

# SOME CONJECTURES AND OPEN PROBLEMS ON PARTITION HOOK LENGTHS

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ABSTRACT. We present some conjectures and open problems on partition hook lengths, which are all motivated by known results on the subject. The conjectures are suggested by extensive experimental calculations using a computer algebra system. The first conjecture unifies two classical results on the number of standard Young tableaux and the number of pairs of standard Young tableaux of the same shape. The second unifies the classical hook formula and the marked hook formula. The third includes the long standing Lehmer conjecture which says that the Ramanujan tau-function never takes the zero value. The fourth is a more precise version of the third one in the case of 3-cores. We also list some open problems on partition hook lengths.

## 1. INTRODUCTION

The hook lengths of partitions are widely studied in the Theory of Partitions, in Algebraic Combinatorics and Group Representation Theory. In this paper we present some conjectures and open problems on partition hook lengths, which are all motivated by known results on the subject. The conjectures are suggested by extensive experimental calculations using a computer algebra system.

The basic notions needed here can be found in [Macdonald 1995, p. 1], [Stanley 1999, p. 287], [Lascoux 2001, p. 1], [Knuth 1998, p. 59] and [Andrews 1976, p. 1]. A *partition*  $\lambda$  is a sequence of positive integers  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$  such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ . The integers  $(\lambda_i)_{i=1,2,\dots,\ell}$  are called the *parts* of  $\lambda$ , the number  $\ell$  of parts being the *length* of  $\lambda$  denoted by  $\ell(\lambda)$ . The sum of its parts  $\lambda_1 + \lambda_2 + \dots + \lambda_\ell$  is denoted by  $|\lambda|$ . Let  $n$  be an integer, a partition  $\lambda$  is said to be a partition of  $n$  if  $|\lambda| = n$ . We write  $\lambda \vdash n$ . The set of all partitions of  $n$  is denoted by  $\mathcal{P}(n)$ . The set of all partitions is denoted by  $\mathcal{P}$ , so that

$$\mathcal{P} = \bigcup_{n \geq 0} \mathcal{P}(n).$$

Each partition can be represented by its Ferrers diagram. For example,  $\lambda = (6, 3, 3, 2)$  is a partition and its Ferrers diagram is reproduced in Fig.1.1.

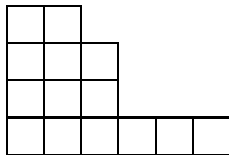


Fig.1.1. Partition

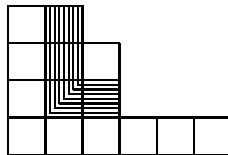


Fig.1.2. Hook length

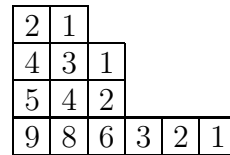


Fig.1.3. Hook lengths

For each box  $v$  in the Ferrers diagram of a partition  $\lambda$ , or for each box  $v$  in  $\lambda$ , for short, define the *hook length* of  $v$ , denoted by  $h_v(\lambda)$  or  $h_v$ , to be the number of boxes  $u$  such that  $u = v$ , or  $u$  lies in the same column as  $v$  and above  $v$ , or in the same row as  $v$  and to the right of  $v$  (see Fig.1.2). The *hook length multi-set* of  $\lambda$ , denoted by  $\mathcal{H}(\lambda)$ , is the multi-set of all hook lengths of  $\lambda$ . In Fig.1.3 the hook lengths of all boxes for the partition  $\lambda = (6, 3, 3, 2)$  have been written in each box. We have  $\mathcal{H}(\lambda) = \{2, 1, 4, 3, 1, 5, 4, 2, 9, 8, 6, 3, 2, 1\}$ .

Let  $t$  be a positive integer. Recall that a partition  $\lambda$  is a  $t$ -core if the *hook length multi-set* of  $\lambda$  does not contain the integer  $t$ . It is known that the hook length multi-set of each  $t$ -core does not contain any *multiple* of  $t$  [Knuth 1998, p. 69, p. 612], [Stanley 1999, p. 468] and [James and Kerber 1981, p. 75].

