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THE q-TANGENT AND q-SECANT NUMBERS VIA BASIC EULERIAN POLYNOMIALS

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Abstract. The classical identity that relates Eulerian polynomials to tangent numbers together with the parallel result dealing with secant numbers is given a q-extension, both analytically and combinatorially. The analytic proof is based on a recent result by Shareshian and Wachs and the combinatorial one on the geometry of alternating permutations.

1. Introduction

Let $(A_n(s,Y))$ $(n\ge 0)$ be the sequence of polynomials in two variables defined by

$$\sum_{n\ge 0}\frac{u^n}{n!}A_n(s,Y)=\frac{1-s}{\exp(su)-s\exp(u)}\exp(Yu). \tag{1.1}$$

It is known (see, e.g. [11], chap. 4) that they are polynomials with positive integral coefficients. For $Y=1$ we recover the *Eulerian polynomials* $(A_n(s,1))$ introduced by Euler [8] for his evaluation of the alternating sum $\sum_{i=1}^m(-1)^i i^n$. Moreover, Euler derived the two identities

$$A_{2n}(-1,1)=0\quad(n\ge 1);\qquad (-1)^nA_{2n+1}(-1,1)=T_{2n+1}\quad(n\ge 0); \tag{1.2}$$

where T_{2n+1} $(n\ge 0)$ are the coefficients of the Taylor expansion of $\tan u$,

$$\begin{aligned}\tan u&=\sum_{n\ge 0}\frac{u^{2n+1}}{(2n+1)!}T_{2n+1}\\ &=\frac{u}{1!}1+\frac{u^3}{3!}2+\frac{u^5}{5!}16+\frac{u^7}{7!}272+\frac{u^9}{9!}7936+\frac{u^{11}}{11!}353792+\cdots\end{aligned} \tag{1.3}$$

The coefficients T_{2n+1} $(n\ge 0)$, called *tangent numbers*, are positive integral numbers, and so are the *secant numbers* E_{2n} $(n\ge 1)$ (see, e.g., [16], p. 177) defined by the Taylor expansion of $\sec u$:

$$\begin{aligned}\sec u=\frac{1}{\cos u}&=1+\sum_{n\ge 1}\frac{u^{2n}}{(2n)!}E_{2n}\\ &=1+\frac{u^2}{2!}1+\frac{u^4}{4!}5+\frac{u^6}{6!}61+\frac{u^8}{8!}1385+\frac{u^{10}}{10!}50521+\cdots\end{aligned} \tag{1.4}$$

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The parallel result to (1.2) involving secant numbers was obtained by Roselle [18] for the polynomials $A_n(s,0)$ in the form

$$A_{2n-1}(-1,0)=0 \quad (n\ge 1);\quad (-1)^n A_{2n}(-1,0)=E_{2n} \quad (n\ge 0). \tag{1.5}$$

For each permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$ the number of *excedances* $\operatorname{exc}\sigma$, (resp. of *descents*, $\operatorname{des}\sigma$,) of σ is defined to be the number of integers i such that $1\le i\le n-1$ and $\sigma(i)>i$ (resp. $\sigma(i)>\sigma(i+1)$). Also, $\operatorname{fix}\sigma$ designates the number of *fixed points* of σ and $\mathfrak{D}_n$ the subset of $\mathfrak{S}_n$ of the permutations without fixed points, called *derangements*. It is known that the two statistics "exc" and "des" have the same distributions over the symmetric group ([15], p. 186). As derived in [11], we have $A_n(s,Y)=\sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma}Y^{\operatorname{fix}\sigma}$. The specializations are due to Riordan ([17], p. 214) for $Y=1$ and to Roselle [18] for $Y=0$.

