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# Hook lengths and shifted parts of partitions

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## 1. Introduction

The hook lengths of partitions are widely studied in the Theory of Partitions, in Algebraic Combinatorics and Group Representation Theory. The basic notions needed here can be found in [St99, p.287; La01, p.1]. A *partition* $\lambda$ is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The integers $(\lambda_i)_{i=1,2,\ldots,\ell}$ are called the *parts* of $\lambda$, the number $\ell$ of parts being the *length* of $\lambda$ denoted by $\ell(\lambda)$. The sum of its parts $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ is denoted by $|\lambda|$. Let $n$ be an integer, a partition $\lambda$ is said to be a partition of $n$ if $|\lambda| = n$. We write $\lambda \vdash n$. Each partition can be represented by its Ferrers diagram. For each box $v$ in the Ferrers diagram of a partition $\lambda$, or for each box $v$ in $\lambda$, for short, define the *hook length* of $v$, denoted by $h_v(\lambda)$ or $h_v$, to be the number of boxes $u$ such that $u = v$, or $u$ lies in the same column as $v$ and above $v$, or in the same row as $v$ and to the right of $v$. The product of all hook lengths of $\lambda$ is denoted by $H_\lambda$.

The hook length plays an important role in Algebraic Combinatorics thanks to the famous hook formula due to Frame, Robinson and Thrall [FRT54]

$$f_\lambda = \frac{n!}{H_\lambda}, \tag{1.1}$$

where $f_\lambda$ is the number of standard Young tableaux of shape $\lambda$.

For each partition $\lambda$ let $\lambda \setminus 1$ be the set of all partitions $\mu$ obtained from $\lambda$ by erasing one *corner* of $\lambda$. By the very construction of the standard Young tableaux and (1.1) we have

$$f_\lambda = \sum_{\mu \in \lambda \setminus 1} f_\mu \tag{1.2}$$

and then

$$\frac{n}{H_\lambda} = \sum_{\mu \in \lambda \setminus 1} \frac{1}{H_\mu}. \tag{1.3}$$

In this note we establish the following perturbation of formula (1.3). Define the *g-function* of a partition $\lambda$ of $n$ to be

$$g_\lambda(x) = \prod_{i=1}^{n} (x + \lambda_i - i), \tag{1.4}$$

where $\lambda_i = 0$ for $i \geq \ell(\lambda) + 1$.

**Theorem 1.1.** *Let $x$ be a formal parameter. For each partition $\lambda$ we have*

$$\frac{g_\lambda(x+1) - g_\lambda(x)}{H_\lambda} = \sum_{\mu \in \lambda \setminus 1} \frac{g_\mu(x)}{H_\mu}. \tag{1.5}$$

Theorem 1.1 is proved in Section 2. Some equivalent forms of Theorem 1.1 and remarks are given in Section 4. As an application we prove (see Section 3) the following result due to Stanley [St08].

**Theorem 1.2.** *Let $p, e$ and $s$ be the usual symmetric functions [Ma95, Chap.I]. Then*

$$\sum_{k=0}^{n} \binom{x+k-1}{k} p_1^k e_{n-k} = \sum_{\lambda \vdash n} H_\lambda^{-1} g_\lambda(x+n) s_\lambda. \tag{1.6}$$

Recently, the author stated some conjectures on partition hook lengths [Ha08a], which were discovered by hook length expansion techniques (see [Ha08b]). Later, Conjecture 3.1 in [Ha08a] was proved by Stanley [St08]. One step of his proof is formula (1.6), whose original proof made use of a result of Andrews, Goulden and Jackson [AGJ88]. Stanley asked for a simple and direct proof of (1.6).

## 2. Proof of Theorem 1.1

Let

$$\epsilon(x) = \frac{g_\lambda(x+1) - g_\lambda(x)}{H_\lambda} - \sum_{\mu \in \lambda \setminus 1} \frac{g_\mu(x)}{H_\mu}. \tag{2.1}$$

We see that $\epsilon(x)$ is a polynomial in $x$ whose degree is less than or equal to $n$. Moreover

$$[x^n]\epsilon(x) = [x^n]\frac{g_\lambda(x+1) - g_\lambda(x)}{H_\lambda} = 0.$$

Furthermore,

$$[x^{n-1}]g_\lambda(x+1) = \sum_{i=1}^{n}(\lambda_i - i + 1) = n + \sum_{i=1}^{n}(\lambda_i - i) = n + [x^{n-1}]g_\lambda(x)$$

and

$$[x^{n-1}]\epsilon(x) = [x^{n-1}]\frac{g_\lambda(x+1) - g_\lambda(x)}{H_\lambda} - \sum_{\mu\in\lambda\backslash 1}\frac{1}{H_\mu} = \frac{n}{H_\lambda} - \sum_{\mu\in\lambda\backslash 1}\frac{1}{H_\mu} = 0.$$

