

The cyclic Deligne conjecture for spaces, chain complexes and Hopf algebras¹

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Hochschild cochains

Definition

For a (unital associative) K -algebra A and A -bimodule M , the *Hochschild cochain complex* of A with coefficients in M is given by

$$C^n(A; M) = \text{Hom}_K(A^{\otimes n}, M), \quad n \geq 0,$$

where for $f \in C^n(A; M)$,

$$(\partial_i f)(a_1, \dots, a_{n+1}) = \begin{cases} a_1 f(a_2, \dots, a_n) & i = 0; \\ f(a_1, \dots, a_i a_{i+1}, \dots, a_n) & i = 1, \dots, n; \\ f(a_1, \dots, a_n) a_{n+1} & i = n + 1. \end{cases}$$
$$(s_i f)(a_1, \dots, a_{n-1}) = f(a_1, \dots, a_i, 1_A, a_{i+1}, \dots, a_{n-1}).$$

The Hochschild cohomology $HH^\bullet(A; M)$ is the cohomology of the cochain complex of the cosimplicial K -module $C^\bullet(A; M)$.

Cup and brace operations on $C^\bullet(A; A)$

There is a *cup product*

$$-\cup- : C^m(A; A) \otimes_K C^n(A; A) \rightarrow C^{m+n}(A; A)$$

$$(f \cup g)(a_1, \dots, a_{m+n}) = f(a_1, \dots, a_m)g(a_{m+1}, \dots, a_{m+n})$$

and a *brace operation*

$$-\{-\} : C^m(A; A) \otimes_K C^n(A; A) \rightarrow C^{m+n-1}(A; A)$$

where $f\{g\}(a_1, \dots, a_{m+n-1})$ is defined by

$$\sum_{1 \leq i \leq m} (-1)^{(i-1)(n-1)} f(a_1, \dots, a_{i-1}, g(a_i, \dots, a_{i+n-1}), a_{i+n}, \dots, a_{m+n-1}).$$

The bracket $\{f, g\} = f\{g\} - (-1)^{(|f|-1)(|g|-1)}g\{f\}$ induces a Lie bracket of degree -1 on $HH^\bullet(A; A)$.

Definition

A Gerstenhaber K -algebra $(H, \cup, \{-, -\})$ is a graded-commutative K -algebra with Lie bracket of degree -1 such that

$$\{f, g \cup h\} = \{f, g\} \cup h + (-1)^{|g|(|f|-1)} g \cup \{f, h\}.$$

Proposition (Gerstenhaber '63)

For any algebra A , the Hochschild cohomology $HH^\bullet(A; A)$ is a Gerstenhaber algebra.

Theorem (F. Cohen '72)

For any field K , the homology $H_\bullet(D_2; K)$ of the little disks operad is the operad for Gerstenhaber K -algebras.

Corollary

For any based space $(X, *)$, the homology $H_\bullet(\Omega^2 X; K)$ is a Gerstenhaber K -algebra.

Connes' coboundary on $C^\bullet(A; A^*)$

For $A^* = \text{Hom}_K(A, K)$, the adjunction

$$\text{Hom}_K(A^{\otimes n}, A^*) \cong \text{Hom}_K(A^{\otimes n+1}, K)$$

induces a cyclic operator τ_n on $C^n(A; A^*)$ of order $n + 1$. These cyclic operators are compatible with the simplicial operators:

$$\tau_{n+1}\partial_i = \partial_{i-1}\tau_n \quad i > 0, \quad \tau_{n-1}s_i = s_{i-1}\tau_n \quad i > 0.$$

It results a covariant functor on Connes' *cyclic category*

$$\Delta C \rightarrow \text{Mod}_K : [n] \mapsto C^n(A; A^*).$$

In particular, $C^\bullet(A; A^*)$ is a *mixed complex*

$$C^0(A; A^*) \rightleftarrows C^1(A; A^*) \rightleftarrows C^2(A; A^*) \rightleftarrows \dots$$

and $HH^\bullet(A; A^*)$ has a differential Δ of degree -1 :

$$\Delta^n : HH^n(A, A^*) \rightarrow HH^{n-1}(A; A^*).$$

Definition

A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra $(H, \cup, \{-, -\})$ with a differential Δ of degree -1 such that

$$(-1)^{|f|}\{f, g\} = \Delta(f \cup g) - (\Delta f \cup g) - (-1)^{|f|}(f \cup \Delta g).$$