The combinatorial counterparts of (1.2) and (1.5) can be expressed as

$$\sum_{\sigma\in\mathfrak{S}_{2n}}(-1)^{\operatorname{exc}\sigma}=0\ (n\ge 1); \tag{1.6}$$

$$\sum_{\sigma\in\mathfrak{S}_{2n+1}}(-1)^{n+\operatorname{exc}\sigma}=T_{2n+1}\ (n\ge 0); \tag{1.7}$$

$$\sum_{\sigma\in\mathfrak{D}_{2n-1}}(-1)^{\operatorname{exc}\sigma}=0\ (n\ge 1); \tag{1.8}$$

$$\sum_{\sigma\in\mathfrak{D}_{2n}}(-1)^{n+\operatorname{exc}\sigma}=E_{2n}\ (n\ge 0). \tag{1.9}$$

Let $(t;q)_n=(1-t)(1-tq)\cdots(1-tq^{n-1})$ for $n\ge 1$ and $(t;q)_0=1$ denote the traditional q-ascending factorials. The purpose of this paper is to construct a q-analog of (1.6)–(1.9), as further stated in Theorem 1. This first means that two sequences of polynomials $(T_{2n+1}(q))$, $(E_{2n}(q))$ with *positive* integral coefficients are to be introduced such that $T_{2n+1}(1)=T_{2n+1}$ and $E_{2n}(1)=E_{2n}$. This is achieved by taking the *q-tangent numbers* $T_{2n+1}(q)$ and the *q-secant numbers* $E_{2n}(q)$ (see [3, 4, 9]) occurring in the two expansions:

$$\tan_q(u)=\frac{\sum\limits_{n\ge 0}(-1)^n u^{2n+1}/(q;q)_{2n+1}}{\sum\limits_{n\ge 0}(-1)^n u^{2n}/(q;q)_{2n}}=\sum_{n\ge 0}\frac{u^{2n+1}}{(q;q)_{2n+1}}T_{2n+1}(q);$$

$$\sec_q(u)=\frac{1}{\sum\limits_{n\ge 0}(-1)^n u^{2n}/(q;q)_{2n}}=\sum_{n\ge 0}\frac{u^{2n}}{(q;q)_{2n}}E_{2n}(q).$$

Note that $\tan_q(u)$ (resp. $\sec_q(u)$) is simply the quotient of the q-sine and q-cosine (resp. the inverse of q-cosine), as introduced by Jackson [14] (also see [12], p. 23). The first values are: $T_1(q)=1$; $T_3(q)=q+q^2$; $T_5(q)=q^2+2q^3+3q^4+4q^5+3q^6+2q^7+q^8$; $E_0(q)=E_2(q)=1$, $E_4(q)=q(1+q)^2+q^4$; $E_6(q)=q^2(1+q)^2(1+q^2+q^4)(1+q+q^2+2q^3)+q^{12}$.

Second, another statistic on the symmetric group $\mathfrak{S}_n$ is to be associated with "exc," so that the left-hand sides of the four identities (1.6)–(1.9) will become true *polynomial* identities. The statistic that meets our expectations is the classical *major index* "maj," defined for each permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ as the *sum* of all the i's such that $\sigma(i)>\sigma(i+1)$. The main result of the present paper is then the following theorem.

Theorem 1. *We have:*

$$\sum_{\sigma\in\mathfrak{S}_{2n}} (-1)^{\operatorname{exc}\sigma} q^{(\operatorname{maj}-\operatorname{exc})\sigma} = 0 \quad (n\ge 1); \tag{1.10}$$

$$\sum_{\sigma\in\mathfrak{S}_{2n+1}} (-1)^{n+\operatorname{exc}\sigma} q^{(\operatorname{maj}-\operatorname{exc})\sigma} = T_{2n+1}(q) \quad (n\ge 0); \tag{1.11}$$

$$\sum_{\sigma\in\mathfrak{D}_{2n-1}} (-1)^{\operatorname{exc}\sigma} q^{(\operatorname{maj}-\operatorname{exc})\sigma} = 0 \quad (n\ge 1); \tag{1.12}$$

$$\sum_{\sigma\in\mathfrak{D}_{2n}} (-1)^{n+\operatorname{exc}\sigma} q^{(\operatorname{maj}-\operatorname{exc})\sigma} = E_{2n}(q) \quad (n\ge 0). \tag{1.13}$$