The First Conjecture stated in Section 2 unifies two classical results on the number of standard Young tableaux and the number of pairs of standard Young tableaux of the same shape. The Second Conjecture unifies the classical hook formula and the marked hook formula (see Section 3). The Third Conjecture, presented in Section 4, includes the long standing Lehmer conjecture which says that the Ramanujan tau-function never takes the zero value. The Fourth Conjecture is a more precise version of the third one in the case of 3-cores (see Section 5). Finally, we list some open problems on partition hook lengths in Section 6.

## 2. FIRST CONJECTURE

The hook length plays an important role in Algebraic Combinatorics thanks to the famous hook formula due to Frame, Robinson and Thrall [Frame, Robinson and Thrall 1954]

$$(2.1) \quad f_\lambda = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h},$$

where  $f_\lambda$  is the number of standard Young tableaux of shape  $\lambda$  (see [Stanley 1999, p. 376], [Knuth 1998, p. 59], [Greene, Nijenhuis and Wilf 1979, Gessel and Viennot 1985, Zeilberger 1984, Novelli, Pak and Stoyanovskii 1997, Krattenthaler 1999]).

Recall that the Robinson-Schensted-Knuth correspondence is a bijection between the set of ordered pairs of standard Young tableaux of  $\{1, 2, \dots, n\}$  of the same shape and the set of permutations of order  $n$  [Knuth 1970] (see also [Knuth 1998, p. 49-59], [Stanley 1999, p. 324]). It provides a combinatorial proof of the following identity

$$(2.2) \quad \sum_{\lambda \in \mathcal{P}_n} f_\lambda^2 = n!$$

By using (2.1) identity (2.2) can be written in the following generating function form

$$(2.3) \quad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = e^x.$$

The Robinson-Schensted-Knuth correspondence also proves the fact that the number of standard Young tableaux of  $\{1, 2, \dots, \}$  is equal to the number of involutions of order  $n$

(see [Knuth 1998, p. 47]). In the generating function form this means that

$$(2.4) \quad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h} = e^{x+x^2/2}.$$

Our first conjecture may be regarded as a hook length formula that interpolates formulas (2.3) and (2.4) holding for permutations and involutions, respectively. It was suggested by the hook length expansion technique developed in [Han 2008a].

**Conjecture 2.1** (First). *We have*

$$(2.5) \quad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho(z; h) = e^{x+zx^2/2},$$

where the weight function  $\rho(z; n)$  is defined by

$$(2.6) \quad \rho(z; n) = \frac{\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} z^k}{n \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} z^k}.$$

The first values of the weight function  $\rho(z, n)$  are listed below.

$$\begin{aligned} \rho(z; 1) &= 1; \\ \rho(z; 2) &= \frac{1+z}{4}; \\ \rho(z; 3) &= \frac{3z+1}{9+3z}; \\ \rho(z; 4) &= \frac{z^2+6z+1}{16+16z}; \\ \rho(z; 5) &= \frac{5z^2+10z+1}{5z^2+50z+25}; \\ \rho(z; 6) &= \frac{z^3+15z^2+15z+1}{120z+36z^2+36}, \\ \rho(z; 7) &= \frac{7z^3+35z^2+21z+1}{7z^3+147z^2+245z+49}. \end{aligned}$$

In fact, formula (2.6) has been verified up to  $n \leq 20$ .

Using the real part  $\Re$  and imaginary part  $\Im$  operators of complex numbers, Conjecture 2.1 can be re-written as the following equivalent form.

**Conjecture 2.2.** *We have*

$$(2.7) \quad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{z\Re(1+iz)^h}{h\Im(1+iz)^h} = e^{x-z^2x^2/2}.$$

In the rest of this section we discuss some specializations of Conjecture 2.1. When  $z = 1$ , then  $\rho(1; n) = 1/n$ ; we recover identity (2.4). When  $z = 0$ , then  $\rho(0; n) = 1/n^2$ ; we recover identity (2.3). However we cannot prove any other special cases of Conjecture 2.1, except the above two values. Now select the coefficients of  $[zx^n]$  on both sides of (2.5). Since

$$(2.8) \quad \rho(z; n) = \frac{1 + \binom{n}{2}z + O(z^2)}{n^2 + n\binom{n}{3}z + O(z^2)} = \frac{1}{n^2} \left(1 + \frac{n^2 - 1}{3}z\right) + O(z^2),$$

the coefficient of  $[zx^n]$  on the left-hand side of (2.5) is

$$(2.9) \quad \begin{aligned} & [z] \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \left(1 + \frac{h^2 - 1}{3}z\right) \\ &= \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} \left(\frac{h^2 - 1}{3}\right) \\ &= \frac{1}{3} \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^2 - \frac{n}{3} \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \\ &= \frac{1}{3} \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^2 - \frac{n}{3n!}. \end{aligned}$$

