Homotopy formulas for cyclic groups acting on rings

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Abstract. *The positive cohomology groups of a finite group acting on a ring vanish when the ring has a norm-one element. In this note we give explicit homotopies on the level of cochains when the group is cyclic, which allows us to express any cocycle of a cyclic group as the coboundary of an explicit cochain. The formulas in this note are closely related to the effective problems considered in previous joint work with Eli Aljadeff.*

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Given a cyclic group acting on a (nonnecessarily commutative) ring, we exhibit in this note homotopy formulas implying the vanishing of the cohomology of the group with coefficients in the ring when the latter has a norm-one element. Using our formulas, we can express any cocycle of a cyclic group as the coboundary of an explicit cochain.

Such formulas are closely related to the effective problems considered in [1] and [2]. Actually we discovered them when we tried to find an effective version of [2, Proposition 3.1]. Our formulas may be added to the already long list(1) of homotopy formulas that are useful in many areas of homological algebra, see [4] for an extensive use of explicit homotopies in cyclic homology.

Let R be a ring (with unit 1), t be a ring automorphism of R, and $n \geq 2$ be a natural number such that $t^n = \mathrm{id}_R$ (the identity of R). Define two $\mathbf{Z}$-linear endomorphisms $T, N : R \to R$ by

$$T = t - \mathrm{id}_R \quad \text{and} \quad N = \mathrm{id}_R + t + \cdots + t^{n-1}.$$

In the literature, the endomorphism N is called the *norm map* or the *trace map*. We have

$$t \circ N = N \circ t = N \quad \text{and} \quad T \circ N = N \circ T = 0. \tag{1}$$

Let $R^t = \mathrm{Ker}(T) = \{a \in R \mid t(a) = a\}$ be the subgroup of R consisting of all t-invariant elements of R. It is a subring of R and the endomorphisms T, N are left and right R^t-linear.

(1) Mariusz Wodzicki wittily proposed the name *homologbook* for such a list

For any element x of R define $\mathbf{Z}$-linear endomorphisms j_x, j'_x, h_x, h'_x of R by

$$j_x(a) = xa, \qquad j'_x(a) = xt(a),$$

$$h_x(a) = -\sum_{i=1}^{n-1} t^i(x)\big(\mathrm{id}_R + t + \cdots + t^{i-1}\big)(a),$$

$$h'_x(a) = \sum_{i=1}^{n-1} (\mathrm{id}_R + t + \cdots + t^{i-1})\big(xt^{-i}(a)\big),$$

where $a \in R$. (By convention, $t^0 = \mathrm{id}_R$.)

Lemma 1.— *For all x, $a \in R$ we have*

$$j'_x(a) - j_x(a) = xT(a) \quad \textit{and} \quad h'_x(a) - h_x(a) = N(x)N(a) - N(xa).$$

PROOF.— The first identity follows immediately from the definitions. For the second one we have

$$\begin{aligned} h'_x(a) - h_x(a) &= \sum_{k=1}^{n-1}\sum_{i=0}^{k-1} t^i(x)t^{i-k}(a) + \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} t^i(x)t^j(a) \\ &= \sum_{0\le i<k\le n-1} t^i(x)t^{n+i-k}(a) + \sum_{0\le j<i\le n-1} t^i(x)t^j(a). \end{aligned}$$

Set $j = n + i - k$ in the penultimate sum. Then $i < k \le n-1$ is equivalent to $i < j \le n-1$. Therefore,

$$\begin{aligned} h'_x(a) - h_x(a) &= \sum_{0\le i<j\le n-1} t^i(x)t^j(a) + \sum_{0\le j<i\le n-1} t^i(x)t^j(a) \\ &= \sum_{\substack{i,j\in\{0,\ldots,n-1\} \\ i\ne j}} t^i(x)t^j(a) \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} t^i(x)t^j(a) - \sum_{i=0}^{n-1} t^i(x)t^i(a) \\ &= \left(\sum_{i=0}^{n-1} t^i(x)\right)\left(\sum_{j=0}^{n-1} t^j(a)\right) - \sum_{i=0}^{n-1} t^i(xa) \\ &= N(x)N(a) - N(xa). \end{aligned}$$

□

Corollary 1.— *For all $x \in R$ we have*

$$j_x \circ N = j'_x \circ N, \qquad T \circ h_x = T \circ h'_x,$$

$$N \circ j_x + h_x \circ T = N \circ j'_x + h'_x \circ T.$$

Proof.— By (1) and Lemma 1, for all $a \in R$ we have

$$(j'_x - j_x)\big(N(a)\big) = xT\big(N(a)\big) = 0.$$

Similarly,

$$T\big(h'_x(a) - h_x(a)\big) = t\big(N(x)\big)t\big(N(a)\big) - N(x)N(a) - T\big(N(xa)\big) = 0.$$

Finally,

$$(h'_x - h_x)\big(T(a)\big) = N(x)N\big(T(a)\big) - N\big(xT(a)\big) = -N\big(j'_x(a) - j_x(a)\big). \qquad \square$$

We now state the main result of this note.

Proposition.— *For all $x \in R$ we have*

$$N(x)\,\mathrm{id}_R = N \circ j_x + h_x \circ T, \tag{2}$$

$$N(x)\,\mathrm{id}_R = j_x \circ N + T \circ h_x, \tag{3}$$

$$N(x)\,\mathrm{id}_R = N \circ j'_x + h'_x \circ T, \tag{4}$$

$$N(x)\,\mathrm{id}_R = j'_x \circ N + T \circ h'_x \tag{5}$$

where $N(x)\,\mathrm{id}_R$ is the left multiplication by $N(x)$ in R.

