  as follows. Under the deformation map any simplex of  $\mathcal{K}_{\text{simp}}^n$  is sent to the union of cells of  $\mathcal{K}^n$ . The image of such a simplex under  $p'$  is the algebraic sum of the cells of the same dimension in the union. For instance, the main simplex is sent to the top cell (indexed by the corolla), and all the other top simplices are sent to 0, since under the deformation they are flattened. From its topological nature it follows that  $p'$  is a chain complex morphism.

In low dimension we get the following. For  $n = 1$ , the map  $p'$  is the identity. For  $n = 2$ , the map  $p'$  is the identity on the 0-simplices and the 1-simplices of the boundary, and on the interior cells, we get:



Suppose that all the simplices are main simplices. Then  $p'(t_j) = t_{i_j}$  for all  $j$  and therefore  $\gamma^{A_\infty}(p'(t_0); p'(t_1), \dots, p'(t_n)) = \gamma^{A_\infty}(t_{i_0}; t_{i_1}, \dots, t_{i_n})$ . On the other hand  $\omega$  contains the main simplex, therefore  $p'(\omega) = \gamma^{A_\infty}(t_{i_0}; t_{i_1}, \dots, t_{i_n})$  and we are done.  $\square$

**Corollary 3.6** *The composite*

$$A_\infty \xrightarrow{q'} AA_\infty \xrightarrow{\Delta} AA_\infty \otimes AA_\infty \xrightarrow{p' \otimes p'} A_\infty \otimes A_\infty .$$

is a diagonal for the operad  $A_\infty$ .

*Proof.* It is immediate to check that this composite of dgns operad morphisms sends  $\mu_2$  to  $\mu_2 \otimes \mu_2$ , since  $\mu_2$  corresponds to the 0-cell of  $\mathcal{K}^0$ .  $\square$

**Proposition 3.7** *If  $A$  is an associative algebra and  $B$  an  $A_\infty$  algebra, then the  $A_\infty$ -structure on  $A \otimes B$  is given by*

$$\mu_n(a_1 \otimes b_1, \dots, a_n \otimes b_n) = a_1 \cdots a_n \otimes \mu_n(b_1, \dots, b_n).$$

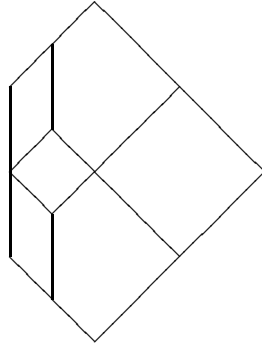
*Proof.* In the formula for  $\Delta$  we have  $\mu_n = 0$  for all  $n \geq 3$ , that is, any tree with a  $k$ -valent vertex for  $k \geq 3$  is 0 on the left side. Hence the only term which is left is *comb*  $\otimes$  *corolla*, whence the assertion.  $\square$

### 3.8 The first formulas

Let us give the explicit form of  $\Delta(\mu_n)$  for  $n = 2, 3, 4$ :

$$\begin{aligned} \Delta(\text{Y}) &= \text{Y} \otimes \text{Y} , \\ \Delta(\text{V}) &= \text{Y} \otimes \text{V} + \text{V} \otimes \text{Y} , \\ \Delta(\text{W}) &= \text{Y} \otimes \text{W} + \text{W} \otimes \text{Y} \\ &+ \text{V} \otimes \text{V} - \text{V} \otimes \text{Y} - \text{Y} \otimes \text{V} + \text{Y} \otimes \text{Y} . \end{aligned}$$

