

Extreme Value Theory for Multivariate Data

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Extreme events with inherent **multivariate** character

Example : Coastal flooding

European Union project *Neptune* 1995–1997

de Haan & de Ronde (1998), **Bruun & Tawn** (1998)

Extreme		high		high
sea	involve	wave	and	still water
conditions		heights		level (surge)
		HmO		SWL

Data : 828 storm events spread over 13 years in front of
the Dutch coast (near Petten)

Problem :

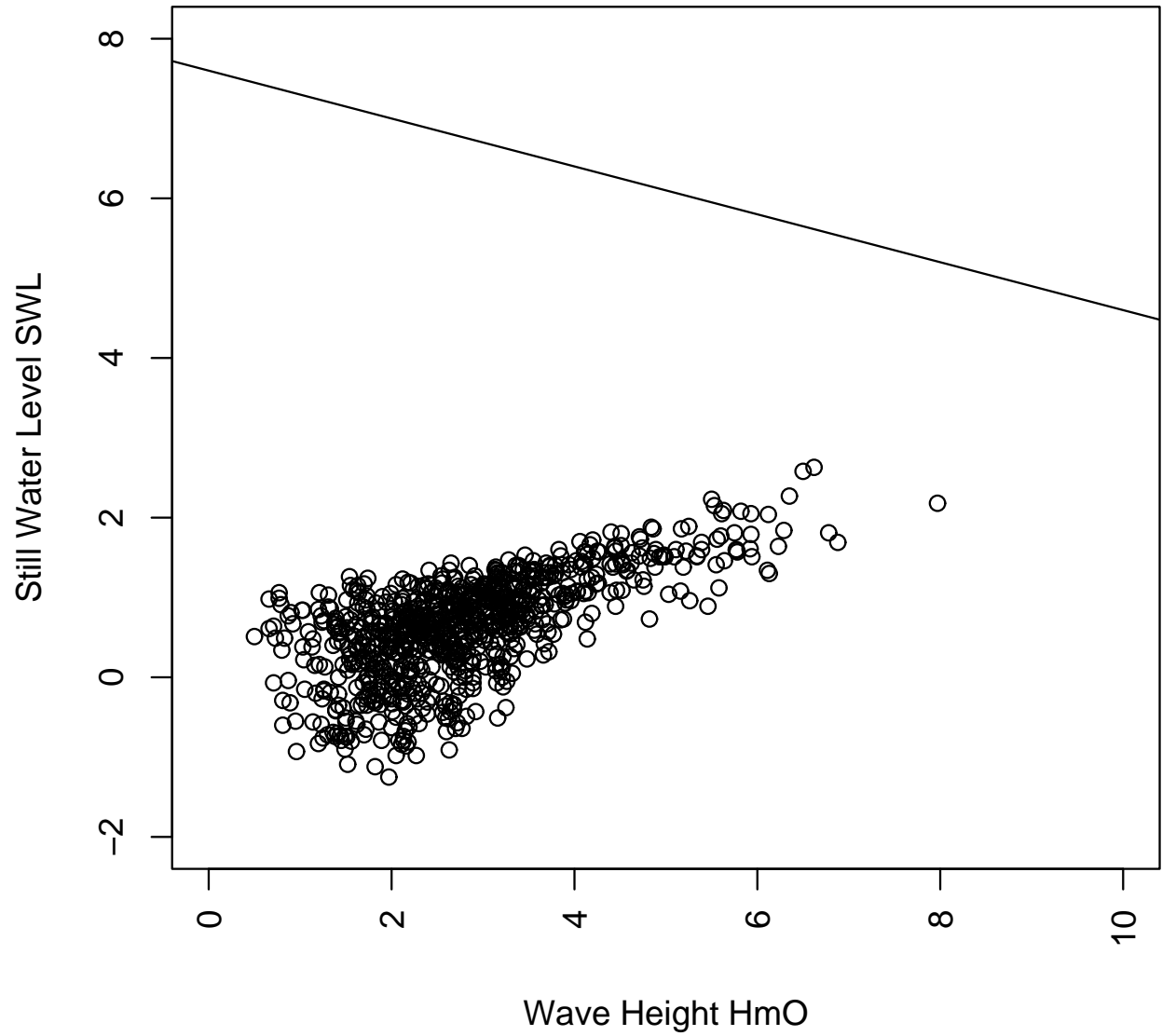
Protection of a dike : Focus on a structure variable

$$\Delta(\mathbf{X}, \nu) = 0.3 \text{ HmO} + \text{SWL} - \nu, \quad \text{for } \mathbf{X} = (\text{HmO}, \text{SWL})$$

Evaluation of $P(\mathbf{X} \in A_\nu)$,

$$\text{where } A_\nu = \{\mathbf{x} \in \mathbb{R}^2 : \Delta(\mathbf{x}, \nu) > 0\}.$$

**Wave Height (Hm0) and Still Water Level (SWL)
recorded during 828 storm events for the Dutch coast**



Other examples :

pollutant concentrations - **Joe, Smith & Weissman** (1992)

reservoir safety - **Anderson & Nadarajah** (1993)

rainfall regime - **Coles & Tawn** (1996)

air quality - **Heffernan & Tawn** (2005)

... among others!

Multivariate context arises from :

- different **variables**
- one variable, at different **sites**
- one variable, at different **times**

Aim of the talk

Survey of the existing models for multivariate extremes

1. Modeling componentwise maxima

Structure of the Multivariate Extreme Value family

Examples - Inference

Limits of the model

2. Modeling under asymptotic independence

Main references

1. Modeling componentwise maxima

Tiago de Oliveira (1958), **Sibuya** (1960), **de Haan & Resnick** (1977), **Deheuvels** (1978), **Pickands** (1981), ...

Books : **Resnick** (1987, 2007), **Coles** (2001),

Beirlant, Goegebeur, Teugels & Segers (2004)

de Haan & Ferreira (2006), ...