The last equality is guaranteed by (1.3), so that $\epsilon(x)$ is a polynomial in $x$ whose degree is less than and equal to $n-2$. To prove that $\epsilon(x)$ is actually zero, it suffices to find $n-1$ distinct values for $x$ such that $\epsilon(x) = 0$. In the following we prove that $\epsilon(i-\lambda_i) = 0$ for $i - \lambda_i$ for $i = 1, 2, \ldots, n-1$.

If $\lambda_i = \lambda_{i+1}$, or if the $i$-th row has no corner, the factor $x + \lambda_i - i$ lies in $g_\lambda(x)$ and also in $g_\mu(x)$ for all $\mu \in \lambda\backslash 1$ . The factor $(x+1)+\lambda_{i+1}-(i+1) = x + \lambda_i - i$ is furthermore in $g_\lambda(x+1)$, so that $\epsilon(i-\lambda_i) = 0$.

Next, if $\lambda_i \geq \lambda_{i+1}+1$, or if the $i$-th row has a corner, the factor $x+\lambda_i-i$ lies in $g_\lambda(x)$ and $g_\mu(x)$ for all $\mu \in \lambda \setminus 1$, except for $\mu = \lambda'$, which is the partition obtained from $\lambda$ by erasing the corner from the $i$-th row. In this case equality (2.1) becomes

$$\epsilon(i-\lambda_i) = \frac{g_\lambda(i-\lambda_i+1)}{H_\lambda} - \frac{g_{\lambda'}(i-\lambda_i)}{H_{\lambda'}}.$$

For proving Theorem 1.1, it remains to prove $\epsilon(i-\lambda_i) = 0$ or

$$\frac{H_\lambda}{H_{\lambda'}} = \frac{g_\lambda(i-\lambda_i+1)}{g_{\lambda'}(i-\lambda_i)}. \tag{2.2}$$

Consider the following product

$$\frac{g_\lambda(x+1)}{g_{\lambda'}(x)} = \frac{\prod_{j=1}^{n}(x+\lambda_j-j+1)}{\prod_{j=1}^{n-1}(x+\lambda'_j-j)}. \tag{2.3}$$

The set of all $1 \leq j \leq n-1$ such that $\lambda_j > \lambda_{j+1}$ is denoted by $\mathcal{T}$. For $1 \leq j \leq n-1$ and $j \notin \mathcal{T}$ (which implies that $j \neq i$ and $\lambda'_j = \lambda_j = \lambda_{j+1}$), the numerator contains $x + \lambda_{j+1} - (j+1) + 1 = x + \lambda_j - j$ and the denominator also contains $x + \lambda'_j - j = x + \lambda_j - j$. After cancellation of those common factors, (2.3) becomes

$$\frac{g_\lambda(x+1)}{g_{\lambda'}(x)} = \frac{\prod_{j\in\mathcal{B}}(x+\lambda_j-j+1)}{\prod_{j\in\mathcal{T}}(x+\lambda'_j-j)} \tag{2.4}$$

where $\mathcal{B} = \{1\} \cup \{i+1 \mid i \in \mathcal{T}\}$. Letting $x = i - \lambda_i$ in (2.4) yields

$$\frac{g_\lambda(i-\lambda_i+1)}{g_{\lambda'}(i-\lambda_i)} = \frac{\prod_{j\in\mathcal{B}}(i-\lambda_i+\lambda_j-j+1)}{\prod_{j\in\mathcal{T}}(i-\lambda_i+\lambda'_j-j)}. \tag{2.5}$$

We distinguish the factors in the right-hand side of (2.5) as follows.

(C1) For $j \in \mathcal{B}$ and $j > i$, $i-\lambda_i+\lambda_j-j+1 = -(\lambda_i-\lambda_j+j-i-1) = -h_v(\lambda)$, where $v$ is the box $(i, \lambda_j+1)$ in $\lambda$.

(C2) For $j \in \mathcal{B}$ and $j \le i$, $i-\lambda_i+\lambda_j-j+1 = h_v(\lambda)$, where $v$ is the box $(j, \lambda_i)$ in $\lambda$.