The coefficient of  $[zx^n]$  on the right-hand side of (2.5) is

$$(2.10) \quad \begin{aligned} [zx^n]e^{x+zx^2/2} &= [zx^n] \sum_{k \geq 1} \frac{(x + zx^2/2)^k}{k!} \\ &= [zx^n] \sum_{k \geq 1} \frac{kx^{k-1}(zx^2/2)}{k!} \\ &= [x^n] \sum_{k \geq 1} \frac{x^{k-1}(x^2)}{2(k-1)!} \\ &= \frac{1}{2(n-2)!}. \end{aligned}$$

By comparing (2.9) and (2.10) we obtain the next marked hook formula, which has been proved in [Han 2008b, Han 2008c].

**Theorem 2.3** (marked hook formula). *We have*

$$(2.11) \quad \sum_{\lambda \vdash n} f_\lambda^2 \sum_{h \in \mathcal{H}(\lambda)} h^2 = \frac{n(3n-1)}{2} n!$$

We can also select the coefficients of  $[z^2x^n]$  on both sides of (2.5). Since

$$\begin{aligned}\rho(z; n) &= \frac{1 + \binom{n}{2}z + \dots}{n^2 + n\binom{n}{3}z + \dots} \\ &= \frac{1}{n^2} \left( 1 + \frac{n^2 - 1}{3}z - \frac{(n^2 - 1)(n^2 - 4)}{45}z^2 \right) + O(z^3),\end{aligned}$$

the coefficient of  $[zx^n]$  on the left-hand side of (2.5) is

$$[z^2] \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \left( 1 + \frac{h^2 - 1}{3}z - \frac{(h^2 - 1)(h^2 - 4)}{45}z^2 \right) = A + B,$$

with

$$A = \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{v \in \lambda} \frac{-(h_v^2 - 1)(h_v^2 - 4)}{45}$$

and

$$B = \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{\{u, v\}, u \neq v} \left( \frac{h_u^2 - 1}{3} \frac{h_v^2 - 1}{3} \right),$$

where the second sum in  $B$  ranges over all unordered pairs  $\{u, v\}$  such that  $u, v \in \lambda$  and  $u \neq v$ . Let us evaluate the two quantities  $A$  and  $B$ . We have

$$A = -\frac{1}{45} \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} (h^4 - 5h^2 + 4) = -\frac{1}{45} (A_1 + A_2 + A_3),$$

with

$$(2.12) \quad A_1 = \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^4,$$

$$A_2 = \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} (-5h^2) = -\frac{5n(3n-1)}{2n!} \quad [\text{by (2.11)}]$$

and

$$A_3 = \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} 4 = \frac{4n}{n!}.$$

We also have

$$B = \frac{1}{9} \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{\{u, v\}} (h_u^2 h_v^2 - (h_v^2 + h_u^2) + 1) = \frac{1}{9} (B_1 + B_2 + B_3),$$

with

$$\begin{aligned}
B_1 &= \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{\{u,v\}} h_u^2 h_v^2 \\
&= \frac{n(n-1)(27n^2 - 67n + 74)}{24n!}, \quad [\text{by Prop. 6.11 in [Han 2008b]}] \\
B_2 &= \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{\{u,v\}} -(h_v^2 + h_u^2), \\
&= (n-1) \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_v (-h_v^2), \\
&= -(n-1) \frac{n(3n-1)}{2n!}, \\
B_3 &= \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{\{u,v\}} 1 = \frac{1}{n!} \binom{n}{2}.
\end{aligned}$$