2. Modeling under asymptotic independence

Ledford & Tawn (1996, 1997), **Heffernan & Tawn** (2004),

Balkema & Embrechts (2007), **Fougères & Soulier** (2008),...

1. Models for componentwise maxima

Main hypothesis :

Existence of a multivariate domain of attraction

Let $\mathbf{X}_i = (X_{i,1}, \dots, X_{i,d})$, $i = 1, \dots, n$, be i.i.d. random vectors of dimension d with d.f. F . We assume that

$$P \left\{ \frac{\max_i X_{i,1} - b_{n,1}}{a_{n,1}} \leq x_1, \dots, \frac{\max_i X_{i,d} - b_{n,d}}{a_{n,d}} \leq x_d \right\}$$
$$= F^n(a_{n,1} x_1 + b_{n,1}, \dots, a_{n,d} x_d + b_{n,d}) = F^n(\mathbf{a}_n \mathbf{x} + \mathbf{b}_n) \rightarrow G(\mathbf{x}),$$

when $n \rightarrow \infty$, with G d.f. with non-degenerate margins G_1, \dots, G_d .

Univariate margins : parametric structure (GEV)

$$G_j(x) = \exp \left\{ - \left(1 + \frac{x - 1}{\xi} \right)_+^{-1/\xi} \right\} \quad j \in \{1; \dots ; d\}:$$

Multivariate law : no more parametric structure.

However : The possible limits G are the **max-stable** distributions (with non degenerate margins), and they admit nice representations.

Let assume for simplicity that the univariate EV margins are unit Fréchet distributed ($P\{Y_j \leq y\} = e^{-1/y}; \forall y > 0$).

Denote Ω the unit sphere on \mathbb{R}_+^2 .

Representations for the EV d.f. G

(de Haan & Resnick, 1977)

(1) : $G(\mathbf{x}) = \exp(-t^{1^*}\{(0; \mathbf{x}]^c\})$; with $t^{1^*}(tB) = t^{1^*}(B)$,
for all $t > 0$ and B Borel set of $E = [0; \infty]^d \setminus \{\mathbf{0}\}$.

(2) : $G(\mathbf{x}) = \exp\left\{-\int_{\Omega} \bigvee_{j=1}^d \frac{!_j}{x_j} dS(\mathbf{w})\right\}$; with $\int_{\Omega} !_j dS(\mathbf{w}) = 1$:

The *exponent measure* 1^* and the *spectral measure* S are related via :

$$1^* \left\{ \mathbf{y} \in E : \|\mathbf{y}\| > r ; \frac{\mathbf{y}}{\|\mathbf{y}\|} \in A \right\} = \frac{S(A)}{r}; \quad (1)$$

(1) : $G(\mathbf{x}) = \exp(-t^{1*}\{(0; \mathbf{x}]^c\})$; with $t^{1*}(tB) = t^{1*}(B)$.

(2) : $G(\mathbf{x}) = \exp\left\{-\int_{\Omega} \prod_{j=1}^d \frac{!_j}{x_j} dS(\mathbf{w})\right\}$; with $\int_{\Omega} !_j dS(\mathbf{w}) = 1$:

(3) : $G(\mathbf{x}) = P\left(\bigvee_{t_k \leq 1} \mathbf{j}_k \leq \mathbf{x}\right)$; where $\sum_k \mathbf{1}_{(t_k, \mathbf{j}_k) \in \cdot}$ is a non homogeneous Poisson process with intensity $\Lambda([0; t] \times B) = t^{1*}(B)$.

Some arguments : Defining $T : \mathbf{y} \mapsto (\|\mathbf{y}\|; \frac{\mathbf{y}}{\|\mathbf{y}\|})$,

$$\text{Equation (1) is } \tau^* \circ T^{-1}\{(r; \infty) \times A\} = \frac{S(A)}{r}:$$

$$\text{Thus } \tau^*\{(0; \mathbf{x}]^c\} = \tau^* \circ T^{-1}(T\{(0; \mathbf{x}]^c\}) = \int_{T\{(0, \mathbf{x}]^c\}} \frac{1}{r^2} dS(\mathbf{w}) dr:$$

Moreover, write

$$\begin{aligned} T((0; \mathbf{x}]^c) &= \{(r; !) \in (0; \infty) \times \Omega : r\mathbf{w} \in (0; \mathbf{x}]^c\} \\ &= \{(r; !) \in (0; \infty) \times \Omega : r > \bigwedge_{j=1}^d \frac{x_j}{!_j}\}: \end{aligned}$$

This leads therefore to

$$\tau^*\{(0; \mathbf{x}]^c\} = \int_{T\{(0, \mathbf{x}]^c\}} \frac{1}{r^2} dS(\mathbf{w}) dr = \int_{\Omega} \bigvee_{j=1}^d \frac{!_j}{x_j} dS(\mathbf{w}):$$

Several representations exist for the EV d.f. G .

? How to formulate them in terms of a d.f. F in the domain of attraction of G $F \in DA(G)$?

Let consider n i.i.d. observations $\mathbf{X}_i = (X_{i,1}, \dots, X_{i,d})$, $i = 1, \dots, n$, with unit Fréchet margins.

[In practice : $X_{i,j} \tilde{A} Z_{i,j} = 1 - \log\{n - (R_{i,j} - 1)\}$,

where $R_{i,j}$ is the rank of $X_{i,j}$ among $X_{1,j}, \dots, X_{n,j}$.]