We provide two proofs of Theorem 1. The *analytic* proof, given in Section 2, is based on a recent result due to Shareshian and Wachs [19] who made an explicit study of the statistic "maj − exc". The *combinatorial* proof, given in Section 4, is based on the geometry of *alternating* permutations, as introduced by André [1, 2] and on an *equidistribution* property between two three-variable statistics (exc, fix, maj) and (lec, pix, inv) established in [10]. However, to make this paper self-contained, we give a new proof of that equidistribution property by directly calculating the distribution of the latter statistic (see Theorem 4, Section 3). Our combinatorial proof consists of reducing the four alternating sums (1.10)–(1.13) by means of two explicit *sign-reversing involutions.* In contrast to the combinatorial proofs of (1.6) and (1.7) given in [11], chap. 5, these involutions naturally lead to the alternating permutation model.

2. The analytic proof

The q-analogs of the Eulerian polynomials, which have been derived, are the generating polynomials for $\mathfrak{S}_n$ by the pair (des, maj) (Carlitz [5, 6]), by (des, inv) (Stanley [20]) and by (exc, maj) (Shareshian and Wachs [19]). Let us recall the latest q-extension.

Theorem 2 (Shareshian-Wachs)**.** *Let*

$$A_n(s,Y,q) = \sum_{\sigma\in\mathfrak{S}_n} s^{\operatorname{exc}\sigma} Y^{\operatorname{fix}\sigma} q^{\operatorname{maj}\sigma}. \tag{2.1}$$

Then

$$\sum_{n\ge 0} \frac{u^n}{(q;q)_n} A_n(s,Y,q) = \frac{1-sq}{e_q(squ) - sq e_q(u)} e_q(Yu), \tag{2.2}$$

where $e_q(u) = \sum_{n\ge 0} u^n/(q;q)_n$ *is the first q-exponential.*

To prove Theorem 1 it then suffices to establish the following equivalent theorem.

Theorem 3. *Let* $(A_n(s,Y,q))$ $(n\ge 0)$ *be the sequence of polynomials defined by* (2.1). *Then*

$$A_{2n}(-q^{-1},1,q)=0 \ (n\ge 1); \quad (-1)^n A_{2n+1}(-q^{-1},1,q)=T_{2n+1}(q) \ (n\ge 0);$$
$$A_{2n-1}(-q^{-1},0,q)=0 \ (n\ge 1); \quad (-1)^n A_{2n}(-q^{-1},0,q)=E_{2n}(q) \ (n\ge 0).$$

Proof. With the substitutions $Y := 0$ and $s := -q^{-1}$ in (2.2) we get

$$\sum_{n\ge 0} \frac{u^n}{(q;q)_n} A_n(-q^{-1},0,q) = \Bigl(1+\sum_{n\ge 1}\frac{u^{2n}}{(q;q)_{2n}}\Bigr)^{-1}.$$

As the fraction on the right involves only even powers of u, we deduce:

$$A_{2n-1}(-q^{-1},0,q)=0 \quad (n\ge 1).$$

By replacing u by iu we obtain

$$\sum_{n\ge 0} \frac{u^{2n}}{(q;q)_{2n}}(-1)^n A_{2n}(-q^{-1},0,q) = \Bigl(1+\sum_{n\ge 1}(-1)^n\frac{u^{2n}}{(q;q)_{2n}}\Bigr)^{-1} = \frac{1}{\cos_q(u)}.$$

Consequently, $(-1)^n A_{2n}(-q^{-1},0,q) = E_{2n}(q) \quad (n\ge 0)$.

From (2.2) we can write:

$$\sum_{n\ge 0}\frac{u^n}{(q;q)_n}A_n(s,1,q) = e_q(u)\sum_{n\ge 0}\frac{u^n}{(q;q)_n}A_n(s,0,q)$$

so that by replacing s by $-q^{-1}$ and u by iu

$$\sum_{n\ge 0}\frac{(iu)^n}{(q;q)_n}A_n(-q^{-1},1,q) = \frac{e_q(iu)}{\cos_q(u)} = \frac{\cos_q(u)+i\sin_q(u)}{\cos_q(u)}.$$