A *symmetric K -algebra* A is a K -algebra equipped with an isomorphism of A -bimodules $A \cong A^*$, i.e. a symmetric exact pairing $\langle -, - \rangle: A \otimes_K A \rightarrow K$ such that $\langle ab, c \rangle = \langle a, bc \rangle$.

Proposition (Menichi '04)

For any symmetric algebra A , the Hochschild cohomology $HH^\bullet(A, A)$ is a Batalin-Vilkovisky algebra.

Theorem (Getzler '94)

For any field K , the homology $H_\bullet(fD_2, K)$ of the framed little disks operad is the operad for Batalin-Vilkovisky K -algebras.

The dg- Deligne conjecture

Theorem (MS '02, KS '02, Vo '02, Ta '04, BF '04)

The Hochschild cochain complex of an algebra A admits a $C_\bullet(D_2)$ -action inducing the Gerstenhaber structure on $HH^\bullet(A; A)$.

Theorem (KS '06, TZ '06, Ka '07, BB '09)

The Hochschild cochain complex of a symmetric algebra A admits a $C_\bullet(fD_2)$ -action inducing the BV -structure on $HH^\bullet(A; A)$.

Proposition (Gerstenhaber-Voronov '95, Menichi '04)

The Hochschild cochain complex of A is isomorphic to the *deformation complex* of the *endomorphism operad* End_A of A .
If A is symmetric, then End_A is multiplicative cyclic.

Proof.

$C^n(A; A) = \text{Hom}(A^{\otimes n}, A) = \text{End}_A(n) \ni \mu_n$. For $f \in \text{End}_A(n)$,
 $\partial_0 f = \mu_2 \circ_1 f$, $\partial_n f = \mu_2 \circ_0 f$, $\partial_i f = f \circ_i \mu_2$ if $0 < i < n$.

If A is symmetric then End_A is cyclic and $\tau_n(\mu_n) = \mu_n$.



Multiplicative operads

Definition

A *multiplicative (cyclic) operad* is a non-symmetric (cyclic) operad \mathcal{O} equipped with a map of (cyclic) operads $\mathcal{A}ss \rightarrow \mathcal{O}$.

A multiplicative (cyclic) operad \mathcal{O} has an underlying cosimplicial (cocyclic) object \mathcal{O}^\bullet . In a closed monoidal category \mathcal{E} equipped with $\delta : \Delta \rightarrow \mathcal{E}$, the *deformation complex* of \mathcal{O} is $\underline{\text{Hom}}_\Delta(\delta^\bullet, \mathcal{O}^\bullet)$.

Example

For $\mathcal{E} = \text{Ch}(\mathbb{Z})$ and $\delta_{\mathbb{Z}} : \Delta \rightarrow \text{Ch}(\mathbb{Z}) : [n] \mapsto N_*(\Delta[n]; \mathbb{Z})$ we get

$$C^\bullet(A; A) = \underline{\text{Hom}}_\Delta(\delta_{\mathbb{Z}}^\bullet, \text{End}_A^\bullet).$$

Theorem (Kaufmann '07, BB '09)

For any multiplicative chain operad \mathcal{O} , the deformation complex of \mathcal{O} admits a $C_\bullet(D_2)$ -action. If \mathcal{O} is multiplicative cyclic, this action extends to a $C_\bullet(fD_2)$ -action.

The coloured operad for multiplicative operads

Let $\mathcal{L}_2(n_1, \dots, n_k; n)$ be the set of iso-classes of planar rooted trees with n leaves and a bipartite vertex-set such that:

1. one part of the vertex-set is in bijection with $\{1, \dots, k\}$;
2. the vertex with label i has arity n_i ;
3. each edge has at least one labelled extremity;
4. unlabelled vertices have arity $\neq 1$.