Representations for $F \in \mathbf{DA}(G)$

(de Haan & Resnick, 1977)

$$(1) : \lim_{t \rightarrow \infty} \frac{-\log F(t\mathbf{x})}{-\log F(t\mathbf{1})} = \lim_{t \rightarrow \infty} \frac{1 - F(t\mathbf{x})}{1 - F(t\mathbf{1})} = \frac{-\log G(\mathbf{x})}{-\log G(\mathbf{1})} = \frac{1^*([\mathbf{0}; \mathbf{x}]^c)}{1^*([\mathbf{0}; \mathbf{1}]^c)}$$

$$(2) : \lim_{t \rightarrow \infty} tP \left\{ \|\mathbf{X}_i\| > t; \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|} \in A \right\} = S(A);$$

(3) : The point process associated with $\{\mathbf{X}_1=n; \dots; \mathbf{X}_n=n\}$ converges weakly to a non homogeneous Poisson process on E with intensity measure 1^* .

Examples : (i) **Bivariate Cauchy** with density

$$f(x; y) = \frac{1}{2^{1/4}}(1 + x^2 + y^2)^{-3/2}; (x; y) \in \mathbb{R}^2;$$

Polar transformation : $\mathbf{X} \mapsto (\|\mathbf{X}\|; \Theta(\mathbf{X})) := (R; \Theta)$

$$(X; Y) \mapsto \left((X^2 + Y^2)^{1/2}; \text{Arctan}\left(\frac{Y}{X}\right) \right) :$$

Then $R \perp \Theta$, $\Theta \sim \mathcal{U}_{[0, 2\pi)}$ and R has density $\frac{r}{(1 + r^2)^{3/2}}$.

Hence,

$$\lim_{t \rightarrow \infty} P[(R=t; \Theta) \in (\xi; \infty) \times (\mu_1; \mu_2)] = \int_{(\xi, \infty) \times (\theta_1, \theta_2)} \frac{dr d\mu}{r^2 2^{1/4}}.$$

so that

$$\frac{1^*([\mathbf{0}; \mathbf{x}]^c)}{1^*([\mathbf{0}; \mathbf{1}]^c)} = \int_{(x_1, \infty) \times (x_2, \infty)} \frac{dx dy}{(x^2 + y^2)^{3/2}}.$$

(ii) **Multivariate normal** d.f. F_N , with all univariate margins equal to $\mathcal{N}(0, 1)$, and with all its correlations less than 1 ($\mathbb{E}X_i X_j < 1$, for all i, j).

Sibuya, 1960 :

$$F_N^n(a_n \mathbf{x} + b_n) \rightarrow G(\mathbf{x}) = \prod_{j=1}^d \exp\{e^{-x_j}\}.$$

Domain of attraction of the **independence**, with univariate Gumbel margins.

[with $a_n = (2 \log n)^{-1/2}$, and $b_n = (2 \log n)^{1/2} - 1 = 2(\log \log n + \log 4^{1/4}) = (2 \log n)^{1/2}$].

Then S is concentrated on $\{e_i, i = 1, \dots, d\}$, vectors of the canonical basis of \mathbb{R}^d .

Inference

From (2) : $\lim_{t \uparrow \infty} t P \left\{ \|\mathbf{X}_i\| > t ; \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|} \in A \right\} = S(A),$

a candidate to estimate S is the **empirical measure** of the $\left(\frac{\|\mathbf{X}_i\|}{t}, \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|} \right)$'s, for $t = t(n)$ ensuring convergence.

Resnick (1986) : $t = \frac{n}{k_n}$, where $k_n \rightarrow \infty$ and $\frac{n}{k_n} \rightarrow \infty$.

$$\Rightarrow S_n(A) = \frac{1}{k_n} \sum_{i=1}^n \mathbf{1} \left\{ \|\mathbf{X}_i\| > \frac{n}{k_n}, \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|} \in A \right\}.$$

$$S_n(A) = \frac{1}{k_n} \sum_{i=1}^n \mathbf{1} \left\{ \|\mathbf{X}_i\| > \frac{n}{k_n}, \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|} \in A \right\}.$$

More convenient in practice, and asympt. equivalent :

$$S_n(A) = \frac{1}{k_n} \sum_{i=1}^n \mathbf{1} \left\{ \|\mathbf{X}_i\| > \|\mathbf{X}\|_{[k_n]}, \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|} \in A \right\},$$

where $\|\mathbf{X}\|_{[k_n]}$ is the $(n - k_n + 1)$ th order statistic of the $\|\mathbf{X}\|_i$'s.

Various *nonparametric* threshold estimation methods :

Initiated by **de Haan** (1985).

See for a review **Abdous & Ghoudi** (2005).

Different choices of norm :

$\|\mathbf{Z}\| = |Z_1 + Z_2|$: **Joe, Smith & Weissman** (1992),

Capéraà & Fougères (2000) ;

$\|\mathbf{Z}\| = (Z_1^2 + Z_2^2)^{1/2}$: **Einmahl, de Haan & Huang** (1993),

de Haan & de Ronde (1998) ;

$\|\mathbf{Z}\| = Z_1 \vee Z_2$: **Einmahl, de Haan & Sinha** (1997),

de Haan & de Ronde (1998),

Einmahl, de Haan & Piterbarg (2001).

Various *parametric* threshold estimation methods :

Using parametric families of multivariate EV distributions :

Based on **(3)** : **Coles & Tawn** (1991, 1994),
Joe, Smith & Weissman (1992) ;

Based on **(1)** : **Ledford & Tawn** (1996),
Smith, Tawn & Coles (1997).

Remark : In some situations, observations *that can be directly considered from an EV d.f. G* are available.

⇒ Specific techniques (developed in the bivariate case) :

Pickands (1981), **Tawn** (1988), **Tiago de Oliveira** (1989), **Smith, Tawn & Yuen** (1990), **Deheuvels** (1991), **Coles & Tawn** (1991), **Capéreaux, Fougères & Genest** (1997), **Hall & Tajvidi** (2000), **Fils, Guillou & Segers** (2005), among others.