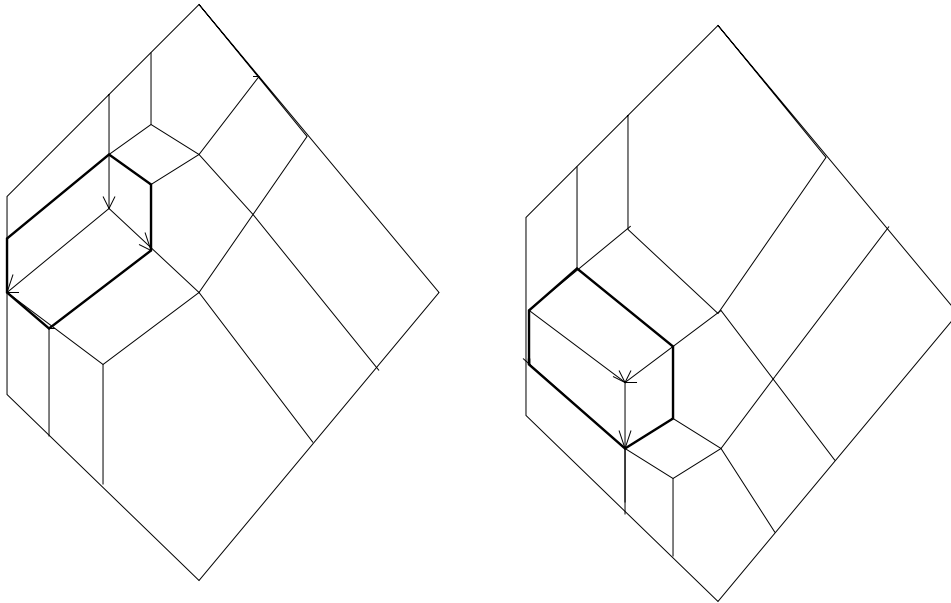
In this last formula the first three summands comes from the triangle (123, 141, 321), the next two summands come from the triangle (123, 213, 321) and the last summand comes from the last triangle (213, 312, 321). It is exactly the same formula as the one obtained by Sanedlidze and Umble (cf. [13] example 1, [11] exercise 12). Topologically the diagonal of the pentagon is approximated as a union of products of cells as follows:



Each cell of this decomposition corresponds to a summand of the above formula, which indicates where the cell goes in the product  $\mathcal{K}^2 \times \mathcal{K}^2$ .

### 3.9 On the non-coassociativity of the diagonal

Though the diagonal of  $AA_\infty$  that we constructed is coassociative, the diagonal of  $A_\infty$  is not. In fact it has been shown in [11] that there does not exist any coassociative diagonal on  $A_\infty$ . The obstruction to coassociativity can be seen topologically on the picture “Iterated diagonal”.



**Fig. 1.** Iterated diagonal

Both pictures are the same combinatorially, except for an hexagon (highlighted on the pictures), which is the union of 3 squares one way on the left and the other way on the right. This is the obstruction to coassociativity. Of course there is a way to reconcile these two decompositions via a homotopy which is given by the cube.

Exercise 1. Show that the image of this cube in  $\mathcal{K}^2 \times \mathcal{K}^2 \times \mathcal{K}^2$  is indexed

by  .

Exercise 2. Compare the five iterated diagonals of the next step (some nice pictures to draw).

## 4 Comparing the operads $AA_\infty$ and $\Omega BAs$

We first give a brief account of [11, 13] where a diagonal of the operad  $A_\infty$  is constructed by using a coassociative diagonal on the dgns operad  $\Omega BAs$ . Then we compare the two operads  $AA_\infty$  and  $\Omega BAs$ .

### 4.1 Cubical decomposition of the associahedron [1]

The associahedron can be decomposed into cubes as follows.

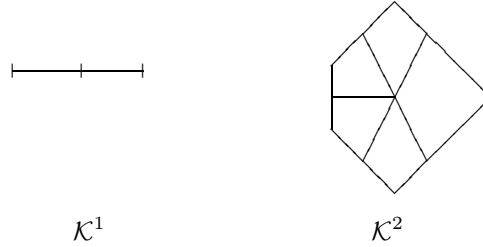
For each tree  $t \in PBT_{n+2}$  we take a copy of the cube  $I^n$  (where  $I = [0, 1]$  is the interval) which we denote by  $I_t^n$ . Then the associahedron  $\mathcal{K}^n$  is the quotient

$$\mathcal{K}^n := \bigsqcup_t I_t^n / \sim$$

where the equivalence relation is as follows. We think of an element  $\tau = (t; \lambda_1, \dots, \lambda_n) \in I_t^n$  as a tree of type  $t$  where the  $\lambda_i$ 's are the lengths of the internal edges. If some of the  $\lambda_i$ 's are 0, then the geometric tree determined by  $\tau$  is not binary anymore (since some of its internal edges have been shrunk to a point). We denote the new tree by  $\bar{\tau}$ . For instance, if none of the  $\lambda_i$ 's is zero, then  $\bar{\tau} = t$ ; if all the  $\lambda_i$ 's are zero, then the tree  $\bar{\tau}$  is the corolla (only one vertex). The equivalence relation  $\tau \sim \tau'$  is defined by the following two conditions:

- $\bar{\tau} = \bar{\tau}'$ ,
  - the lengths of the nonzero-length edges of  $\tau$  are the same as those of  $\tau'$ .
- Hence  $\mathcal{K}^n$  is obtained as a cubical realization denoted  $\mathcal{K}_{\text{cub}}^n$ .

Examples:



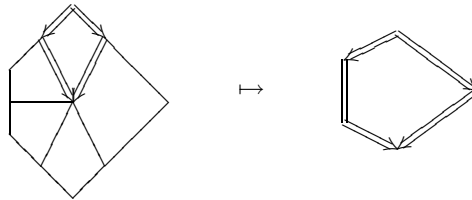
**4.2 Markl-Shnider version of Saneblidze-Umble diagonal [11, 13]**

In [1] Boardman and Vogt showed that the bar-cobar construction on the operad  $As$  is a dgns operad  $\Omega BAs$  whose chain complex in arity  $n$  can be identified with the chain complex of the cubical decomposition of the associahedron:

$$(\Omega BAs)_n = C_*(\mathcal{K}_{\text{cub}}^{n-2}) .$$

In [11] (where  $\mathcal{K}_{\text{cub}}^{n-2}$  is denoted  $W_n$  and  $\mathcal{K}^{n-2}$  is denoted  $K_n$ ) Markl and Shnider use this result to construct a coassociative diagonal on the operad  $\Omega BAs$ . There is a quasi-isomorphism  $q : A_\infty \rightarrow \Omega BAs$  induced by the cubical decomposition of the associahedron (the image of the top cell is the the algebraic sum of the  $c_{n-1}$  cubes). They construct an inverse quasi-isomorphism  $p : \Omega BAs \rightarrow A_\infty$  by giving explicit algebraic formulas. At the chain level the map  $p : C_*(\mathcal{K}_{\text{cub}}^n) \rightarrow C_*(\mathcal{K}^n)$  has a topological interpretation using a deformation of the cubical associahedron as follows. The cube indexed by the left comb is called the *main cube* of the decomposition. The deformation sends the main cube to the top cell of the associahedron and flatten all the other ones.

Example:



The exact way the main cube is deformed is best explained by drawing the associahedron on the cube. This is recalled in the Appendix. In [4] Kadeishvili and Saneblidze give a general method for constructing a diagonal on some polytopes admitting a cubical decomposition along the same principle (inflating the main cube).

Markl and Shnider claim that the composite

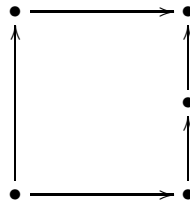
$$A_\infty \xrightarrow{q} \Omega BAs \rightarrow \Omega BAs \otimes \Omega BAs \xrightarrow{p \otimes p} A_\infty \otimes A_\infty$$

is the Saneblidze-Umble diagonal.

### 5 Appendix 1: Drawing a Stasheff polytope on a cube

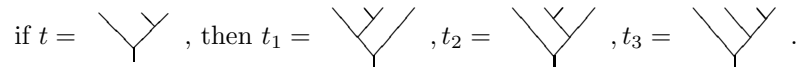
This is an account of some effort to construct the Stasheff polytope that I did in 2002 while visiting Northwestern University. During this visit I had the opportunity to meet Samson Saneblidze and Ron Umble, who were drawing the same kind of figures for different reasons (explained above). It makes the link between Markl and Shnider algebraic description of the map  $p$ , the pictures appearing in Saneblidze and Umble paper, and some algebraic properties of the planar binary trees.