Let $C_{[n]} = \mathbb{Z}/(n+1)\mathbb{Z}$ and put

$$\mathcal{L}_2^{\text{cyc}}(n_1, \dots, n_k; n) = \mathcal{L}_2(n_1, \dots, n_k; n) \times C_{[n_1]} \times \cdots \times C_{[n_k]}.$$

\mathcal{L}_2 and $\mathcal{L}_2^{\text{cyc}}$ are \mathbb{N} -coloured operads for an evident substitution of trees into trees; in $\mathcal{L}_2^{\text{cyc}}$, the cyclic permutations distinguish for each labelled vertex one of its incident edges, the neutral element stands for the edge closest to the root of the tree.

Lemma

\mathcal{L}_2 -algebras are multiplicative operads; $\mathcal{L}_2^{\text{cyc}}$ -algebras are multiplicative cyclic operads. The category of unary operations of \mathcal{L}_2 (resp. $\mathcal{L}_2^{\text{cyc}}$) is Δ (resp. ΔC).

Condensation of coloured operads

Unary operations of a coloured operad act covariantly on inputs and contravariantly on the output; therefore:

$$\mathcal{L}_2(-, \dots, -; -) : \Delta^{\text{op}} \times \dots \times \Delta^{\text{op}} \times \Delta \rightarrow \text{Sets}.$$

Given $\delta_{\mathbb{Z}} : \Delta \rightarrow \text{Ch}(\mathbb{Z})$ we can *realize* multisimplicially, and *totalize* the resulting cosimplicial chain complex. This yields

$$\xi(\mathcal{L}_2, \delta_{\mathbb{Z}})(k) := \underline{\text{Hom}}_{\Delta}(\delta^{\bullet}, |\mathcal{L}_2(\overbrace{-, \dots, -}^k; \bullet)|_{\delta_{\mathbb{Z}}^{\otimes k}}), \quad k \geq 0.$$

Proposition (Day-Street '03, McClure-Smith '04, BB '09)

$\xi(\mathcal{L}_2, \delta_{\mathbb{Z}})$ (resp. $\xi(\mathcal{L}_2^{\text{cyc}}, \delta_{\mathbb{Z}}^{\text{cyc}})$) is a chain operad acting on the deformation complex of any multiplicative (cyclic) operad.

Theorem (BB '09)

As chain operads we have $\xi(\mathcal{L}_2, \delta_{\mathbb{Z}}) \sim C_{\bullet}(D_2)$ and $\xi(\mathcal{L}_2^{\text{cyc}}, \delta_{\mathbb{Z}}^{\text{cyc}}) \sim C_{\bullet}(fD_2)$.

The cobar complex of a bialgebra

Theorem (cf. Gerstenhaber-Schack '92, Menichi '04)

The cobar complex ΩA of a bialgebra (resp. involutive Hopf algebra) A has an action by $\xi(\mathcal{L}_2, \delta_{\mathbb{Z}})$ (resp. $\xi(\mathcal{L}_2^{cyc}, \delta_{\mathbb{Z}}^{cyc})$). Its homology $H_{\bullet}(\Omega A; \mathbb{Z})$ is a Gerstenhaber (resp. BV-) algebra.

Proof.

The bialgebra A is a comonoid in the monoidal category of A -modules. Therefore: $(\Omega A)_n = A^{\otimes n} \cong \text{Hom}_A(A, A^{\otimes n})$. This \mathbb{Z} -linear operad is multiplicative via the diagonal of A . If A has an involutive antipode then the operad is multiplicative cyclic. \square

Remark

(a) $\Omega C_{\bullet}(\Omega X; \mathbb{Z}) \sim C_{\bullet}(\Omega^2 X; \mathbb{Z})$ (Adams). The $\xi(\mathcal{L}_2, \delta_{\mathbb{Z}})$ -action on $\Omega C_{\bullet}(\Omega X; \mathbb{Z})$ corresponds to the $C_{\bullet}(D_2)$ -action on $C_{\bullet}(\Omega^2 X; \mathbb{Z})$.