Summary in a very simple case :

Given a sample $(\mathbf{X}_i; i = 1; \dots; n)$, with d.f. F , how to estimate $P(\mathbf{X} \in A)$, where A is a exceptional set ?

Hyp : $F \in \text{DA}(G)$. $F_j(y) = e^{-1/y}$.

If $A = (0; n\mathbf{u}]^c$, using for example **(1)** :

$$P(\mathbf{X} \in A) = 1 - F(n\mathbf{u}) \approx -\frac{1}{n} \log G(\mathbf{u}) = \frac{1}{n} \int_{\Omega} \bigvee_{j=1}^d \frac{!j}{u_j} dS(\mathbf{w}):$$

Making use of the empirical measure S_n , an estimator of $P(\mathbf{X} \in A)$ is then given by

$$\frac{1}{nk_n} \sum_{i=1}^n \left(\bigvee_{j=1}^d \frac{X_{i,j}}{u_j \|X_{i,j}\|} \right) \mathbf{1}\{\|\mathbf{X}_i\| > \|\mathbf{X}\|_{[k_n]}\}:$$

Remarks :

- The choice of the proportion of data k_n used for the estimation of S is a delicate point in practice.
- Dealing with any form of extreme event A is of course not so straightforward, and needs care!

Refer for example to **de Haan & de Ronde** (1998), **Bruun & Tawn** (1998), or **de Haan & Ferreira** (2006) for complete application and evaluation of failure probabilities.

- Most work done with $d = 2$ or 3 .

Limits of the EV model

Asymptotic Independence : $F \in \text{DA}(\text{independence})$.

\Rightarrow • Pb of regularity for MLE.

- Less satisfying results for nonparametric methods.

In this case, the probability mass of **joint tails**

$$\{(X_1 - b_{n,1})/a_{n,1} > x_1, (X_2 - b_{n,2})/a_{n,2} > x_2\}$$

is **of lower order** than that for sets like

$$\{(X_1 - b_{n,1})/a_{n,1} > x_1 \text{ or } (X_2 - b_{n,2})/a_{n,2} > x_2\}.$$

\Rightarrow **No satisfying way to estimate such joint tails using EV distributions.**

Another case where EV models do not provide a satisfying answer under asympt. independence :

Problem : Evaluate $P\{(X, Y) \in A | X > x\}$, for x large.

An answer is given as soon as there exist a, α, b, β, μ non degenerate such that, when $u \rightarrow \infty$,

$$P \left\{ \frac{X - b(u)}{a(u)} > x; \frac{Y - b(u)}{a(u)} \leq y | X > b(u) \right\} \rightarrow P\{(x; +\infty] \times [-\infty; y]\}:$$

Assume for simplicity that X and Y have same margins.

(MDA) : There exist a non degenerate $G, a_n > 0; b_n$, s.t.

$$\lim_{n \rightarrow \infty} P \left\{ \frac{\max_i X_i - b_n}{a_n} \leq x; \frac{\max_i Y_i - b_n}{a_n} \leq y \right\} = G(x; y):$$

(MDA) :

$$\lim_{n \rightarrow \infty} P \left\{ \frac{\max_i X_i - b_n}{a_n} \leq x; \frac{\max_i Y_i - b_n}{a_n} \leq y \right\} = G(x; y):$$

$$\Leftrightarrow \lim_{n \rightarrow \infty} nP\{X > b_n + a_n x \text{ or } Y > b_n + a_n y\} = -\log G(x; y):$$

Reparametrization :

$$n \tilde{A} \frac{1}{P(X > u)}; \quad b_n \tilde{A} u; \quad \text{and} \quad a_n \tilde{A} \tilde{A}(u):$$

$$\Leftrightarrow \lim_{u \rightarrow \infty} \frac{P(X > u + x\tilde{A}(u) \text{ or } Y > u + y\tilde{A}(u))}{P(X > u)} = -\log G(x; y)$$

$$\Leftrightarrow \lim_{u \rightarrow \infty} P \left(\frac{X - u}{\tilde{A}(u)} > x; \frac{Y - u}{\tilde{A}(u)} \leq y \mid X > u \right) = \log F^*(y) - \log G(x; y);$$

Recall the problem : Does there exist $a; \mathbb{R}; b; \mathbb{R}; \mathbb{1}$ non degenerate such that, when $u \rightarrow \infty$,

$$P \left\{ \frac{X - b(u)}{a(u)} > x; \frac{Y - \mathbb{R}(u)}{\mathbb{R}(u)} \leq y \mid X > b(u) \right\} \rightarrow \mathbb{1}\{(x; +\infty] \times [-\infty; y]\}?$$

We obtained via **(MDA)** :

$$\lim_{u \rightarrow \infty} P \left(\frac{X - u}{\tilde{A}(u)} > x; \frac{Y - u}{\tilde{A}(u)} \leq y \mid X > u \right) = \log F^*(y) - \log G(x; y);$$

i.e. answer for $a(u) = \mathbb{R}(u) = u$ and $b(u) = \mathbb{R}(u) = \tilde{A}(u) \dots$ which is unsatisfying under asymptotic independence

$$\mathbf{(AI)} \quad G(x; y) = F^*(x)F^*(y),$$

since $\mathbb{1}\{(x; +\infty] \times [-\infty; y]\} = -\log F^*(x)$ is degenerate in y .

Consequence : Under **Asymptotic Independence**, using extreme value rates yields degenerate distributions for the conditional events :

$$\lim_{u \rightarrow \infty} P(Y \leq u + y\tilde{A}(u) | X > u) = 1;$$

$$\lim_{u \rightarrow \infty} P(X > u + x\tilde{A}(u); Y > u + y\tilde{A}(u) | X > u) = 0:$$

Conclusion : Checking presence or absence of **Asymptotic Independence** might be important !

See additional details in **de Haan & Ferreira** (2006) and **Resnick** (2007).