(C3) For $j \in \mathcal{T}$ and $j > i$, $i-\lambda_i+\lambda_j-j = -(\lambda_i-\lambda_j+j-i) = -h_u(\lambda')$, where $u$ is the box $(i, \lambda_j)$ in $\lambda'$.

(C4) For $j \in \mathcal{T}$ and $j < i$, $i-\lambda_i+\lambda_j-j = h_u(\lambda')$, where $u$ is the box $(j, \lambda_i)$ in $\lambda'$.

(C5) For $j \in \mathcal{T}$ and $j = i$, $i-\lambda_i+\lambda'_j-j = i-\lambda_i+\lambda'_i-i = -1$. See Fig. 2.3 and 2.4 for an example.

Since each $j \in \mathcal{B}$ such that $j > i$ is associated with $j-1 \in \mathcal{T}$ and $j-1 \ge i$, the right-hand side of (2.5) is positive and can be re-written

$$\frac{g_\lambda(i-\lambda_i+1)}{g_{\lambda'}(i-\lambda_i)} = \frac{\prod_v h_v(\lambda)}{\prod_u h_u(\lambda')}, \tag{2.6}$$

where $v, u$ range over the boxes described in (C1)-(C4). Finally $H_\lambda/H_{\lambda'}$ is equal to the right-hand side of (2.6), since the hook lengths of all other boxes cancel. We have completed the proof of (2.2). ∎

For example, consider the partition $\lambda = 55331$ and $i = 4$. We have $\lambda' = 55321$ and

$$\frac{H_\lambda}{H_{\lambda'}} = \frac{4\cdot 2\cdot 1\cdot 2\cdot 5\cdot 6}{3\cdot 1\cdot 1\cdot 4\cdot 5}. = \frac{4\cdot 2\cdot 2\cdot 6}{3\cdot 4}.$$

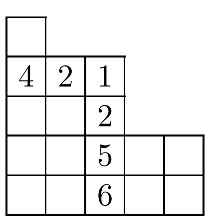

Fig. 2.1. Hook lengths of $\lambda$

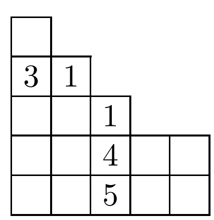

Fig. 2.2. Hook lengths of $\lambda'$

On the other hand, $\mathcal{T} = \{2,4,5\}$, $\mathcal{B} = \{1,3,5,6\}$ and

$$\frac{g_\lambda(x+1)}{g_{\lambda'}(x)} = \frac{(x+5)(x+1)(x-3)(x-5)}{(x+3)(x-2)(x-4)}.$$

Letting $x = i - \lambda_i = 4 - 3 = 1$ yields

$$\frac{g_\lambda(2)}{g_{\lambda'}(1)} = \frac{(6)(2)(-2)(-4)}{(4)(-1)(-3)} = \frac{6\cdot 2\cdot 2\cdot 4}{4\cdot 3}.$$

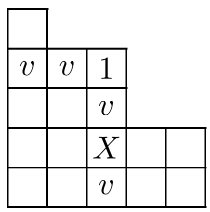

Fig. 2.3. The boxes $v$ in $\lambda$

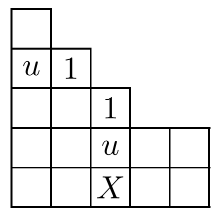

Fig. 2.4. The boxes $u$ in $\lambda'$

## 3. Proof of Theorem 1.2

Let $R_n(x)$ be the right-hand side of (1.6). By Theorem 1.1

$$\begin{aligned} R_n(x) &= \sum_{\lambda\vdash n}\Big(\frac{g_\lambda(x+n-1)}{H_\lambda} + \sum_{\mu\in\lambda\backslash 1}\frac{g_\mu(x+n-1)}{H_\mu}\Big)s_\lambda \\ &= R_n(x-1) + \sum_{\lambda\vdash n}\sum_{\mu\in\lambda\backslash 1}\frac{g_\mu(x+n-1)}{H_\mu}s_\lambda \\ &= R_n(x-1) + \sum_{\mu\vdash n-1}\sum_{\lambda\,:\,\mu\in\lambda\backslash 1}\frac{g_\mu(x+n-1)}{H_\mu}s_\lambda \\ &= R_n(x-1) + \sum_{\mu\vdash n-1}\frac{g_\mu(x+n-1)}{H_\mu}p_1 s_\mu, \end{aligned}$$

where the next to last equality is

$$\sum_{\lambda\,:\,\mu\in\lambda\backslash 1} s_\lambda = p_1 s_\mu$$

by using Pieri's rule [Ma95, p.73]. We obtain the following recurrence for $R_n(x)$.