By selecting the real and imaginary parts we get

$$\sum_{n \text{ even}}\frac{(iu)^n}{(q;q)_n}A_n(-q^{-1},1,q) = 1$$

and

$$\sum_{n\ge 0}\frac{u^{2n+1}}{(q;q)_{2n+1}}(-1)^n A_{2n+1}(-q^{-1},1,q) = \frac{\sin_q(u)}{\cos_q(u)} = \tan_q(u).$$

We then conclude:

$$A_{2n}(-q^{-1},1,q) = 0 \quad (n\ge 1),$$
$$(-1)^n A_{2n+1}(-q^{-1},1,q) = T_{2n+1}(q) \quad (n\ge 0).$$ □

3. A direct derivation

A word $w = x_1x_2\cdots x_m$ is called a *hook* if $x_1 > x_2$ and either $m=2$, or $m\ge 3$ and $x_2 < x_3 < \cdots < x_m$. As proved by Gessel [13], each permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ admits a unique factorization, called its *hook factorization*, $p\tau_1\tau_2\cdots\tau_k$, where p is an *increasing* word and each factor $\tau_1, \tau_2, \ldots, \tau_k$ is a hook. To derive the hook factorization of a permutation, it suffices to start from the right and at each step determine the right factor which is a hook. For each i let $\operatorname{inv}\tau_i$ denote the *number of inversions* of τ_i and define:

(3.1) $$\operatorname{lec}\sigma := \sum_{1\le i\le k}\operatorname{inv}\tau_i;$$

(3.2) $$\operatorname{pix}\sigma := \text{length of the factor } p.$$

Those two statistics have been introduced and used in [10]. Furthermore, let Desar_n be the subset of $\mathfrak{S}_n$ of the permutations σ, called *desarrangements*, having the property that $\mathrm{pix}\,\sigma = 0$ [7].

For instance, the hook factorization of the following permutation is indicated by vertical bars: $\sigma = 1\,3\,4\,14 \mid 12\,2\,5\,11\,15 \mid 8\,6\,7 \mid 13\,9\,10$. We have $p = 1\,3\,4\,14$, so that $\mathrm{pix}\,\sigma = 4$, $\mathrm{inv}(12\,2\,5\,11\,15) = 3$, $\mathrm{inv}(8\,6\,7) = 2$, $\mathrm{inv}(13\,9\,10) = 2$, so that $\mathrm{lec}\,\sigma = 7$. The next theorem, already derived in [10], is given a new direct proof.

Theorem 4. *Let*

$$A_n^{\mathrm{lec,pix,inv}}(s, Y, q) := \sum_{\sigma \in \mathfrak{S}_n} s^{\mathrm{lec}\,\sigma} Y^{\mathrm{pix}\,\sigma} q^{\mathrm{inv}\,\sigma}. \tag{3.3}$$

Then

$$\sum_{n \geq 0} \frac{u^n}{(q;q)_n} A_n^{\mathrm{lec,pix,inv}}(s, Y, q) = \frac{1 - sq}{e_q(squ) - sq e_q(u)} e_q(Yu). \tag{3.4}$$

Proof. Let $p\tau_1\tau_2 \cdots \tau_r$ be the hook factorization of a permutation σ from $\mathfrak{S}_n$. Let A_0 (resp. A_i $(1 \leq i \leq r)$) denote the *set* of all letters in the word p (resp. in the hook τ_i) and call *content* of σ the sequence $\mathrm{Cont}\,\sigma = (A_0, A_1, \ldots, A_r)$. Note that $\#A_i \geq 2$ for $i = 1, \ldots, r$ and $(A_0, A_1, \ldots, A_r)$ is an ordered partition of $[n] = \{1, 2, \ldots, n\}$. The statistic $(\mathrm{inv} - \mathrm{lec})\sigma$ is equal to the number of pairs (k, l) such that $k \in A_i$, $l \in A_j$, $k > l$ and $i < j$, a number we shall denote by $\mathrm{inv}(A_0, A_1 \ldots, A_r)$.

