

HOOK LENGTHS AND 3-CORES

GUO-NIU HAN AND KEN ONO

ABSTRACT. Recently, the first author generalized a formula of Nekrasov and Okounkov which gives a combinatorial formula, in terms of hook lengths of partitions, for the coefficients of certain power series. In the course of this investigation, he conjectured that $a(n) = 0$ if and only if $b(n) = 0$, where integers $a(n)$ and $b(n)$ are defined by

$$\begin{aligned} \sum_{n=0}^{\infty} a(n)x^n &:= \prod_{n=1}^{\infty} (1-x^n)^8, \\ \sum_{n=0}^{\infty} b(n)x^n &:= \prod_{n=1}^{\infty} \frac{(1-x^{3n})^3}{1-x^n}. \end{aligned}$$

The numbers $a(n)$ are given in terms of hook lengths of partitions, while $b(n)$ equals the number of 3-core partitions of n . Here we prove this conjecture.

1. INTRODUCTION AND STATEMENT OF RESULTS

In their work on random partitions and Seiberg-Witten theory, Nekrasov and Okounkov [8] proved the following striking formula:

$$(1.1) \quad F_z(x) := \sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{n=1}^{\infty} (1-x^n)^{z-1}.$$

Here the sum is over integer partitions λ , $|\lambda|$ denotes the integer partitioned by λ , and $\mathcal{H}(\lambda)$ denotes the multiset of classical hooklengths associated to a partition λ . In a recent preprint, the first author [3] has obtained an extension of (1.1), one which has a specialization which gives the classical generating function

$$(1.2) \quad C_t(x) := \sum_{n=0}^{\infty} c_t(n)x^n = \prod_{n=1}^{\infty} \frac{(1-x^{tn})^t}{1-x^n}$$

for the number of t -core partitions of n . Recall that a partition is a t -core if none of its hook lengths are multiples of t .

In the course of his work, the first author [4] formulated a number of conjectures concerning hook lengths of partitions. One of these conjectures is related to classical identities of Jacobi. For positive integers t , he compared the functions $F_{t^2}(x)$ and $C_t(x)$. If $t = 1$, we obviously have that

$$F_1(x) = C_1(x) = 1.$$

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For $t = 2$, by two famous identities of Jacobi, we have

$$F_4(x) = \prod_{n=1}^{\infty} (1 - x^n)^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1) x^{(k^2+k)/2},$$

$$C_2(x) = \prod_{n=1}^{\infty} \frac{(1 - x^{2n})^2}{1 - x^n} = \sum_{k=0}^{\infty} x^{(k^2+k)/2}.$$

In both pairs of power series one sees that the non-zero coefficients are supported on the same terms. For $t = 3$, we then have

$$(1.3) \quad F_9(x) = \sum_{n=0}^{\infty} a(n)x^n := \prod_{n=1}^{\infty} (1 - x^n)^8$$

$$= 1 - 8x + 20x^2 - 70x^4 + \cdots - 520x^{14} + 57x^{16} + 560x^{17} + 182x^{20} + \cdots$$

and

$$(1.4) \quad C_3(x) = \sum_{n=0}^{\infty} b(n)x^n := \prod_{n=1}^{\infty} \frac{(1 - x^{3n})^3}{1 - x^n}$$

$$= 1 + x + 2x^2 + 2x^4 + \cdots + 2x^{14} + 3x^{16} + 2x^{17} + 2x^{20} + \cdots .$$

Remark. It is clear that $b(n) = c_3(n)$.

In accordance with the elementary observations when $t = 1$ and 2, one notices that the non-zero coefficients of $F_9(x)$ and $C_3(x)$ appear to be supported on the same terms. Based on substantial numerical evidence, the first author made the following conjecture.

Conjecture 4.6. (Conjecture 4.6 of [4])

Assuming the notation above, we have that $a(n) = 0$ if and only if $b(n) = 0$.

Remark. The obvious generalization of Conjecture 4.6 and the examples above is not true for $t = 4$. In particular, one easily finds that

$$F_{16}(x) = 1 - 15x + 90x^2 - \cdots + 641445x^{52} + 1537330x^{54} + \cdots ,$$

$$C_4(x) = 1 + x + 2x^2 + 3x^3 + \cdots + 5x^{52} + 8x^{53} + 10x^{54} + \cdots .$$

The coefficient of x^{53} vanishes in $F_{16}(x)$ and is non-zero in $C_4(x)$.

Here we prove that Conjecture 4.6 is true. We have the following theorem.