(3.1) $$R_n(x) = R_n(x-1) + p_1 R_{n-1}(x).$$

Let $L_n(x)$ be the left-hand side of (1.6). Using elementary properties of binomial coefficients

$$\begin{aligned}L_n(x) &= \sum_{k=0}^{n}\binom{x+k-1}{k}p_1^k e_{n-k}\\ &= e_n + \sum_{k=1}^{n}(\binom{x+k-2}{k}+\binom{x+k-2}{k-1})p_1^k e_{n-k}\\ &= L_n(x-1) + p_1\sum_{k=1}^{n}\binom{x+k-2}{k-1}p_1^{k-1}e_{n-k}\\ &= L_n(x-1)+p_1 L_{n-1}(x). \qquad (3.2)\end{aligned}$$

We verify that $L_1(x) = R_1(x)$ and $L_n(0) = R_n(0)$, so that $L_n(x) = R_n(x)$ by (3.1) and (3.2). ∎

## 4. Equivalent forms and further remarks

Let $\lambda = \lambda_1\lambda_2\cdots\lambda_\ell$ be a partition of $n$. The set of all $1 \le j \le n$ such that $\lambda_j > \lambda_{j+1}$ is denoted by $\mathcal{T}$ and let $\mathcal{B} = \{1\} \cup \{i+1 \mid i \in \mathcal{T}\}$. Those two sets can be viewed as the *in-corner* and *out-corner* index sets, respectively. Notice that $\#\mathcal{B} = \#\mathcal{T}+1$. For each $i \in \mathcal{T}$ we define $\lambda^{i-}$ to be the partition of $n-1$ obtained form $\lambda$ by erasing the right-most box from the $i$-th row. Hence

$$\lambda \setminus 1 = \{\lambda^{i-} \mid i \in \mathcal{T}\}. \qquad (4.1)$$

We verify that

$$g_{\lambda^{i-}}(x) = \frac{g_\lambda(x)(x+\lambda_i-i-1)}{(x+\lambda_i-i)(x-n)}. \qquad (4.2)$$

From Theorem 1.1

$$\frac{g_\lambda(x+1)-g_\lambda(x)}{H_\lambda} = \sum_{i\in\mathcal{T}}\frac{g_\lambda(x)(x+\lambda_i-i-1)}{(x+\lambda_i-i)(x-n)}\frac{1}{H_\lambda^{i-}}$$

or

$$\sum_{i\in\mathcal{T}}\frac{H_\lambda}{H_{\lambda^{i-}}}\times(1-\frac{1}{x+\lambda_i-i}) = n-x+\frac{(x-n)g_\lambda(x+1)}{g_\lambda(x)}. \qquad (4.3)$$

Let us re-write (1.3)

$$\sum_{\mu\in\lambda\setminus 1}\frac{H_\lambda}{H_\mu} = n. \qquad (4.4)$$

By subtracting (4.3) from (4.4) we obtain the following equivalent form of Theorem 1.1.

**Theorem 4.1.** *We have*

$$\sum_{i\in\mathcal{T}} \frac{H_\lambda}{H_{\lambda^{i-}}} \times \frac{1}{x+\lambda_i-i} = x - \frac{(x-n)g_\lambda(x+1)}{g_\lambda(x)}. \tag{4.5}$$

By the definitions of $\mathcal{T}$ and $\mathcal{B}$ we have

$$\frac{(x-n)g_\lambda(x+1)}{g_\lambda(x)} = \frac{\prod_{i\in\mathcal{B}}(x+\lambda_i-i+1)}{\prod_{i\in\mathcal{T}}(x+\lambda_i-i)}, \tag{4.6}$$

so that Theorem 1.1 is also equivalent to the following result.

**Theorem 4.2.** *We have*

$$\sum_{i\in\mathcal{T}} \frac{H_\lambda}{H_{\lambda^{i-}}} \times \frac{1}{x+\lambda_i-i} = x - \frac{\prod_{i\in\mathcal{B}}(x+\lambda_i-i+1)}{\prod_{i\in\mathcal{T}}(x+\lambda_i-i)}. \tag{4.7}$$

For example, take $\lambda = 55331$. Then $\mathcal{T} = 2,4,5$ and $\mathcal{B} = 1,3,5,6 = \{1, 2+1, 4+1, 5+1\}$.