If $A_i = \{a_1 < a_2 < \cdots < a_m\}$, then the hooks with content A_i are the $(m-1)$ words: $w_1 = a_m a_1 \cdots a_{m-2} a_{m-1}$, $w_2 = a_{m-1} a_1 \cdots a_{m-2} a_m$, $\ldots$, $w_{m-1} = a_2 a_1 a_3 \cdots a_{m-1} a_m$, whose "lec" statistic and inversion number are both equal to $(m-1)$, $(m-2)$, $\ldots$, 1, respectively. The generating polynomial for those $(m-1)$ hooks by the pair $(\mathrm{lec}, \mathrm{inv})$ is then equal to

$$P_m(s, q) := sq + (sq)^2 + \cdots + (sq)^{m-1} = \frac{sq - (sq)^m}{1 - sq}. \tag{3.5}$$

The identity $\sum_{\substack{(A_0, A_1, \ldots, A_r) \\ \#A_i = a_i}} q^{\mathrm{inv}(A_0, A_1, \ldots, A_r)} = \left[{n \atop a_0, a_1, \ldots, a_r}\right]_q$, where the right-hand side is the q-multinomial coefficient $(q;q)_n / ((q;q)_{a_0} (q;q)_{a_1} \cdots (q;q)_{a_r})$ is easy to verify. We then have:

$$\begin{aligned}
A_n^{\mathrm{lec,pix,inv}}(s, Y, q) &= \sum_{\sigma \in \mathfrak{S}_n} q^{(\mathrm{inv} - \mathrm{lec})\sigma} (sq)^{\mathrm{lec}\,\sigma} Y^{\mathrm{pix}\,\sigma} \\
&= \sum_{(A_0, A_1, \ldots, A_r)} q^{\mathrm{inv}(A_0, A_1, \ldots, A_r)} Y^{\#A_0} \sum_{\mathrm{Cont}\,\sigma = (A_0, A_1, \ldots, A_r)} (sq)^{\mathrm{lec}\,\sigma} \\
&= \sum_{(A_0, A_1, \ldots, A_r)} q^{\mathrm{inv}(A_0, A_1, \ldots, A_r)} Y^{\#A_0} \prod_{1 \leq i \leq r} P_{\#A_i}(s, q) \\
&= \sum_{\substack{a_0 + a_1 + \cdots + a_r = n \\ a_i \geq 2\ (1 \leq i \leq r)}} Y^{a_0} \prod_{1 \leq i \leq r} P_{a_i}(s, q) \sum_{\substack{(A_0, A_1, \ldots, A_r) \\ \#A_i = a_i}} q^{\mathrm{inv}(A_0, A_1, \ldots, A_r)} \\
&= \sum_{\substack{a_0 + a_1 + \cdots + a_r = n \\ a_i \geq 2\ (1 \leq i \leq r)}} \begin{bmatrix} n \\ a_0, a_1, \ldots, a_r \end{bmatrix}_q Y^{a_0} \prod_{1 \leq i \leq r} P_{a_i}(s, q).
\end{aligned}$$

The rest of the calculation is routine:

$$\begin{aligned}\sum_{n\ge 0} A_n^{\mathrm{lec,pix,inv}}(s,Y,q)\frac{u^n}{(q;q)_n} &= \sum_{n\ge 0}\sum_{\substack{a_0+a_1+\cdots+a_r=n\\ a_i\ge 2\ (1\le i\le r)}} Y^{a_0}\frac{u^{a_0}}{(q;q)_{a_0}}\prod_{1\le i\le r} P_{a_i}(s,q)\frac{u^{a_i}}{(q;q)_{a_i}}\\ &= \Bigl(\sum_{a_0\ge 0}\frac{(Yu)^{a_0}}{(q;q)_{a_0}}\Bigr)\Bigl(1-\sum_{b\ge 2} P_b(s,q)\frac{u^b}{(q;q)_b}\Bigr)^{-1}\\ &= e_q(Yu)\Bigl(1-\sum_{b\ge 2}\frac{sq-(sq)^b}{1-sq}\frac{u^b}{(q;q)_b}\Bigr)^{-1}\\ &= e_q(Yu)\frac{1-sq}{e_q(squ)-sqe_q(u)}. \quad\square\end{aligned}$$