Theorem 1.1. *Assuming the notation above, we have that $a(n) = 0$ if and only if $b(n) = 0$. Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $\text{ord}_p(3n+1)$ is odd for some prime $p \equiv 2 \pmod{3}$.*

Remark. As usual, $\text{ord}_p(N)$ denotes the power of a prime p dividing an integer N .

Remark. Theorem 1.1 shows that $a(n) = b(n) = 0$ in a systematic way. The vanishing coefficients are associated to primes $p \equiv 2 \pmod{3}$. If $n \equiv 1 \pmod{3}$ has the property that $\text{ord}_p(n)$ is odd, then we have

$$a\left(\frac{n-1}{3}\right) = b\left(\frac{n-1}{3}\right) = 0.$$

For example, since $\text{ord}_5(10) = 1 \equiv 1 \pmod{2}$, we have that $a(3) = b(3) = 0$.

As an immediate corollary, we have the following.

Corollary 1.2. *For positive integers N , we have that*

$$\sum_{\lambda \vdash N} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{9}{h^2}\right) = 0$$

if and only if there are no 3-core partitions of N .

Theorem 1.1 implies that “almost all” of the $a(n)$ and $b(n)$ are 0. More precisely, we have the following.

Corollary 1.3. *Assuming the notation above, we have that*

$$\lim_{X \rightarrow +\infty} \frac{\#\{0 \leq n \leq X : a(n) = b(n) = 0\}}{X} = 1.$$

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2. PROOFS

It is convenient to renormalize the functions $a(n)$ and $b(n)$ using the series

$$\begin{aligned} (2.1) \quad \mathcal{A}(z) &= \sum_{n=1}^{\infty} a^*(n)q^n := \sum_{n=0}^{\infty} a(n)q^{3n+1} \\ &= q - 8q^4 + 20q^7 - 70q^{13} + 64q^{16} + 56q^{19} - 125q^{25} - 160q^{28} + \dots \end{aligned}$$

and

$$\begin{aligned} (2.2) \quad \mathcal{B}(z) &= \sum_{n=1}^{\infty} b^*(n)q^n := \sum_{n=0}^{\infty} b(n)q^{3n+1} \\ &= q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + q^{25} + 2q^{28} + \dots \end{aligned}$$

Here we have that $z \in \mathbb{H}$, the upper-half of the complex plane, and we let $q := e^{2\pi iz}$. We make these changes since $\mathcal{A}(z)$ and $\mathcal{B}(z)$ are examples of two special types of modular forms (for background on modular forms, see [1, 6, 7, 9]). The modularity of these two series follows easily from the properties of Dedekind’s eta-function

$$(2.3) \quad \eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

The proofs of Theorem 1.1 and Corollary 1.3 shall rely on exact formulas we derive for the numbers $a^*(n)$ and $b^*(n)$.

2.1. **Exact formulas for $a^*(n)$.** The modular form $\mathcal{A}(z)$ given by

$$\mathcal{A}(z) = \eta(3z)^8 = \sum_{n=1}^{\infty} a^*(n)q^n$$

is in $S_4(\Gamma_0(9))$, the space of weight 4 cusp forms on $\Gamma_0(9)$. This space is one dimensional (see Section 1.2.3 in [9]). Therefore, every cusp form in the space is a multiple of $\mathcal{A}(z)$. It turns out that $\mathcal{A}(z)$ is a form with *complex multiplication*.

We now briefly recall the notion of a newform with complex multiplication (for example, see Chapter 12 of [6] or Section 1.2 of [9], [10]). Let $D < 0$ be the fundamental discriminant of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. Let O_K be the ring of integers of K , and let $\chi_K := \left(\frac{D}{\cdot}\right)$ be the usual Kronecker character associated to K . Let $k \geq 2$, and let c be a Hecke character of K with exponent $k - 1$ and conductor \mathfrak{f}_c , a non-zero ideal of O_K . By definition, this means that

$$c : I(\mathfrak{f}_c) \longrightarrow \mathbb{C}^\times$$

is a homomorphism, where $I(\mathfrak{f}_c)$ denotes the group of fractional ideals of K prime to \mathfrak{f}_c . In particular, this means that

$$c(\alpha O_K) = \alpha^{k-1}$$

for $\alpha \in K^\times$ for which $\alpha \equiv 1 \pmod{\mathfrak{f}_c}$. To c we naturally associate a Dirichlet character ω_c defined, for every integer n coprime to \mathfrak{f}_c , by

$$\omega_c(n) := \frac{c(nO_K)}{n^{k-1}}.$$

Given this data, we let

$$(2.4) \quad \Phi_{K,c}(z) := \sum_{\mathfrak{a}} c(\mathfrak{a})q^{N(\mathfrak{a})},$$

where \mathfrak{a} varies over the ideals of O_K prime to \mathfrak{f}_c , and where $N(\mathfrak{a})$ is the usual ideal norm. It is known that $\Phi_{K,c}(z) \in S_k(|D| \cdot N(\mathfrak{f}_c), \chi_K \cdot \omega_c)$ is a normalized newform.