On the other hand, the coefficient of  $[z^2 x^n]$  on the right-hand side of (2.5) is

$$\begin{aligned}
(2.13) \quad [z^2 x^n] e^{x+zx^2/2} &= [z^2 x^n] \sum_{k \geq 1} \frac{(x + zx^2/2)^k}{k!} \\
&= [z^2 x^n] \sum_{k \geq 2} \frac{k(k-1)/2 \times x^{k-2} (zx^2/2)^2}{k!} \\
&= [x^n] \sum_{k \geq 2} \frac{k(k-1)/2 \times x^{k-2} (x^2/2)^2}{k!} \\
&= [x^n] \sum_{k \geq 2} \frac{x^{k+2}}{8(k-2)!} \\
&= \frac{1}{8(n-4)!}.
\end{aligned}$$

By Conjecture 2.1 and (2.13) we have

$$\frac{1}{8(n-4)!} = -\frac{1}{45}(A_1 + A_2 + A_3) + \frac{1}{9}(B_1 + B_2 + B_3).$$

The values of  $A_2, A_3, B_1, B_2, B_3$  being explicitly calculated, the expression of  $A_1$  shown in (2.12) leads to the following Conjecture.

**Conjecture 2.4.** *We have*

$$(2.14) \quad \sum_{\lambda \in \mathcal{P}(n)} \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^4 = \frac{n(40n^2 - 75n + 41)}{6n!}.$$

## 3. SECOND CONJECTURE

The next conjecture is suggested by the fact that formulas (2.2), (2.11) and (2.14) have the same form.

**Conjecture 3.1** (Second). *Let  $k$  be a positive integer. Then*

$$P_k(n) = (k+1)!(n-1)! \sum_{\lambda \vdash n} \left( \prod_{v \in \lambda} \frac{1}{h_v^2} \right) \left( \sum_{u \in \lambda} h_u^{2k} \right)$$

*is a polynomial in  $n$  of degree  $k$  with integral coefficients.*

Notice that the classical hook formula (2.2), the marked hook formula (2.11) and Conjecture 2.4 are all consequences of Conjecture 3.1 (the cases  $k = 0, 1, 2$ ), because if we know that  $P_k(n)$  is a polynomial in  $n$  of degree  $k$ , we can determinate the polynomial  $P_k(n)$  by taking  $(k+1)$  numerical values of  $P_k(n)$  using the Lagrange interpolation formula. Let us go one more step by looking at case  $k = 3$ .

**Conjecture 3.2.** *We have*

$$\sum_{\lambda \vdash n} f_\lambda^2 \sum_{v \in \lambda} h_v^6 = \frac{n}{24} (1050n^3 - 4060n^2 + 5586n - 2552) n!$$

The first values of the polynomials  $P_k(n)$  ( $0 \leq k \leq 9$ ), suggested by extensive experimental calculations using a computer algebra system, are shown in the next table.

$$P_0(n) = 1,$$

$$P_1(n) = 3n - 1,$$

$$P_2(n) = 40n^2 - 75n + 41,$$

$$P_3(n) = 1050n^3 - 4060n^2 + 5586n - 2552,$$

$$P_4(n) = 42336n^4 - 265860n^3 + 654360n^2 - 721800n + 291084,$$

$$P_5(n) = 2328480n^5 - 20956320n^4 + 77962500n^3 - 146671800n^2 + 136808100n - 49470240,$$

$$P_6(n) = 163088640n^6 - 1941619680n^5 + 9851665824n^4 - 26869883040n^3 + 41020980000n^2 - 32822800920n + 10598574216,$$

$$P_7(n) = 13913499600n^7 - 206918712000n^6 + 1332526235040n^5 - 4753759570560n^4 + 10023914300400n^3 - 12352918032000n^2 + 8158628953440n - 2215386633600,$$

$$P_8(n) = 1401656256000n^8 - 24914439950400n^7 + 192568162026240n^6 - 830326365348480n^5 + 2134506603220992n^4 - 3232434128152320n^3 + 2628227513681280n^2 - 860196155051520n - 8832846318912.$$

From Conjecture 2.4, we derive the following formula.