In view of (2.1), (2.2), (3.3) and (3.4) the following identity holds

$$\sum_{\sigma\in\mathfrak{S}_n} s^{\mathrm{exc}\,\sigma}Y^{\mathrm{fix}\,\sigma}q^{\mathrm{maj}\,\sigma} = \sum_{\sigma\in\mathfrak{S}_n} s^{\mathrm{lec}\,\sigma}Y^{\mathrm{pix}\,\sigma}q^{\mathrm{inv}\,\sigma} \tag{3.6}$$

and Theorem 1 is proved if we establish the four following identities:

$$\sum_{\sigma\in\mathfrak{S}_{2n}} (-1)^{\mathrm{lec}\,\sigma}q^{(\mathrm{inv}\,-\,\mathrm{lec})\sigma} = 0 \quad (n\ge 1); \tag{3.7}$$

$$\sum_{\sigma\in\mathfrak{S}_{2n+1}} (-1)^{n+\mathrm{lec}\,\sigma}q^{(\mathrm{inv}\,-\,\mathrm{lec})\sigma} = T_{2n+1}(q) \quad (n\ge 0); \tag{3.8}$$

$$\sum_{\sigma\in\mathrm{Desar}_{2n-1}} (-1)^{\mathrm{lec}\,\sigma}q^{(\mathrm{inv}\,-\,\mathrm{lec})\sigma} = 0 \quad (n\ge 1); \tag{3.9}$$

$$\sum_{\sigma\in\mathrm{Desar}_{2n}} (-1)^{n+\mathrm{lec}\,\sigma}q^{(\mathrm{inv}\,-\,\mathrm{lec})\sigma} = E_{2n}(q) \quad (n\ge 0). \tag{3.10}$$

This is the object of the next section.

4. The combinatorial proof

An *alternating* (resp. *falling alternating*) permutation is defined to be a permutation $\sigma=\sigma(1)\cdots\sigma(n)$ having the following properties: $\sigma(1)<\sigma(2)$, $\sigma(2)>\sigma(3)$, $\sigma(3)<\sigma(4)$, etc. (resp. $\sigma(1)>\sigma(2)$, $\sigma(2)<\sigma(3)$, $\sigma(3)>\sigma(4)$, etc.) in an alternating way. The set of alternating (resp. falling alternating) permutations of order n is denoted by $\mathfrak{T}_n$ (resp. by $\mathfrak{T}'_n$). The combinatorial interpretations $\#\mathfrak{T}_{2n+1}=\#\mathfrak{T}'_{2n+1}=T_{2n+1}$, $\#\mathfrak{T}_{2n}=\#\mathfrak{T}'_{2n}=E_{2n}$ are due to Désiré André [1, 2]. For each permutation σ let $\mathrm{inv}\,\sigma$ denote the *number of its inversions*. The following theorem is of common knowledge today, once we know how to q-transpose the calculation made by André in his memoirs (see, e.g., [3], Proposition 4.1.)

Theorem 5. *For each $n\ge 0$ we have*

$$\sum_{\sigma\in\mathfrak{T}_{2n+1}} q^{\mathrm{inv}\,\sigma} = \sum_{\sigma\in\mathfrak{T}'_{2n+1}} q^{\mathrm{inv}\,\sigma} = T_{2n+1}(q),$$

and

$$\sum_{\sigma\in\mathfrak{T}_{2n}} q^{\mathrm{inv}\,\sigma} = E_{2n}(q).$$

We first prove identities (3.9) and (3.10). With the notations of the previous section the content of each desarrangement is of the form $(\emptyset, A_1, \ldots, A_r)$, so that

$$\sum_{\sigma\in\mathrm{Desar}_n}(-1)^{\mathrm{lec}\,\sigma}q^{(\mathrm{inv}\,-\,\mathrm{lec})\sigma} = A_n^{\mathrm{lec,pix,inv}}(-q^{-1},0,q)$$

$$= \sum_{\substack{a_1+\cdots+a_r=n\\ a_i\ge 2}}\ \prod_{1\le i\le r} P_{a_i}(-q^{-1},q)\sum_{\substack{(A_1,\ldots,A_r)\\ \#A_i=a_i}} q^{\mathrm{inv}(A_1,\ldots,A_r)}.$$