Using this theory, we obtain the following theorem.

Theorem 2.1. *Assume the notation above. Then the following are true:*

- (1) *If $p = 3$ or $p \equiv 2 \pmod{3}$ is prime, then $a^*(p) = 0$.*
- (2) *If $p \equiv 1 \pmod{3}$ is prime, then*

$$a^*(p) = 2x^3 - 18xy^2,$$

where x and y are integers for which $p = x^2 + 3y^2$ with $x \equiv 1 \pmod{3}$.

Remark. It is a classical fact that every prime $p \equiv 1 \pmod{3}$ is of the form $x^2 + 3y^2$. Moreover, there is a unique pair of positive integers x and y for which $x^2 + 3y^2 = p$. Therefore, the formula for $a^*(p)$ is well defined.

Proof. There is a form with complex multiplication in $S_4(\Gamma_0(9))$. Following the recipe above, it is obtained by letting $k = 4$, $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{-3})$ and $f_c := (\sqrt{-3})$. For primes p , the coefficients of q^p in this form agree with the claimed formulas. Since $S_4(\Gamma_0(9))$ is one dimensional, this form must be $\mathcal{A}(z)$. \square

Using this theorem, we obtain the following immediate corollary.

Corollary 2.2. *The following are true about $a^*(n)$.*

(1) *If m and n are coprime positive integers, then*

$$a^*(mn) = a^*(m)a^*(n).$$

(2) *For every positive integer s , we have that $a^*(3^s) = 0$.*

(3) *If $p \equiv 2 \pmod{3}$ is prime and s is a positive integer, then*

$$a^*(p^s) = \begin{cases} 0 & \text{if } s \text{ is odd,} \\ (-1)^{s/2} p^{3s/2} & \text{if } s \text{ is even.} \end{cases}$$

(4) *If $p \equiv 1 \pmod{3}$ is prime and s is a positive integer, then $a^*(p^s) \neq 0$. Moreover, we have that*

$$a^*(p^s) \equiv (8x^3)^s \pmod{p},$$

where $p = x^2 + 3y^2$ with $x \equiv 1 \pmod{3}$.

Proof. Since $S_4(\Gamma_0(9))$ is one dimensional and since $a^*(1) = 1$, it follows that $\mathcal{A}(z)$ is a normalized Hecke eigenform. Claim (1) is well known to hold for all normalized Hecke eigenforms.

Claim (2) follows by inspection since $a^*(n) = 0$ if $n \equiv 0, 2 \pmod{3}$.

To prove claims (3) and (4), we note that since $\mathcal{A}(z)$ is a normalized Hecke eigenform on $\Gamma_0(9)$, it follows, for every prime $p \neq 3$, that

$$(2.5) \quad a^*(p^s) = a^*(p)a^*(p^{s-1}) - p^3 a^*(p^{s-2}).$$

If $p \equiv 2 \pmod{3}$ is prime, then Theorem 2.1 implies that

$$a^*(p^s) = -p^3 a^*(p^{s-2}).$$

Claim (3) now follows by induction since $a^*(1) = 1$ and $a^*(p) = 0$.

Suppose that $p \equiv 1 \pmod{3}$ is prime. By Theorem 2.1, we know that $a^*(p) \neq 0$. More importantly, we have that

$$a^*(p) \equiv 8x^3 \pmod{p},$$

where $p = x^2 + 3y^2$ with $x \equiv 1 \pmod{3}$. To see this, one merely observes that

$$2x^3 - 18xy^2 = 2x(x^2 - 9y^2) = 2x(x^2 - 3(p - x^2)) \equiv 8x^3 \pmod{p}.$$

Since $|x| \leq \sqrt{p}$ and is non-zero, it follows that $a^*(p) \equiv 8x^3 \not\equiv 0 \pmod{p}$. By (2.5), we then have that

$$a^*(p^s) \equiv a^*(p)a^*(p^{s-1}) \equiv 8x^3 a^*(p^{s-1}) \pmod{p}.$$

By induction, it follows that $a^*(p^s) \equiv (8x^3)^s \pmod{p}$, which is non-zero modulo p . This proves claim (4). \square

Example 2.3. Here we give some numerical examples of the formulas for $a^*(n)$.