Short numerical data excursion

Wave data set : 828 storm events on Dutch coast.

X = Water height (HmO), Y = Still water level (SWL).

Two steps : – Marginal analysis

– Analysis of the dependence structure.

See [De Haan & Ferreira \(2006\)](#) for more!

A. Marginal analysis : Univariate extreme fit

Wave data set (n=828)

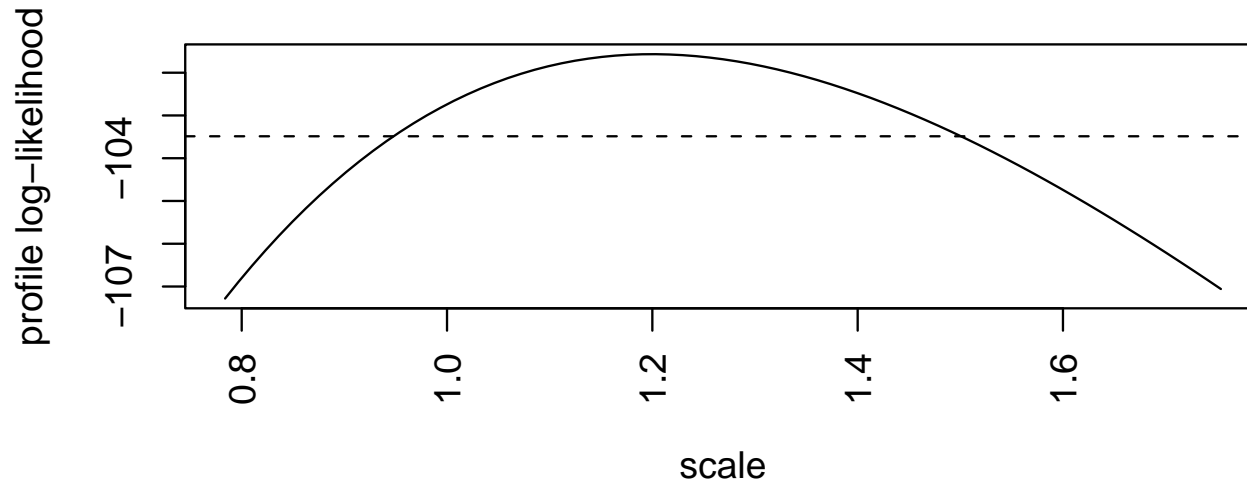
MLE in GPD model and Moment-type estimator both give :

$$\hat{\rho}_{HmO} = -0.22 ; \quad \hat{\rho}_{SWL} = 0:$$

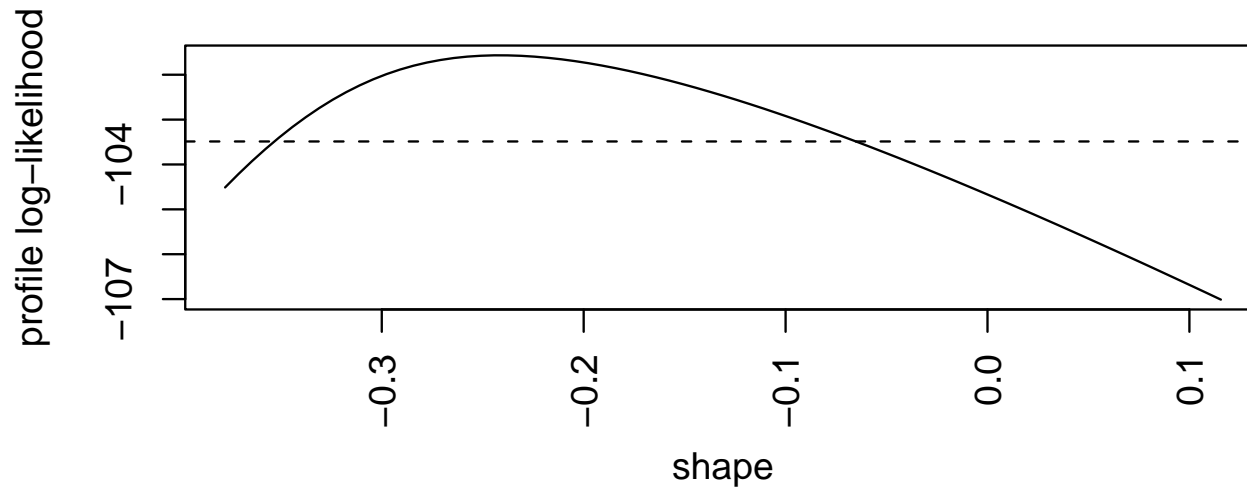
Next pages :

Profile log-likelihood and 95%-confidence intervals for the maximum likelihood parameters of the GPD distributions of HmO and SWL respectively.