**Conjecture 3.3.** *Let  $n$  be an positive integer. We have*

$$\sum_{\lambda \vdash n} \left( \prod_{v \in \lambda} \frac{1}{h_v^2} \right) \left( \sum_{u \in \lambda} h_u^2 \right)^2 = \frac{1}{12(n-1)!} (27n^3 - 14n^2 - 9n + 8).$$

#### 4. THIRD CONJECTURE

Let us state our third conjecture, followed by some specializations and remarks.

**Conjecture 4.1** (Third). *Let  $n, s, t$  be positive integers such that  $t \neq 4, 10$  and  $s \mid t$ . Then the coefficient of  $x^n$  in*

$$\prod_{k \geq 1} \frac{(1 - x^{sk})^{t^2/s}}{1 - x^k}$$

*is equal to zero, if and only if the coefficient of  $x^n$  in*

$$\prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k}$$

*is also equal to zero.*

Conjecture 4.1 has been verified by the author for all pairs  $(t, n)$  such that  $t \leq 13$  and  $n \leq 4000$ .

*Remark.* Even if the conjecture is stated with the exceptions  $t \neq 4, 10$ , it is almost true in the latter cases. For example, up to  $n = 4000$ , there are only four exceptions  $n = 53, 482, 1340, 2627$  for  $s = 1, t = 4$ ; five exceptions  $n = 35, 320, 890, 1745, 2885$  for  $s = 2, t = 4$  and two exceptions  $n = 24, 49$  for  $s = 5, t = 10$ . Ken Ono [Ono 2008] has pointed out that there are infinitely many exceptions for  $s = 1, t = 4$ .

Let  $\mathcal{P}(n; t)$  denote the set of all  $t$ -cores of  $n$ . The generating function for  $t$ -cores is given by the following formula

$$(4.1) \quad \sum_{\lambda} x^{|\lambda|} = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k},$$

where the sum ranges over all  $t$ -cores [Knuth 1998, p. 69, p. 612], [Stanley 1999, p. 468], [Garvan, Kim and Stanton 1990].

In [Han 2008c, Corollary 5.3] we proved the following result.

**Theorem 4.2.** *We have*

$$(4.2) \quad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{v \in \lambda, s \mid h_v} \left( 1 - \frac{sz}{h_v^2} \right) = \prod_{k \geq 1} \frac{(1 - x^{sk})^z}{1 - x^k}.$$

Hence, Conjecture 4.1 can be re-written by using (4.1) and (4.2) as follows.

**Conjecture 4.3.** *Let  $n, s, t$  be positive integers such that  $t \neq 4, 10$  and  $s \mid t$ . The expression*

$$(4.3) \quad \sum_{\lambda \in \mathcal{P}(n; t)} \prod_{v \in \lambda, s \mid h_v} \left( 1 - \frac{t^2}{h_v^2} \right)$$

*is equal to zero if and only if  $\mathcal{P}(n; t) = \emptyset$ .*

Conjecture 4.1 is true for  $s = 1$  and  $t = 2$ , thanks to the following two well-known formulas due to Jacobi (see [Andrews 1976, p. 21], [Knuth 1998, p. 20]) and Gauss [Stanley 1999, p. 518], [Andrews 1976, p. 23], respectively.

**Theorem 4.4** (Jacobi). *We have*

$$(4.4) \quad \prod_{m \geq 1} (1 - x^m)^3 = \sum_{m \geq 0} (-1)^m (2m + 1) x^{m(m+1)/2}.$$

**Theorem 4.5** (Gauss). *We have*

$$(4.5) \quad \prod_{m \geq 1} \frac{(1 - x^{2m})^2}{1 - x^m} = \sum_{m \geq 0} x^{m(m+1)/2}.$$

Consider the specialization  $s = 1$  and  $t = 3$ . Let  $(a(n))$  be the coefficients in the expansion of the product

$$(4.6) \quad \begin{aligned} \prod_{m \geq 1} (1 - x^m)^8 &= \sum_{n \geq 0} a(n) x^n \\ &= 1 - 8x + 20x^2 - 70x^4 + 64x^5 + 56x^6 - 125x^8 + \\ &\quad \dots - 20482x^{220} + 24050x^{224} - 21624x^{225} + \dots \end{aligned}$$

and  $(b(n))$  the coefficients in the expansion of the product

$$(4.7) \quad \begin{aligned} \prod_{m \geq 1} \frac{(1 - x^{3m})^3}{1 - x^m} &= \sum_{b \geq 0} b(n) x^n \\ &= 1 + x + 2x^2 + 2x^4 + x^5 + 2x^6 + x^8 + \\ &\quad \dots + 2x^{220} + 2x^{224} + 3x^{225} + \dots \end{aligned}$$

Notice that the coefficients  $b(n)$  are rather small and  $a(n)$  are rather large. Conjecture 4.1 may be restated as follows.

