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Yet another generalization of Postnikov's hook length formula for binary trees

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Abstract. — We discover another one-parameter generalization of Postnikov's hook length formula for binary trees. The particularity of our formula is that the hook length h_v appears as an exponent. As an application, we derive another simple hook length formula for binary trees when the underlying parameter takes the value 1/2.

1. Introduction

Consider the set $\mathcal{B}(n)$ of all binary trees with n vertices. For each vertex v of $T \in \mathcal{B}(n)$ the *hook length* of v, denoted by h_v, or just h for short, is the number of descendants of v (including v). The following hook length formula for binary trees

$$\sum_{T\in\mathcal{B}(n)} \prod_{v\in T} \bigl(1+\frac{1}{h_v}\bigr) = \frac{2^n}{n!}(n+1)^{n-1} \tag{1}$$

was discovered by Postnikov [Po04]. Further combinatorial proofs and extensions have been proposed by several authors [CY08, GS06, MY07, Se08]. In particular, Lascoux conjectured the following one-parameter generalization

$$\sum_{T\in\mathcal{B}(n)} \prod_{v\in T} \bigl(x+\frac{1}{h_v}\bigr) = \frac{1}{(n+1)!}\prod_{k=0}^{n-1}((n+1+k)x+n+1-k), \tag{2}$$

which was subsequently proved by Du-Liu [DL08]. The latter generalization appears to be very natural, because the *left-hand side* of (2) can be obtained from the left-hand side of (1) by replacing 1 by x.

It is also natural to look for an extension of (1) by introducing a new variable z in the *right-hand side*, namely by replacing $2^n(n+1)^{n-1}/n!$ by $2^n z(n+z)^{n-1}/n!$ It so happens that the corresponding left-hand side is also a sum on binary trees, but this time the hook length h_v appears as an exponent. The purpose of this Note is to prove the following Theorem.

Theorem 1. *For each positive integer n we have*

$$\sum_{T\in\mathcal{B}(n)} \prod_{v\in T} \frac{(z+h)^{h-1}}{h(2z+h-1)^{h-2}} = \frac{2^n z}{n!}(n+z)^{n-1}. \tag{3}$$

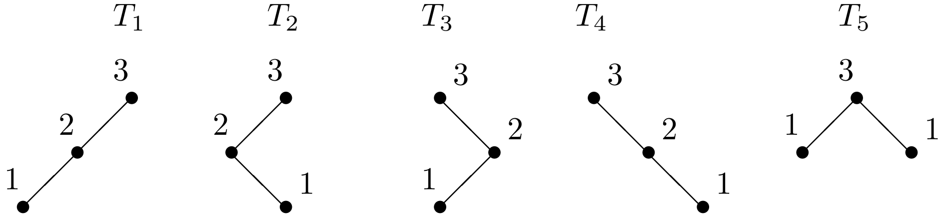
With $z=1$ in (3) we recover Postnikov's identity (1). The following corollary is derived from our identity (3) by taking $z=1/2$.

Corollary 2. *For each positive integer n we have*

$$\sum_{T\in\mathcal{B}(n)}\prod_{v\in T}\big(1+\frac{1}{2h}\big)^{h-1}=\frac{(2n+1)^{n-1}}{n!}. \tag{4}$$

2. Proof of the Theorem

Let us take an example before proving the Theorem. There are five binary trees with $n=3$ vertices:



The hook lengths of T_1, T_2, T_3, T_4 are all the same $1,2,3$; but the hook lengths of T_5 are $1,1,3$. The left-hand side of (3) is then equal to

$$4\times\frac{1}{(2z)^{-1}}\cdot\frac{(z+2)^1}{2}\cdot\frac{(z+3)^2}{3(2z+1)}+\frac{1}{(2z)^{-1}}\cdot\frac{1}{(2z)^{-1}}\cdot\frac{(z+3)^2}{3(2z+1)}=\frac{2^3z(z+3)^2}{3!}.$$

Let $y(x)$ be a formal power series in x such that

$$y(x)=e^{xy(x)}. \tag{5}$$

By the Lagrange inversion formula $y(x)^z$ has the following explicit expansion:

$$y(x)^z=\sum_{n\ge 0}z(n+z)^{n-1}\frac{x^n}{n!}. \tag{6}$$

Since $y^{2z}=(y^z)^2$ we have

$$\sum_{n\ge 0}2z(n+2z)^{n-1}\frac{x^n}{n!}=\big(\sum_{n\ge 0}z(n+z)^{n-1}\frac{x^n}{n!}\big)^2. \tag{7}$$

Comparing the coefficients of x^n on both sides of (7) yields the following Lemma.