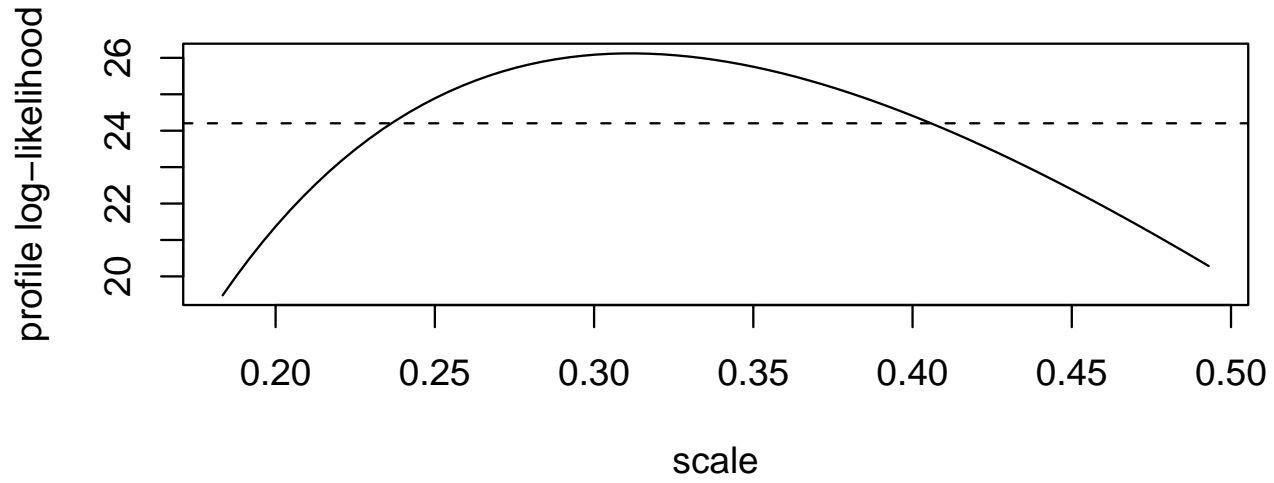
Profile Log-likelihood of Scale



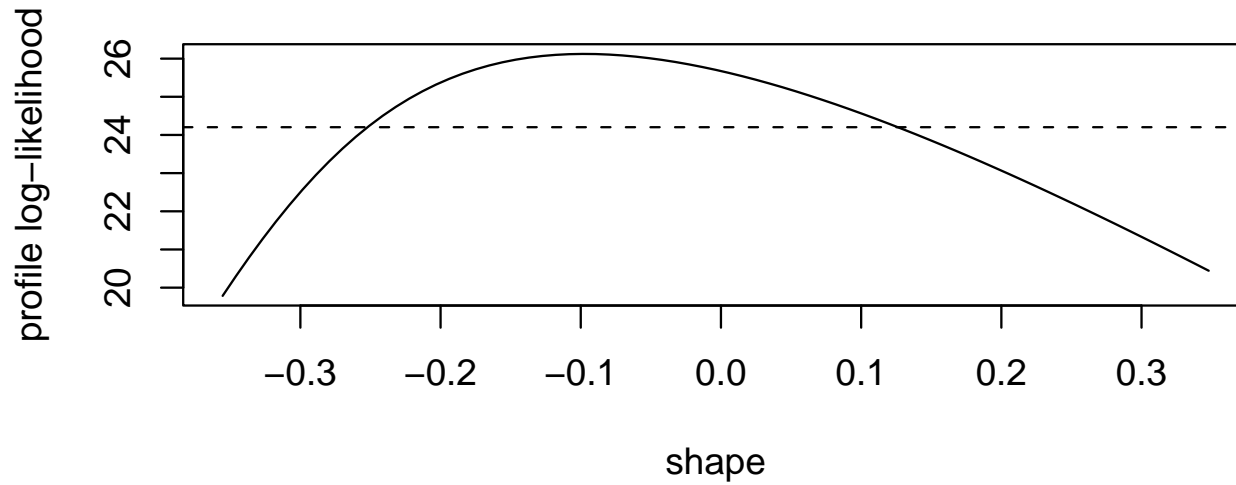
Profile Log-likelihood of Shape



Profile Log-likelihood of Scale



Profile Log-likelihood of Shape



B. Analysis of the dependence structure

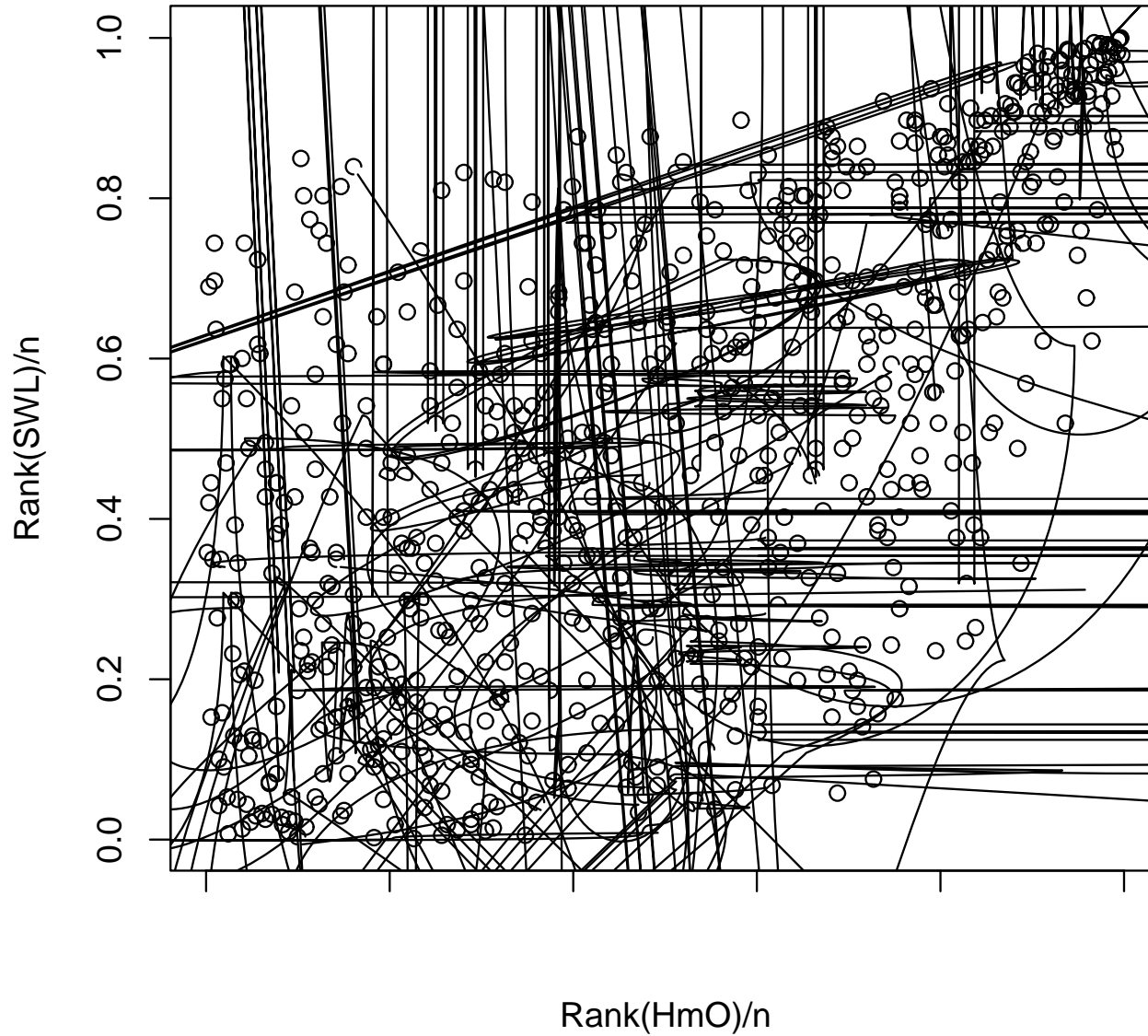
Some tools :