(b) If A is involutive, the $\xi(\mathcal{L}_2^{cyc}, \delta_{\mathbb{Z}}^{cyc})$ -action induces a cocyclic structure on ΩA yielding $HC^{\bullet}(A)$ of Connes-Moscovici '99.

(c) $\xi(\mathcal{L}_2, \delta_{\mathbb{Z}})$ contains the second filtration stage of the *surjection operad* of MS '03, BF '04 as a suboperad. Cyclic extension ?

The topological Deligne conjecture

There is a cosimplicial resp. cocyclic space

$$\delta_{top} : \Delta \rightarrow \text{Top} : [n] \mapsto \Delta^n \text{ resp. } \delta_{top}^{cyc} : \Delta C \rightarrow \text{Top} : [n] \mapsto \Delta^n \times S^1.$$

Theorem (McClure-Smith '04, Salvatore '09, BB '09)

The operad $\xi(\mathcal{L}_2, \delta_{top})$ is weakly equivalent to D_2 and acts on the deformation complex of multiplicative operads in spaces.

The operad $\xi(\mathcal{L}_2^{cyc}, \delta_{top}^{cyc})$ is weakly equivalent to fD_2 and acts on the deformation complex of multiplicative cyclic operads in spaces.

Remark (cf. Markl '99, Salvatore-Wahl '03, Salvatore '09)

$$fD_2(k) \cong D_2(k) \times (S^1)^k, \quad \xi(\mathcal{L}_2^{cyc}, \delta_{top}^{cyc})(k) \cong \xi(\mathcal{L}_2, \delta_{top})(k) \times (S^1)^k.$$

For $n = 1$:

$$fD(1) \cong D(1) \times S^1, \quad \underline{\text{Hom}}_{\Delta C}(\delta_{top}^{cyc}, \delta_{top}^{cyc}) \cong \underline{\text{Hom}}_{\Delta}(\delta_{top}, \delta_{top}) \boxtimes S^1.$$

Proposition (Sinha '06)

The simplicial 2-sphere $S^2 = \Delta[2]/\partial\Delta[2]$ is an \mathcal{L}_2 -coalgebra in finite pointed sets. For a based space $(X, *)$, $\Omega^2 X$ is the deformation complex of the multiplicative operad $(X, *)^{(S^2, *)}$.

Braid and ribbon-braid groups

\mathfrak{S}_k denotes the *permutation group* on k letters. \mathfrak{S}_k^\pm denotes the *signed permutation group* on k letters.

$\mathfrak{S}_k^\pm = \mathfrak{S}_k \wr \mathfrak{S}_2 = \mathfrak{S}_k \rtimes (\mathfrak{S}_2)^k$ acts on $fD_2(k) = D_2(k) \times (S^1)^k$.

Definition (Braid and ribbon-braid groups on k strands)

$$\begin{aligned} B_k &= \pi_1(D_2(k)/\mathfrak{S}_k) & RB_k &= \pi_1(fD_2(k)/\mathfrak{S}_k^\pm) \\ PB_k &= \pi_1(D_2(k)) & PRB_k &= \pi_1(fD_2(k)) \end{aligned}$$

Proposition (Asphericity of $D_2(k)$ and $fD_2(k)$)

$$\begin{aligned} D_2(k)/\mathfrak{S}_k &= K(B_k, 1) & fD_2(k)/\mathfrak{S}_k^\pm &= K(RB_k, 1) \\ D_2(k) &= K(PB_k, 1) & fD_2(k) &= K(PRB_k, 1) \end{aligned}$$

Corollary

The coverings $D_2(k) \rightarrow D_2(k)/\mathfrak{S}_k$ and $fD_2(k) \rightarrow fD_2(k)/\mathfrak{S}_k^\pm$ are classified by the short exact sequences $1 \rightarrow PB_k \rightarrow B_k \rightarrow \mathfrak{S}_k \rightarrow 1$ and $1 \rightarrow PRB_k \rightarrow RB_k \rightarrow \mathfrak{S}_k^\pm \rightarrow 1$.