1) One easily finds that $a^*(13) = -70$. The prime $p = 13$ is of the form $x^2 + 3y^2$ where $x = 1$ and $y = 2$. Obviously, $x = 1 \equiv 1 \pmod{3}$, and so Theorem 2.1 asserts that $a^*(13) = 2 \cdot 1^3 - 18 \cdot 1 \cdot 2^2 = -70$.

2) We have that $a^*(13) = -70$ and $a^*(16) = 64$. One easily checks that $a^*(13 \cdot 16) = a^*(208) = -70 \cdot 64 = -4408$. This is an example of Corollary 2.2 (1).

3) If $p = 5$ and $s = 3$, then Corollary 2.2 (3) asserts that $a^*(5^3) = 0$. If $p = 5$ and $s = 4$, then it asserts that $a^*(5^4) = 5^6 = 15625$. One easily checks both evaluations numerically.

4) Now we consider the prime $p = 13 \equiv 1 \pmod{3}$. Since $x = 1$ and $y = 2$ for $p = 13$, Corollary 2.2 (4) asserts that $a^*(13^s) \equiv 8^s \pmod{13}$. One easily checks that

$$\begin{aligned} a^*(13) &= -70 \equiv 8 \pmod{13}, \\ a^*(13^2) &= 2703 \equiv 8^2 \pmod{13}, \\ a^*(13^3) &= -35420 \equiv 8^3 \pmod{13}. \end{aligned}$$

2.2. Proof of Theorem 1.1 and Corollary 1.3. Before we prove Theorem 1.1, we recall a known formula for $b(n)$ (also see Section 3 of [2]), the number of 3-core partitions of n .

Lemma 2.4. *Assuming the notation above, we have that*

$$\mathcal{B}(z) = \sum_{n=1}^{\infty} b^*(n)q^n = \sum_{n=0}^{\infty} b(n)q^{3n+1} = \sum_{n=0}^{\infty} \sum_{d|3n+1} \left(\frac{d}{3}\right) q^{3n+1},$$

where $\left(\frac{\bullet}{3}\right)$ denotes the usual Legendre symbol modulo 3.

Proof. We have that $\mathcal{B}(z) = \eta(9z)^3/\eta(3z)$ is in $M_1(\Gamma_0(9), \chi)$, where $\chi := \left(\frac{-3}{\bullet}\right)$. The lemma follows easily from this fact. One may implement the theory of weight 1 Eisenstein series to obtain the desired formulas.

Alternatively, one may use the weight 1 form

$$\Theta(z) = \sum_{n=0}^{\infty} c(n)q^n := \sum_{x,y \in \mathbb{Z}} q^{x^2+xy+y^2} = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + \dots$$

Using the theory of twists, we find that

$$\begin{aligned} \tilde{\Theta}(z) &= \sum_{n \equiv 1 \pmod{3}} c(n)q^n = 6q + 6q^4 + 12q^7 + 12q^{13} + 6q^{16} + 12q^{19} + 6q^{25} + \dots \\ &= 6(q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + q^{25} + \dots). \end{aligned}$$

By dimensionality (see Section 1.2.3 of [9]) we have that $\mathcal{B}(z) = 6\tilde{\Theta}(z)$. The claimed formulas for the coefficients follows easily from the fact that $x^2 + xy + y^2$ corresponds to the norm form on the ring of integers of $\mathbb{Q}(\sqrt{-3})$. □

Example 2.5. The only divisors of primes $p \equiv 1 \pmod{3}$ are 1 and p , and so we have that $b^*(p) = 1 + \binom{p}{3} = 1 + \binom{1}{3} = 2$.

Proof of Theorem 1.1. The theorem follows immediately from Theorems 2.1, 2.2 and Lemma 2.4. One sees that the only $n \equiv 1 \pmod{3}$ for which $a^*(n) = 0$ are those n for which $\text{ord}_p(n)$ is odd for some prime $p \equiv 2 \pmod{3}$. The same conclusion holds for $b^*(n)$. Using the fact that

$$a(n) = a^*(3n + 1) \quad \text{and} \quad b(n) = b^*(3n + 1),$$

the theorem follows. □

Proof of Corollary 1.3. In a famous paper [11], Serre proved that “almost all” of the coefficients of a modular form with complex multiplication are zero. This implies that almost all of the $a^*(n)$ are zero. The result now follows thanks to Theorem 1.1. □

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I.R.M.A., UMR 7501, UNIVERSITÉ LOUIS PASTEUR ET CNRS, 7 RUE RENÉ-DESCARTES, F-67084 STRASBOURG, FRANCE

E-mail address: guoniu@math.u-strasbg.fr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

E-mail address: ono@math.wisc.edu