2008/03/26

New hook length formulas for binary trees

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Abstract. — We find two new hook length formulas for binary trees. The particularity of our formulas is that the hook length h_v appears as an exponent.

Consider the set $\mathcal{B}(n)$ of all binary trees with n vertices. It is well-known that the cardinality of $\mathcal{B}(n)$ is equal to the Catalan number (see, e.g., [9, p.220]):

$$\sum_{T\in\mathcal{B}(n)} 1 = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

For each vertex v of a binary tree $T \in \mathcal{B}(n)$ the *hook length* of v, denoted by $h_v(T)$ or h_v, is the number of descendants of v (including v). It is also well-known [4, p.67] that the number of ways to label the vertices of T with $\{1, 2, \ldots, n\}$, such that the label of each vertex is less than that of its descendants, is equal to $n!$ divided by the product of the h_v's ($v \in T$). On the other hand, each labeled binary tree with n vertices is in bijection with a permutation of order n [8, p.24], so that

$$\sum_{T\in\mathcal{B}(n)} n! \prod_{v\in T} \frac{1}{h_v} = n! \tag{2}$$

The following hook length formula for binary trees

$$\sum_{T\in\mathcal{B}(n)} \frac{n!}{2^n} \prod_{v\in T} \bigl(1+\frac{1}{h_v}\bigr) = (n+1)^{n-1} \tag{3}$$

is due to Postnikov [6]. Further combinatorial proofs and extensions have been proposed by several authors [1, 2, 3, 5, 7].

In the present Note we obtain the following two new hook length formulas for binary trees. The particularity of our formulas is that the hook length h_v appears as an exponent. Their proofs are based on the induction principle. It would be interesting to find simple bijective proofs.

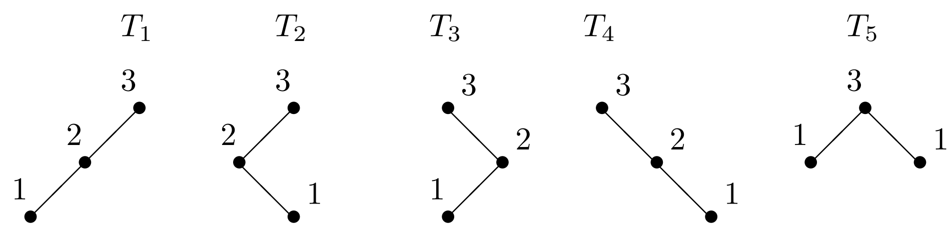
Theorem. *For each positive integer n we have*

$$\sum_{T\in\mathcal{B}(n)}\prod_{v\in T}\frac{1}{h_v 2^{h_v-1}}=\frac{1}{n!} \tag{4}$$

and

$$\sum_{T\in\mathcal{B}(n)}\prod_{v\in T}\frac{1}{(2h_v+1)2^{2h_v-1}}=\frac{1}{(2n+1)!}. \tag{5}$$

Example. There are five binary trees with $n=3$ vertices:



The hook lengths of T_1, T_2, T_3, T_4 are all the same $1, 2, 3$; but the hook lengths of T_5 are $1, 1, 3$. The left-hand side of (4) is then equal to

$$\frac{4}{1\cdot 2^0\cdot 2\cdot 2^1\cdot 3\cdot 2^2}+\frac{1}{1\cdot 2^0\cdot 1\cdot 2^0\cdot 3\cdot 2^2}=\frac{1}{3!}$$

and the left-hand side of (5) to

$$\frac{4}{3\cdot 2^1\cdot 5\cdot 2^3\cdot 7\cdot 2^5}+\frac{1}{3\cdot 2^1\cdot 3\cdot 2^1\cdot 7\cdot 2^5}=\frac{1}{7!}.$$

Proof. Let

$$P(n)=\sum_{T\in\mathcal{B}(n)}\prod_{v\in T}\frac{1}{h_v 2^{h_v-1}}.$$

With each binary tree $T\in\mathcal{B}(n)$ $(n\geq 1)$ we can associate a triplet (T', T'', u), where $T'\in\mathcal{B}(k)$ $(0\leq k\leq n-1)$, $T''\in\mathcal{B}(n-1-k)$ and u is a vertex of hook length $h_u=n$. Hence

$$P(n)=\sum_{k=0}^{n-1}P(k)P(n-1-k)\times\frac{1}{n\cdot 2^{n-1}}. \tag{6}$$

It is routine to verify that $P(n) = 1/n!$ for $n = 1, 2, 3$. Suppose that $P(k) = 1/k!$ for $k \le n-1$. Then

$$P(n) = \sum_{k=0}^{n-1} \frac{1}{k!(n-1-k)!n \cdot 2^{n-1}} = \frac{1}{2^{n-1}n!} \sum_{k=0}^{n-1} \binom{n-1}{k} = \frac{1}{n!}.$$

By induction, formula (4) is true for any positive integer n.

In the same manner, let

$$Q(n) = \sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \frac{1}{(2h_v + 1)2^{2h_v - 1}}.$$

Using the previous decomposition

$$Q(n) = \sum_{k=0}^{n-1} Q(k)Q(n-1-k) \times \frac{1}{(2n+1) \cdot 2^{2n-1}}. \tag{7}$$

It is routine to verify that $Q(n) = 1/(2n+1)!$ for $n = 1, 2, 3$. Suppose that $Q(k) = 1/(2k+1)!$ for $k \le n-1$. Then

$$\begin{aligned} Q(n) &= \sum_{k=0}^{n-1} \frac{1}{(2k+1)!(2n-2k-1)!(2n+1) \cdot 2^{2n-1}} \\ &= \frac{2}{2^{2n}(2n+1)!} \sum_{k=0}^{n-1} \binom{2n}{2k+1}. \end{aligned}$$

Since

$$\sum_{k=0}^{n-1} \binom{2n}{2k+1} = \sum_{k=0}^{n} \binom{2n}{2k} = \frac{1}{2} \sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n-1},$$

so that

$$Q(n) = \frac{2}{2^{2n}(2n+1)!} 2^{2n-1} = \frac{1}{(2n+1)!}.$$

By induction, formula (5) is true for any positive integer n. ∎

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