Problem

Describe the operad structure of D_2 (resp. fD_2) in terms of the pure braid (resp. ribbon-braid) groups.

Coxeter geometry of permutation groups

The braid group B_k is an *Artin group* with presentation $\langle s_1, \dots, s_{k-1} \mid s_i s_j = s_j s_i \text{ if } |i-j| > 1 \text{ and } s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle$.
The *pure Artin group* $PB_k = \text{Ker}(B_k \rightarrow \mathfrak{S}_k) \cong \pi_1(\mathbb{C}^k - \mathcal{A}_{\mathfrak{S}_k})$
where $\mathcal{A}_{\mathfrak{S}_k}$ is the complexified *braid arrangement*.

The *Salvetti complex* $Sal_{\mathfrak{S}_k}$ is a partially ordered set of the same equivariant homotopy type as $\mathbb{C}^k - \mathcal{A}_{\mathfrak{S}_k}$.

$$Sal_- : (\text{Coxeter groups}) \rightarrow (\text{posets})$$

is a functor commuting with finite products. Thus, $(PB_k)_{k \geq 0}$ is a categorical operad. Similarly, $(PRB_k)_{k \geq 0}$ is a categorical operad.

Proposition

$D_2 \sim K(PB, 1)$ and $fD_2 \sim K(PRB, 1)$ as operads. Moreover, PB -algebras are braided strict monoidal categories; PRB -algebras are ribbon-braided (i.e. balanced) strict monoidal categories.

Corollary (B '98, Salvatore-Wahl '03)

The nerve of a braided (resp. ribbon-braided) strict monoidal category is E_2 (resp. framed E_2).

The categorical Deligne conjecture

Consider the cosimplicial category

$$\delta_{\text{Cat}} : \Delta \rightarrow \text{Cat} : [n] \mapsto [n][n]^{-1}$$

Proposition

There are weak equivalences of categorical operads

$$PB \xrightarrow{\sim} \xi(\mathcal{L}_2, \delta_{\text{Cat}}) \quad \text{and} \quad PRB \xrightarrow{\sim} \xi(\mathcal{L}_2^{\text{cyc}}, \delta_{\text{Cat}}^{\text{cyc}}).$$

Definition

A central element of a monoidal category \mathcal{E} is a pair (A, c_A) where $c_{A,-} : A \otimes - \cong - \otimes A$ and $c_{A,B \otimes C} = (1_B \otimes c_{A,C}) \circ (c_{A,B} \otimes 1_C)$. The center $\mathcal{Z}\mathcal{E}$ is the category of central elements.

Proposition

For $\mathcal{E} = \text{Mod}_H$, $\mathcal{Z}\mathcal{E} \simeq \text{Mod}_{DH}$ where DH is the Drinfeld double of the Hopf algebra H .

The Drinfeld double of a Hopf algebra

Proposition (Street '04)

$$\mathcal{Z}\mathcal{E} = \underline{\text{Hom}}_{\Delta}(\delta_{\text{Cat}}, \text{End}_{\mathcal{E}})$$

Corollary

The center of a monoidal category is braided monoidal; in particular, the Drinfeld double of a Hopf algebra is “braided”.

Definition

An involutive category is a closed monoidal category \mathcal{E} such that the duality functor $(-)^* = \underline{\text{Hom}}(-, I)$ is self-adjoint. A Hopf algebra H is called quasi-involutive if Mod_H is involutive.

Proposition

The category \mathcal{E}_f of symmetric duality objects of an involutive category \mathcal{E} has a multiplicative cyclic endomorphism-operad $\text{End}_{\mathcal{E}_f}$.

Corollary

The center of \mathcal{E}_f is ribbon-braided; in particular, the Drinfeld double of a quasi-involutive Hopf algebra is “ribbon-braided”.