Lemma 3. *We have*

$$\frac{2z(n+2z)^{n-1}}{n!}=\sum_{k=0}^{n}\frac{z(k+z)^{k-1}}{k!}\times\frac{z(n-k+z)^{n-k-1}}{(n-k)!}. \tag{8}$$

Proof of the Theorem. Let

$$P(n)=\sum_{T\in\mathcal{B}(n)}\prod_{v\in T}\frac{(z+h)^{h-1}}{h(2z+h-1)^{h-2}}.$$

With each binary tree $T\in\mathcal{B}(n)$ $(n\geq 1)$ we can associate a triplet (T',T'',u), where $T'\in\mathcal{B}(k)$ $(0\leq k\leq n-1)$, $T''\in\mathcal{B}(n-1-k)$ and u is a vertex of hook length $h_u=n$. Hence

$$P(n)=\sum_{k=0}^{n-1}P(k)P(n-1-k)\times\frac{(z+n)^{n-1}}{n(2z+n-1)^{n-2}}. \tag{9}$$

It is routine to verify that $P(n)=2^n z(z+n)^{n-1}/n!$ for $n=1,2,3$. Suppose that $P(k)=2^k z(z+k)^{k-1}/k!$ for $k\leq n-1$. From identity (9) and Lemma 3 we have

$$\begin{aligned}P(n)&=\sum_{k=0}^{n-1}\frac{2^k z(z+k)^{k-1}}{k!}\times\frac{2^{n-k-1}z(z+n-k-1)^{n-k-2}}{(n-k-1)!}\\&\qquad\qquad\times\frac{(z+n)^{n-1}}{n(2z+n-1)^{n-2}}\\&=\frac{2^n z}{n!}(z+n)^{n-1}.\end{aligned}$$

By induction, formula (3) is true for any positive integer n. ∎

3. Concluding and Remarks

The right-hand sides of (3) and (4) have been studied by other authors [GS06, DL08, MY07], but our formula has the following two major differences: (i) the hook length h_v appears as an exponent; (ii) the underlying set remains the set of binary trees, whereas in the above mentioned papers the summation has been changed to the set of m-ary trees or plane forests. It is interesting to compare Corollary 2 with the following results obtained by Du and Liu [DL08]. Note that the right-hand sides of formulas (4), (10) and (11) are all identical!

Proposition 4. *For each positive integer n we have*

$$\sum_{T\in\mathcal{T}(n)} \prod_{v\in I(T)} \left(\frac{2}{3}+\frac{1}{3h}\right) = \frac{(2n+1)^{n-1}}{n!}, \tag{10}$$

where $\mathcal{T}(n)$ is the set of all 3-ary trees with n internal vertices and $I(T)$ is the set of all internal vertices of T.

Proposition 5. *For each positive integer n we have*

$$\sum_{T\in\mathcal{F}(n)} \prod_{v\in(T)} \left(2-\frac{1}{h}\right) = \frac{(2n+1)^{n-1}}{n!}, \tag{11}$$

where $\mathcal{F}(n)$ is the set of all plane forests with n vertices.

References

[CY08] Chen, William Y.C.; Yang, Laura L.M., On Postnikov's hook length formula for binary trees, *European Journal of Combinatorics*, in press, 2008.

[DL08] Du, Rosena R. X.; Liu, Fu, (k,m)-Catalan Numbers and Hook Length Polynomials for Plane Trees, *European J. Combin*, **28** (2007), pp. 1312–1321.

[GS06] Gessel, Ira M.; Seo, Seunghyun, A refinement of Cayley's formula for trees, *arXiv:math.CO/0507497*.

[MY07] Moon, J. W.; Yang, Laura L. M., Postnikov identities and Seo's formulas, *Bull. Inst. Combin. Appl.*, **49** (2007), pp. 21–31.

[Po04] Postnikov, Alexander, Permutohedra, associahedra, and beyond, *arXiv:math. CO/0507163*, 2004.

[Se08] Seo, Seunghyun, A combinatorial proof of Postnikov's identity and a generalized enumeration of labeled trees, *arXiv:math.CO/0409323*.

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