**Conjecture 4.6.** *Let  $n$  be a positive integer. Then  $a(n) = 0$  if and only if  $b(n) = 0$ .*

Recall the following theorem due to Granville and Ono [Granville and Ono 1996].

**Theorem 4.7.** *Let  $n, t$  be two positive integers such that  $t \geq 4$ . Then  $\mathcal{P}(n; t) \neq \emptyset$ .*

Hence, Conjectures 4.1 can be re-written in the following way.

**Conjecture 4.8.** *Let  $t \geq 5, n, s$  be positive integers such that  $s \mid t$  and  $t \neq 10$ . Then the coefficient of  $x^n$  in*

$$\prod_{k \geq 1} \frac{(1 - x^{sk})^{t^2/s}}{1 - x^k}$$

*is not equal to 0.*

In particular, when  $s = 1$  and  $t = 5$  in Conjecture 4.8, we recover the following long standing Lehmer conjecture (see [Serre 1970]). Recall that the Ramanujan  $\tau$ -function is defined by (see [Serre 1970, p. 156])

$$\begin{aligned} x \prod_{m \geq 1} (1 - x^m)^{24} &= \sum_{n \geq 1} \tau(n) x^n \\ &= x - 24x^2 + 252x^3 - 1472x^4 + 4830x^5 - 6048x^6 + \dots \end{aligned}$$

**Conjecture 4.9** (Lehmer). *For each  $n$  we have  $\tau(n) \neq 0$ .*

## 5. FOURTH CONJECTURE

Recall that  $a(n)$  and  $b(n)$  are defined by (4.6) and (4.7), respectively. The following conjectures characterize the integers  $n$  for which  $a(n) = 0$  or  $b(n) = 0$ . They are suggested by Theorem 5.3 stated later in this section.

**Conjecture 5.1** (Fourth). *Let  $N$  be a positive integer.*

(i) *If there are integers  $n \geq 0, m \geq 1$  such that*

$$N = 4^m n + (10 \cdot 4^{m-1} - 1)/3,$$

*then  $a(N) = 0$ ;*

(ii) *If there are integers  $n \geq 0, m \geq 1, k \geq 1$  with  $m \not\equiv 2k - 1 \pmod{6k - 1}$ , such that*

$$N = (6k - 1)^2 n + (6k - 1)m + 4k - 1,$$

*then  $a(N) = 0$ ;*

(iii) *For all positive integers  $N$  we have  $a(N) \neq 0$ , except those in cases (i) and (ii).*

If the Third Conjecture is true, then Conjecture 5.1 is equivalent to the following conjecture for  $b(n)$ . It is known (see, e.g., [Garvan, Kim and Stanton 1990]) that  $b(n)$  is equal to the number of the integer solutions of the Diophantine equation

$$3(x^2 + xy + y^2) + x + 2y = n.$$

**Conjecture 5.2.** *Let  $N$  be a positive integer.*

(i) *If there are integers  $n \geq 0, m \geq 1$  such that*

$$N = 4^m n + (10 \cdot 4^{m-1} - 1)/3,$$

*then  $b(N) = 0$ ;*

(ii) *If there are integers  $n \geq 0, m \geq 1, k \geq 1$  with  $m \not\equiv 2k - 1 \pmod{6k - 1}$ , such that*

$$N = (6k - 1)^2 n + (6k - 1)m + 4k - 1,$$

*then  $b(N) = 0$ ;*

(iii) *For all positive integers  $N$  we have  $b(N) \neq 0$ , except those in cases (i) and (ii).*

Taking special values for  $m$  and  $k$  in Conjecture 5.1 yields the following relations.