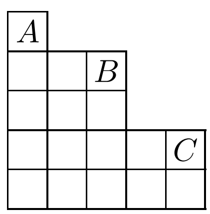


Fig. 4.1. in-corner

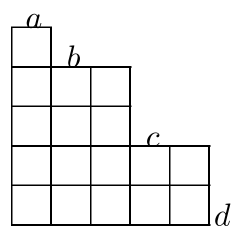


Fig. 4.2. out-corner

Hence $\lambda^{2-} = 54331$, $\lambda^{4-} = 55321$ and $\lambda^{5-} = 55330$. Equality (4.7) becomes

$$\begin{aligned} \frac{H_\lambda}{H_{\lambda^{5-}}} \times \frac{1}{x-4} + \frac{H_\lambda}{H_{\lambda^{4-}}} \times \frac{1}{x-1} + \frac{H_\lambda}{H_{\lambda^{2-}}} \times \frac{1}{x+3} \\ = x - \frac{(x-5)(x-3)(x+1)(x+5)}{(x-4)(x-1)(x+3)}. \\ = \frac{17x^2-38x-75}{(x-4)(x-1)(x+3)}. \end{aligned}$$

Theorems 4.1 and 4.2 can be proved directly using the method used in the proof of Theorem 1.1. First, we must verify that the numerator in the right-hand side of (4.5) is a polynomial in $x$ whose degree is less than $(\le)$ $\#\mathcal{T}-1$. By the partial fraction expansion technique it suffices to verify that (4.7) is true for all $x = i-\lambda_i$ $(i\in\mathcal{T})$. This direct proof contains the main part of the proof of Theorem 1.1. However it does not make use of the fundamental relation (1.3) or (4.4). Thus, the following corollary of Theorem 4.2 makes sense.

**Corollary 4.4.** *We have*

$$\sum_{\mu\in\lambda\setminus 1}\frac{H_\lambda}{H_\mu}=n. \tag{4.8}$$

*Proof.* Let $\#\mathcal{T}=k$. The right-hand side of (4.7) has the following form

$$\frac{Cx^{k-1}+\cdots}{x^k+\cdots}.$$

We now evaluate the coefficient $C$. By (4.6) we can write $C=A-B$ with

$$A=[x^{n-1}]x\prod_{i=1}^{n}(x+\lambda_i-i)=\sum_{1\le i<j\le n}(\lambda_i-i)(\lambda_j-j)$$

and

$$\begin{aligned}B&=[x^{n-1}](x-n)\prod_{i=1}^{n}(x+\lambda_i-i+1)\\&=\sum_{1\le i<j\le n}(\lambda_i-i+1)(\lambda_j-j+1)-n\sum_{1\le i\le n}(\lambda_i-i+1).\\&=B_1-n\sum_{1\le i\le n}(\lambda_i-i+1),\end{aligned}$$

where

$$\begin{aligned}B_1&=\sum_{1\le i<j\le n}(\lambda_i-i+1)(\lambda_j-j+1)\\&=\sum_{1\le i<j\le n}\Big((\lambda_i-i)(\lambda_j-j)+(\lambda_i-i)+(\lambda_j-j)+1\Big)\\&=A+\sum_{1\le i<j\le n}(\lambda_i-i)+\sum_{1\le i<j\le n}(\lambda_j-j)+\binom{n}{2}\\&=A+\sum_{1\le i\le n}(n-i)(\lambda_i-i)+\sum_{1\le j\le n}(j-1)(\lambda_j-j)+\binom{n}{2}\\&=A+\sum_{1\le i\le n}(n-1)(\lambda_i-i)+\binom{n}{2}.\end{aligned}$$

Finally

$$C=A-B$$

$$= -\sum_{1\le i\le n}(n-1)(\lambda_i - i) - \binom{n}{2} + n\sum_{1\le i\le n}(\lambda_i - i + 1)$$
$$= -\sum_{1\le i\le n} n(\lambda_i - i) + \sum_{1\le i\le n}(\lambda_i - i) - \binom{n}{2} + n\sum_{1\le i\le n}(\lambda_i - i) + n^2$$
$$= \sum_{1\le i\le n}(\lambda_i - i) - \binom{n}{2} + n^2$$
$$= n - \binom{n+1}{2} - \binom{n}{2} + n^2 = n. \quad \square$$

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