Proof.— By Corollary 1, Identity (4) (resp. (3)) is equivalent to Identity (2) (resp. (5)). It suffices to prove (2) and (5).

Identity (2): We have

$$\begin{aligned} h_x\big(T(a)\big) &= -\sum_{i=1}^{n-1} t^i(x)\big(\mathrm{id}_R + t + \cdots + t^{i-1}\big)\big(t(a) - a\big) \\ &= -\sum_{i=1}^{n-1} t^i(x)\big(t^i(a) - a\big) \\ &= -\sum_{i=0}^{n-1} t^i(xa) + xa + \sum_{i=0}^{n-1} t^i(x)a - xa \\ &= -N(xa) + N(x)a \\ &= N(x)a - N(j_x(a)). \end{aligned}$$

Identity (5): We have

$$\begin{aligned}
T\big(h'_x(a)\big) &= \sum_{i=1}^{n-1} (t-\mathrm{id}_R)\Big((\mathrm{id}_R + t + \cdots + t^{i-1})\big(xt^{-i}(a)\big)\Big) \\
&= \sum_{i=1}^{n-1} (t^i-\mathrm{id}_R)\big(xt^{-i}(a)\big) \\
&= \sum_{i=1}^{n-1} t^i\big(xt^{-i}(a)\big) - \sum_{i=1}^{n-1} xt^{-i}(a) \\
&= \sum_{i=0}^{n-1} t^i(x)a - xa - \sum_{i=0}^{n-1} xt^{-i}(a) + xa \\
&= N(x)a - xN(a) = N(x)a - x\,t\big(N(a)\big) \\
&= N(x)a - j'_x\big(N(a)\big).
\end{aligned}$$

□

The following *homotopy formulas* are immediate consequences of the proposition.

Corollary 2.— *If $x \in R$ satisfies $N(x) = 1$ (the unit of R), then*

$$N \circ j_x + h_x \circ T = N \circ j'_x + h'_x \circ T = \mathrm{id}_R,$$

$$j_x \circ N + T \circ h_x = j'_x \circ N + T \circ h'_x = \mathrm{id}_R.$$

Corollary 2 has the following interesting consequences. Suppose there is an element x of R such that $N(x) = 1$ (this is equivalent to the image of the norm map $N : R \to R$ being the subring R^t). Under this condition any element $a \in R^t$ (i.e., killed by T) is the image under N of explicit elements of R, namely

$$a = N(xa). \tag{6}$$

Similarly, any element $a \in R$ killed by the norm map (i. e., $N(a) = 0$) is the image under T of explicit elements of R, namely

$$a = T\big(h_x(a)\big) = T\big(h'_x(a)\big). \tag{7}$$

(The identity $a = T\big(h'_x(a)\big)$ already appeared in [1, Lemma 1 and Formula (3)].)

Concluding remarks. (i) Since $t^n = \mathrm{id}_R$, the automorphism t induces a G-module structure on the ring R, where G is the cyclic group of order n. Consider the (co)homology groups $H^i(G,R)$ and $H_i(G,R)$ of G with coefficients in this G-module. As is well known (see, e.g., [3, Chap. XII, § 7]), these groups can be realized as the (co)homology groups of the periodic (co)chain complex

$$\cdots \to R \xrightarrow{T} R \xrightarrow{N} R \xrightarrow{T} R \xrightarrow{N} R \to \cdots . \tag{8}$$

If there is an element $x \in R$ such that $N(x) = 1$, then by Corollary 2 the operators j_x, j'_x, h_x, h'_x are homotopies for the complex (8). Consequently,

$$H^i(G,R) = 0 = H_i(G,R) \tag{9}$$

for all $i > 0$. The vanishing of the (co)homology of a cyclic group with coefficients in a ring R in the presence of a norm-one element $x \in R$ has been observed in [2]. (Actually, similar vanishing results hold for any finite group, see [2, Proposition 3.1].)

With Formulas (6), (7) we can do better than (9), namely we can express any (co)cycle in the complex (8) as the (co)boundary of an explicit (co)chain. Such effective formulas play an important role in [1] and [2, Sections 4–5].

(ii) If R is uniquely n-divisible and $x = 1/n \in R$ is the unique element such that $nx = 1$, then x is t-invariant and $N(x) = nx = 1$. Under this condition the operators h_x and h'_x become (for $a \in R$)

$$h_x(a) = -\frac{1}{n}\sum_{j=0}^{n-1} (n-1-j)\, t^j(a) \quad \text{and} \quad h'_x(a) = \frac{1}{n}\sum_{j=0}^{n-1} j t^j(a). \tag{10}$$

(iii) Let $R_n = \mathbf{Z}\langle t_0(X), t_1(X), t_2(X), \ldots, t_{n-1}(X)\rangle$ be the free ring on n indeterminates $t_i(X)$ indexed by the cyclic group $\mathbf{Z}/n$ of order n. We consider the ring automorphism t of R_n determined by

$$t\big(t_i(X)\big) = t_{i+1}(X)$$

for all $i \in \mathbf{Z}/n$. For any ring R equipped with a ring automorphism t such that $t^n = \mathrm{id}_R$, and any $x \in R$, there is a unique ring map $f : R_n \to R$ that commutes with the automorphisms t and sends X to x. We may consider R_n as the universal ring for the situation considered in this note. The proposition holds in R_n (with x replaced by X).

The conclusion of Corollary 2 holds in the quotient of R_n by the two-sided ideal generated by $1 - \sum_{i\in\mathbf{Z}/n} t_i(X)$; this quotient-ring is universal for all rings equipped with a $\mathbf{Z}/n$-action and an element of norm one.

In this sense the formulas in this note can be considered as universal.

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