**Theorem 5.3.** *We have*

$$\begin{aligned} a(4n+3) &= 0; \\ a(16n+13) &= 0; \\ a(25n+3) = 0, \quad a(25n+13) = 0, \quad a(25n+18) = 0, \quad a(25n+23) &= 0; \\ a(64n+53) &= 0. \end{aligned}$$

*Proof.* In fact, the relations in Theorem 5.3 were *discovered* and automatically *proved* by using a computer algebra program thanks to the next theorem, which asserts that a simple variation of the classical Macdonald identity [Macdonald 1972] holds. For example, each term in identity (5.1) has two parameters  $k$  and  $m$  (or only one parameter  $k$ ). To prove  $a(4n+3) = 0$ , we need only check  $a(4n+3) = 0$  for  $k, m = 0, 1, 2, 3$ , since the coefficient and the exponent in each term are both polynomials in  $k$  and  $m$  with integral coefficients. There are finitely many cases to verify.  $\square$

**Theorem 5.4.** *We have*

$$\begin{aligned} (5.1) \quad \prod_{k \geq 1} (1 - q^k)^8 &= \sum_{k \geq 0} \left( (3k+1)^3 q^{3k^2+2k} - (3k+2)^3 q^{3k^2+4k+1} \right) \\ &+ \sum_{k > m \geq 0} \left( (3k+1)(3m+1)(3k+3m+2) q^{k^2+k+m^2+m+km} \right. \\ &\quad \left. - (3k+2)(3m+2)(3k+3m+4) q^{k^2+k+m^2+m+(k+1)(m+1)} \right). \end{aligned}$$

In principle, any specialization of Conjecture 5.1 can be proved in the same way (if the computer is fast enough!). However, the general case requires a true mathematical investigation.

In the same manner, the following congruence properties were also *discovered* and automatically *proved* by using a computer algebra program. However, we are not able to imagine a global formula similar to Conjecture 5.1.

**Theorem 5.5.** *We have*

$$\begin{aligned} (5.2) \quad a(2n+1) &\equiv 0 \pmod{2}; \\ a(4n+1) &\equiv a(4n+2) \equiv 0 \pmod{4}; \\ a(5n+2) &\equiv a(5n+3) \equiv a(5n+4) \equiv 0 \pmod{5}; \\ a(7n+3) &\equiv a(7n+4) \equiv a(7n+6) \equiv 0 \pmod{7}; \\ a(8n+1) &\equiv a(8n+5) \equiv a(8n+6) \equiv 0 \pmod{8}; \\ a(10n+2) &\equiv a(10n+4) \equiv 0 \pmod{10}; \\ a(11n+7) &\equiv 0 \pmod{11}; \\ a(14n+4) &\equiv a(14n+6) \equiv a(14n+10) \equiv 0 \pmod{14}. \end{aligned}$$

## 6. OPEN PROBLEMS

Is there a combinatorial proof of the marked hook formula (2.11), analogous to the Robinson-Schensted-Knuth correspondence for proving (2.2)? Let  $T$  be a standard

Young tableau of shape  $\lambda$  (see [Knuth 1998, p. 47]),  $u$  be a box in  $\lambda$  and  $m$  an integer such that  $1 \leq m \leq h_u(\lambda)$ . The triplet  $(T, u, m)$  is called a *marked Young tableau* of shape  $(\lambda, u)$ . The number of marked Young tableaux of shape  $(\lambda, u)$  is then  $f_\lambda h_u$ . On the other hand, call *marked permutation* each triplet  $(\sigma, j, k)$  where  $\sigma \in \mathfrak{S}_n$ ,  $1 \leq j \leq n$  and  $1 \leq k \leq n + j - 1$ . We say that the letter  $j$  within the permutation  $\sigma$  is marked  $k$ . The total number of marked permutations of order  $n$  is

$$\sum_{j=1}^n (n+j-1)n! = \frac{n(3n-1)}{2}n!$$

*Example.* The sequence 6 4 9  $5_k$  7 1 2 8 3 with  $1 \leq k \leq 13$  is a marked permutation, whose letter 5 is marked  $k$ . The following two diagrams are two marked Young tableaux of the same shape, where  $1 \leq i, j \leq 3$ .



Fig.6.1. Marked Young tableaux

**Problem 6.1.** Find a marked Robinson-Schensted-Knuth correspondence between pairs of marked Young tableaux and marked permutations that yields a direct proof of the marked hook formula (Theorem 2.3).