From the very definition of $P_m(s,q)$ we have:

$$P_m(-q^{-1},q) = (-1)^{\mathrm{inv}\,w_1}+\cdots+(-1)^{\mathrm{inv}\,w_{m-1}} = \begin{cases} 0, & \text{if } m \text{ odd;}\\ -1, & \text{if } m \text{ even.}\end{cases}$$

Hence, if the ordered partition $(A_1, A_2, \ldots, A_r)$ has at least one block of *odd* cardinality and $\#A_i = a_i$ $(1 \le i \le r)$, the product $\prod_{1\le i\le r} P_{a_i}(-q^{-1},q)$ is null. As each desarrangement from $\mathfrak{S}_{2n-1}$ has at least one hook of odd length, the sum $\sum_{\sigma\in\mathrm{Desar}_{2n-1}}(-1)^{\mathrm{lec}\,\sigma}q^{(\mathrm{inv}\,-\,\mathrm{lec})\sigma}$ is zero. This already proves identity (3.9).

Next, consider the *even* case. If *all* the blocks $A_1, A_2, \ldots, A_r$ of the ordered partition $(A_1, A_2, \ldots, A_r)$ are of *even* cardinality, then

$$\sum_{\sigma\in\mathrm{Cont}(A_1,\ldots,A_r)}(-1)^{n+\mathrm{lec}\,\sigma}q^{(\mathrm{inv}\,-\,\mathrm{lec})\sigma} = (-1)^{n+r}q^{\mathrm{inv}(A_1,\ldots,A_r)}.$$

Hence,

$$\sum_{\sigma\in\mathrm{Desar}_{2n}}(-1)^{n+\mathrm{lec}\,\sigma}q^{(\mathrm{inv}\,-\,\mathrm{lec})\sigma} = \sum_{\substack{(A_1,\ldots,A_r)\\ \#A_i \text{ even}}}(-1)^{n+r}q^{\mathrm{inv}(A_1,\ldots,A_r)}. \tag{4.1}$$

Sign-reversing involution. An ordered partition $\Omega = (A_1, A_2, \ldots, A_r)$ is said to have an *increase* at i if $1 \le i \le r-1$ and $\max A_i < \min A_{i+1}$. If it has no increase and all its blocks A_j are of cardinality 2, then $r = n$ and the corresponding term in (4.1) is equal to $q^{\mathrm{inv}\,\Omega}$. If it is not the case, let i be the integer with the following properties:

(i) $\#A_1 = \cdots = \#A_{i-1} = 2$;
(ii) no increase at $1, 2, \ldots, (i-1)$;
(iii) either $\#A_i \ge 4$, or
(iv) $\#A_i = 2$ and there is an increase at i.

Say that the partition Ω is of class C_i (resp. C_i') if (i), (ii) and (iii) (resp. and (iv)) hold. If Ω is of class C_i, let $A_i = \{a_1 < a_2 < \cdots < a_{2m}\}$ $(m \ge 2)$ and form $\Omega' = (A_1, \ldots, A_{i-1}, \{a_1, a_2\}, \{a_3, \ldots, a_{2m}\}, A_{i+1}, \ldots, A_r)$. Then, Ω' is of class C_i'. Thus $\Omega \mapsto \Omega'$ is an involution of C_i onto C_i', which is sign-reversing since $\mathrm{inv}\,\Omega' = \mathrm{inv}\,\Omega$ and $(-1)^{n+r}q^{\mathrm{inv}\,\Omega} + (-1)^{n+r+1}q^{\mathrm{inv}\,\Omega'} = 0$.