There is a way of constructing an associahedron structure on a cube as follows. For  $n = 0$  and  $n = 1$  there is nothing to do since  $\mathcal{K}^0$  and  $\mathcal{K}^1$  are the cubes  $I^0$  and  $I^1$  respectively. For  $n = 2$ , we simply add one point in the middle of an edge to obtain a pentagon:

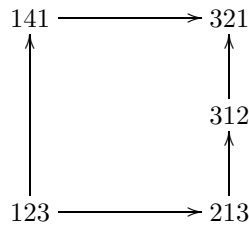


Inductively we draw  $\mathcal{K}^n$  on  $I^n$  out of the drawing of  $\mathcal{K}^{n-1}$  on  $I^{n-1}$  as follows. Any tree  $t \in PBT_{n+1}$  gives rise to an ordered sequence of trees  $(t_1, \dots, t_k)$  in  $PBT_{n+2}$  as follows. We consider the edges which are on the right side of  $t$ , including the root. The tree  $t_1$  is formed by adding a leaf which starts from the middle of the root and goes rightward (see [6] p. 297). The tree  $t_2$  is formed by adding a leaf which starts from the middle of the next edge and goes rightward. And so forth. Obviously  $k$  is the number of vertices lying on the right side of  $t$  plus one (so it is always greater than or equal to 2).

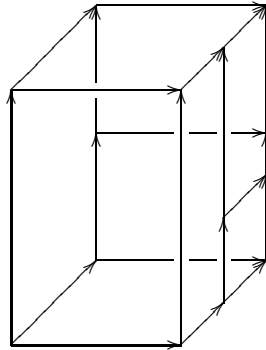
Example:



In  $I^n = I^{n-1} \times I$  we label the point  $\{t\} \times \{0\}$  by  $t_1$ , the point  $\{t\} \times \{1\}$  by  $t_k$ , and we introduce (in order) the points  $t_2, \dots, t_{k-1}$  on the edge  $\{t\} \times I$ . For  $n = 2$  we obtain (with the coding introduced in section 1.1):



For  $n = 3$  we obtain the following picture:



(It is a good exercise to draw the tree at each vertex). Compare with [13], p. 3). The case  $n = 4$  can be found on my home-page. It is important to observe that the order induced on the vertices by the canonical orientation of the cube coincides precisely with the Tamari poset structure. The referee informed me that these pictures already appeared (without any mention of the Stasheff polytope) in [3].

Surprisingly, this way of viewing the associahedron is related to an algebraic structure on the set of planar binary trees  $PBT = \bigcup_{n \geq 1} PBT_n$ , related to dendriform algebras. Indeed there is a non-commutative monoid structure on the set of homogeneous nonempty subsets of  $PBT$  constructed in [6]. It comes from the associative structure of the free dendriform algebra on one generator. This monoid structure is denoted by  $+$ , the neutral element is the tree  $|$ . If  $t \in PBT_p$  and  $s \in PBT_q$ , then  $s + t$  is a subset of  $PBT_{p+q-1}$ . It is proved in [6] that the trees which lie on the edge  $\{t\} \times I \subset I^n$  are precisely the trees of  $t + \begin{array}{c} \diagup \\ \diagdown \end{array}$ . For instance:

$$\begin{array}{c} \diagup \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \diagup \\ \diagdown \end{array} \cup \begin{array}{c} \diagup \\ \diagdown \diagdown \end{array}$$

and

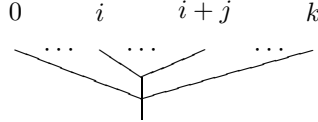
$$\begin{array}{c} \diagup \diagup \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \diagup \diagup \\ \diagdown \end{array} \cup \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} \cup \begin{array}{c} \diagup \diagup \diagdown \\ \diagdown \end{array} .$$

The deformation of the associahedron consisting in inflating the main simplex to the top cell can be performed into two steps by considering a cube inside the associahedron. This cube is determined by the previous construction. First, we inflate the main simplex to the full cube as described in 3.1, then we deform the cube into the associahedron as indicated above.

Finally we remark that the deformation described in 3.3 permits us to draw the associahedron on the simplex.