- Empirical copula
- Independence test (Genest & Rémillard, 2004) ; see R package "Copula", subroutine "empcopu.test".
- Coles, Heffernan & Tawn (1999) functions $\hat{A}(u)$ and $\hat{\bar{A}}(u)$.

$$\hat{A}(u) = 2 - \frac{\log C(u; u)}{\log u} \quad ; \quad \hat{\bar{A}}(u) = \frac{2 \log(1 - u)}{\log \bar{C}(u; u)} - 1$$

- Estimation of the spectral measure.

Empirical copula of (HmO,SWL)



– **Genest & Rémillard (2004) independence test :**

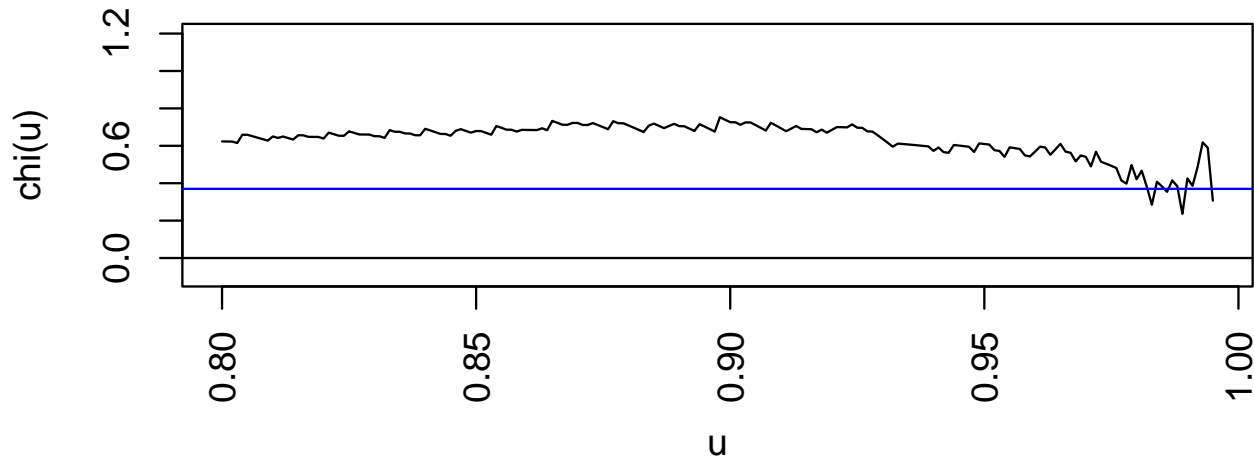
Reject independence $pval = 5 \cdot 10^{-3}$.

When testing all the data, as well as when testing the highest 20% or 10% of the data.

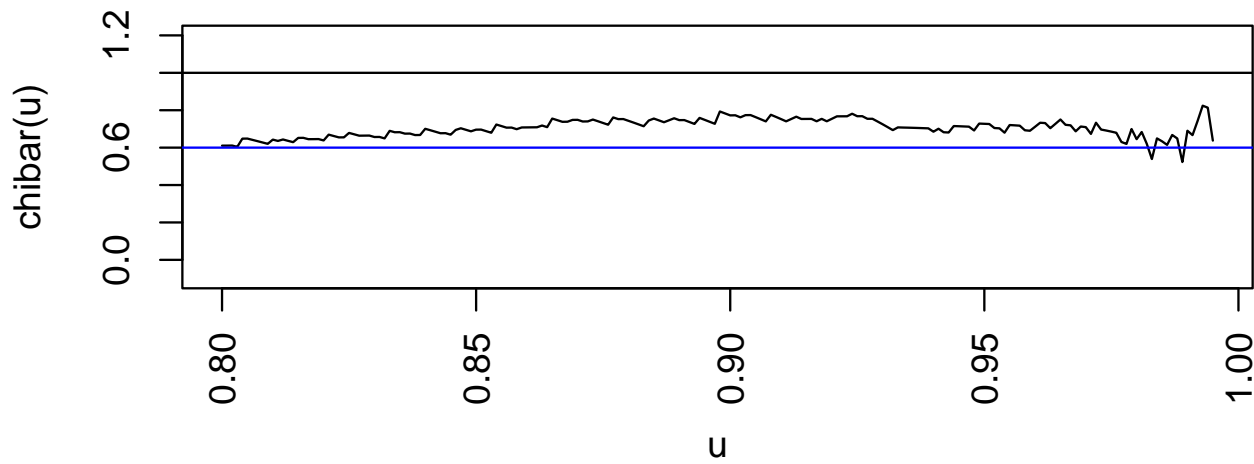
– **Coles, Heffernan & Tawn (1999) measures :**

- $\lim_{u \rightarrow 1} \hat{A}(u) := \hat{A} = 0$ means Asymptotic Independence,
and then $\lim_{u \rightarrow 1} \hat{A}(u) := \hat{A}$ is a second-order dependence parameter.
- $\hat{A} > 0$ means Asymptotic Dependence, and then $\hat{A} = 1$.