Keeping in mind that the number of all standard Young tableaux on  $\{1, 2, \dots, n\}$  is equal to the number of involutions of order  $n$  (see (2.4)), we are led to make the following statement.

**Problem 6.2.** Find a formula for the number of all marked standard Young tableaux (that could be called marked involutions):

$$\sum_{\lambda \vdash n} f_\lambda \sum_{v \in \lambda} h_v = ?$$

More generally, is there a simple formula for

$$\sum_{\lambda \in \mathcal{P}} \prod_{v \in \lambda} \left(1 + \frac{1}{h_v}\right) x = ?$$

Let  $t = 2t' + 1$  be an odd positive integer. In [Han 2008c] we have constructed a bijection  $\phi_V : \lambda \mapsto (v_0, v_1, \dots, v_{t-1})$ , which maps each  $t$ -core onto a  $V$ -coding such that

$$(6.1) \quad |\lambda| = \frac{1}{2t}(v_0^2 + v_1^2 + \dots + v_{t-1}^2) - \frac{t^2 - 1}{24}$$

and

$$(6.2) \quad \prod_{v \in \lambda} \left(1 - \frac{t^2}{h_v^2}\right) = \frac{(-1)^{t'}}{1! \cdot 2! \cdot 3! \cdot \dots \cdot (t-1)!} \prod_{0 \leq i < j \leq t-1} (v_i - v_j).$$

The right-hand side of (6.2) appears in the Macdonald identities for type  $A_\ell^{(a)}$  (see [Macdonald 1972]). Notice that the parameter  $t$  on the right-hand side of (6.2) can only take positive integer values, because  $t$  is a vector length, whereas on the left-hand side  $t$  can be any complex number. For that reason we call formula (6.2) an *indiscretization* analogue of the Macdonald identities for  $A_\ell^{(a)}$ . This indiscretization principle led us to the following Nekrasov-Okounkov formula [Nekrasov and Okounkov 2006, Han 2008a]

$$(6.3) \quad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{k \geq 1} (1 - x^k)^{z-1}.$$

**Problem 6.3.** *Find the indiscretization analogue of the Macdonald identities for the other affine root systems (see [Macdonald 1972]) and deduce other expansion formulas for the powers of the Euler Product.*

The answer to Problem 6.2 will produce a lot of identities for powers of the Euler Product. For example,

$$\begin{aligned} \prod_{m \geq 1} (1 - x^m) &= \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k+1)/2} && \text{(by Euler);} \\ &= \sum_{\lambda \in \mathcal{P}} \prod_{v \in \lambda} \left(1 - \frac{2}{h_v^2}\right) x && \text{(by type } A_l^{(a)}\text{);} \\ &=? && \text{(by type } B_l\text{);} \\ &\dots \end{aligned}$$

In general, it is not easy to convert one identity to another directly.

Taking  $z = 4$  in (6.3) yields the following identity due to Jacobi

$$(6.4) \quad \prod_{m \geq 1} (1 - x^m)^3 = \sum_{m \geq 0} (-1)^m (2m + 1) x^{m(m+1)/2}.$$

In fact, the general form of the Jacobi triple product identity reads:

$$(6.5) \quad \prod_{n \geq 0} (1 + ax^{n+1})(1 + x^n/a)(1 - x^{n+1}) = \sum_{n=-\infty}^{+\infty} a^n x^{n(n+1)/2}.$$

**Problem 6.4.** *Find an  $a$ -analogue of (6.3) that can be transformed to the Jacobi triple product identity (6.5) by specialization.*

#### LAST NEWS AND COMMENTS

Mihai Cipu has proved Conjecture 5.2 [Cipu 2008]. Richard Stanley has recently proved Conjecture 3.1 when  $k = 2$ , which is equivalent to proving Conjecture 2.4 [Stanley 2008]. Ken Ono and the author have proved Conjecture 4.6 [Han and Ono 2008]. Another conjecture referred to as Conjecture 1.7 in [Han 2008b] has the same nature as the conjectures presented in the paper. It is no longer reproduced, since it has just been proved by the author [Han 2008c].

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