### 6 Appendix 2: $\Delta(\mu_5)$

In this appendix we give the computation of  $\Delta(\mu_5)$  and we show that we get the same result as Saneblidze and Umble. In order to compare with their result we adopt their way of indexing the planar trees, which is as follows. Let  $t$  be a tree whose root vertex has  $k + 1$  inputs, that we label (from left to right) by  $0, \dots, k$ . Then, by definition,  $d_{ij}(t)$  is the tree obtained by replacing, locally, the root vertex by the following tree with one internal edge:



The operator  $d_{ij}$  is well-defined for  $0 \leq i \leq k, 1 \leq j \leq k - i$  and  $(i, j) \neq (0, k)$ . So we get:

$ij$	=	01	02	11	12	21
$d_{ij} \left( \begin{array}{c} \diagup \diagdown \diagup \diagdown \diagup \diagdown \end{array} \right)$	=					

and  $d_{01}d_{01} \left( \begin{array}{c} \diagup \diagdown \diagup \diagdown \diagup \diagdown \end{array} \right) = \begin{array}{c} \diagup \diagdown \diagup \diagdown \end{array}$ , etc.

Let us index the sixteen 3-simplices forming  $\mathcal{K}_{\text{simp}}^3$  by the tree indexing the face in  $\text{fat}\mathcal{K}_{\text{simp}}^2$  and either  $a, b, c$  if this face is a pentagon (cf. 1.5) or the shuffle  $\alpha = (1, 2), \beta = (2, 1)$  if this face is a square (cf. 2.5). In the following tableau we indicate the image of the 3-simplices under the map  $p' \otimes p' \circ \Delta^{AA_\infty}$ . In the left column we indicate the information which determines the 3-simplex  $(d_{ij}(\mu_5), x)$ . In the right column we give its image (up to signs) as a sum of four terms, since in the AW morphism there are four terms.

03	$a$	$(01)(01)(01) \otimes \mu_5 + (02)(01) \otimes (- (21) + (22))$ $+ (03) \otimes ((11)(21) + (12)(21) + (11)(22)) + \mu_5 \otimes (11)(21)(31)$
03	$b$	$0 + 0 + 0 + 0$
03	$c$	$0 + 0 + 0 + 0$
02	$\alpha$	$0 + 0 + (02) \otimes (- (11)(31) - (12)(31)) + 0$
02	$\beta$	$0 + (01)(02) \otimes ((11) + (12) + (13)) + 0 + 0$
01	$a$	$0 + (01)(01) \otimes (31) + (01) \otimes (21)(31)$
01	$b$	$0 + 0 + 0 + 0$
01	$c$	$0 + 0 + 0 + 0$
12	$\alpha$	$0 + 0 + (12) \otimes ((12)(21) + (11)(22)) + 0$
12	$\beta$	$0 + 0 + 0 + 0$
11	$a$	$0 + 0 - (11) \otimes (12)(31) + 0$
11	$b$	$0 + (02)(11) \otimes ((13) + (12)) + 0 + 0$
11	$c$	$0 + (11)(11) \otimes (13) + 0 + 0$
21	$a$	$0 + (11)(01) \otimes (22) + (21) \otimes (11)(22) + 0$
21	$b$	$0 + 0 + 0 + 0$
21	$c$	$0 + 0 + 0 + 0$

As a result  $\Delta(\mu_5)$  is the algebraic sum of 22 elements, which are exactly the same as in [13] Example 1. Topologically, it means that  $\mathcal{K}^3$  can be realized as the union of 2 copies of  $\mathcal{K}^3$  (having only one vertex in common), 6 copies of  $\mathcal{K}^1 \times \mathcal{K}^2$ , 6 copies of  $\mathcal{K}^2 \times \mathcal{K}^1$ , 4 copies of  $(\mathcal{K}^1 \times \mathcal{K}^1) \times \mathcal{K}^1$  and 4 copies of  $\mathcal{K}^1 \times (\mathcal{K}^1 \times \mathcal{K}^1)$ .

From this computation it is reasonable to conjecture that the diagonal constructed from the simplicial decomposition of the associahedron is the same as the Saneblidze-Umbel diagonal.

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