By applying the involution $\Omega \mapsto \Omega'$, the only partitions $(A_1, \ldots, A_r)$ remaining in (4.1) are of the form $\Omega = (\{a_1 < b_1\}, \{a_2 < b_2\}, \ldots, \{a_n < b_n\})$ having the property that for each $i = 2, \ldots, n$ the relation $b_{i-1} > a_i$ holds. Hence, the permutation $\omega = a_1b_1a_2b_2\cdots a_nb_n$ is an *alternating* permutation starting with a rise $a_1 < b_1$. We have then proved

$$\sum_{\sigma\in\mathrm{Desar}_{2n}}(-1)^{n+\mathrm{lec}\,\sigma}q^{(\mathrm{inv}\,-\,\mathrm{lec})\sigma} = \sum_{\omega} q^{\mathrm{inv}\,\omega},$$

where the sum is over all alternating permutations of order $2n$, which establishes identity (3.10) by Theorem 5.

Next, consider the left-hand side LHS of (3.8) (resp. of (3.7)). The sum may be regarded as being over all hook factorizations $p\tau_1\tau_2\cdots\tau_r$ of permutations from $\mathfrak{S}_{2n+1}$ (resp. $\mathfrak{S}_{2n}$). Let A_0 be the set of the letters in p and, as before, let A_i be the set of all letters in τ_i $(1\le i\le r)$. As above, we may apply the involution $\Omega\mapsto\Omega'$ to all contents $\Omega=(A_1,A_2,\dots,A_r)$, remembering that this time Ω is an ordered partition of $[2n+1]\setminus A_0$ (resp. of $[2n]\setminus A_0$). After applying $\Omega\mapsto\Omega'$ we get

$$\text{LHS}=\sum_{(A_0,A_1,\dots,A_r)}(-1)^{n+r}q^{\operatorname{inv}(A_0,A_1,\dots,A_r)}, \tag{4.2}$$

where the sum is over all ordered partitions $\Theta=(A_0,A_1,\dots,A_r)$ of $[2n+1]$ (resp. of $[2n]$ with the convention that A_0 may be empty), having the property that $(A_1,\dots,A_r)$ is an ordered partition of a subset of $[2n+1]$ (resp. of $[2n]$) into blocks of cardinality 2 having no increase.

Another sign-reversing involution. Each ordered partition $\Theta=(A_0,A_1,\dots,A_r)$ of $[2n+1]$ (resp. of $[2n]$) is said to be of type D, if $\max A_0<\min A_1$ (by convention, $\max\emptyset=-\infty$). It is of type D'' if $\max A_0<\min A_1$ does not hold and $\#A_0\ge 2$. If Θ is of type D, define $\Theta''=(A_0\cup A_1,A_2,\dots,A_r)$. Then, Θ'' is of type D'' and $\operatorname{inv}\Theta''=\operatorname{inv}\Theta$, so that

$$(-1)^{n+r}q^{\operatorname{inv}\Theta}+(-1)^{n+r-1}q^{\operatorname{inv}\Theta''}=0. \tag{4.3}$$

Thus $\Theta\mapsto\Theta''$ is a sign-reversing involution.

After applying the transformation $\Theta\mapsto\Theta''$ to the summands in (4.2) there remains no term if the sum is made over the ordered partitions of $[2n]$. This proves identity (3.7). When making the sum over ordered partitions of $[2n+1]$, the remaining terms correspond to the ordered partitions $(A_0,A_1,\dots,A_r)$ having no increase and such that A_0 is a singleton. In particular, $r=n$. Such a partition is of the form $(\{a_0\},\{a_1<b_1\},\{a_2<b_2\},\dots,\{a_n<b_n\})$ having the property that $a_0>a_1$ and for each $i=2,\dots,n$ the relation $b_{i-1}>a_i$ holds. Hence $\omega=a_0a_1b_1a_2b_2\cdots a_nb_n$ is a *falling alternating* permutation. We have then proved

$$\sum_{\sigma\in\mathfrak{S}_{2n+1}}(-1)^{n+\operatorname{lec}\sigma}q^{(\operatorname{inv}-\operatorname{lec})\sigma}=\sum_{\omega\in\mathfrak{T}'_{2n+1}}q^{\operatorname{inv}\omega}\quad(n\ge 0).$$

This establishes (3.8) by Theorem 5.

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