Behaviour of extremal dependence via $u \rightarrow \chi(u)$



Behaviour of extremal dependence via $u \rightarrow \chi_{\text{bar}}(u)$



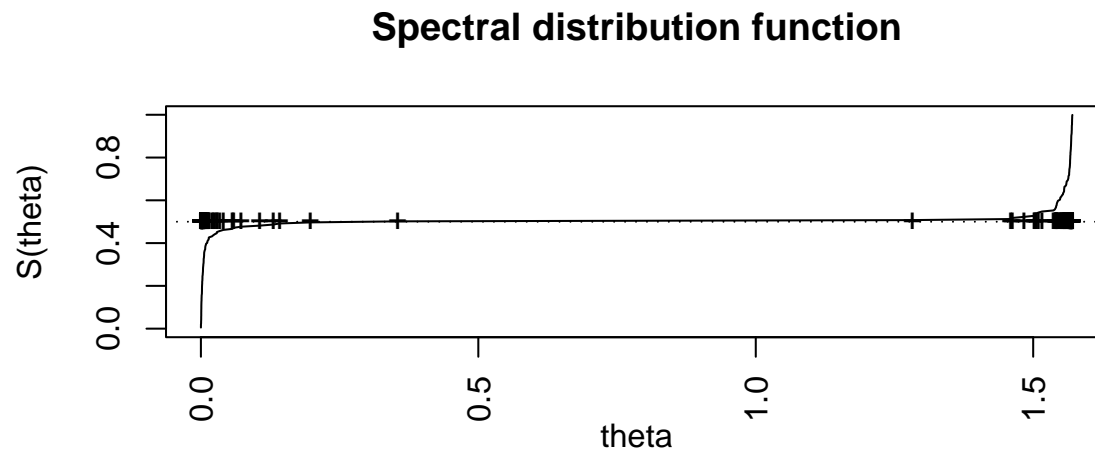
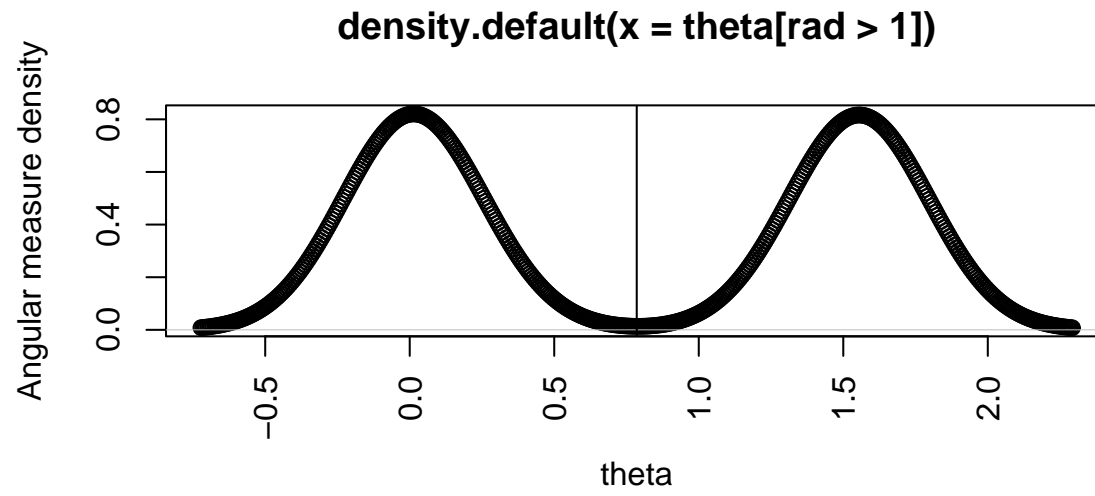
– **Estimating the spectral measure :**

Estimation of the associated distribution function and the density function (assuming it exists).

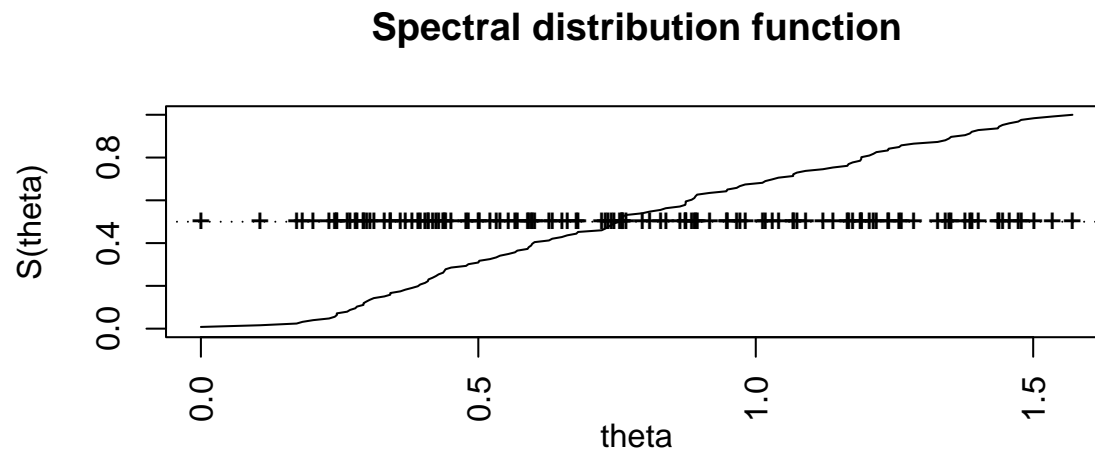
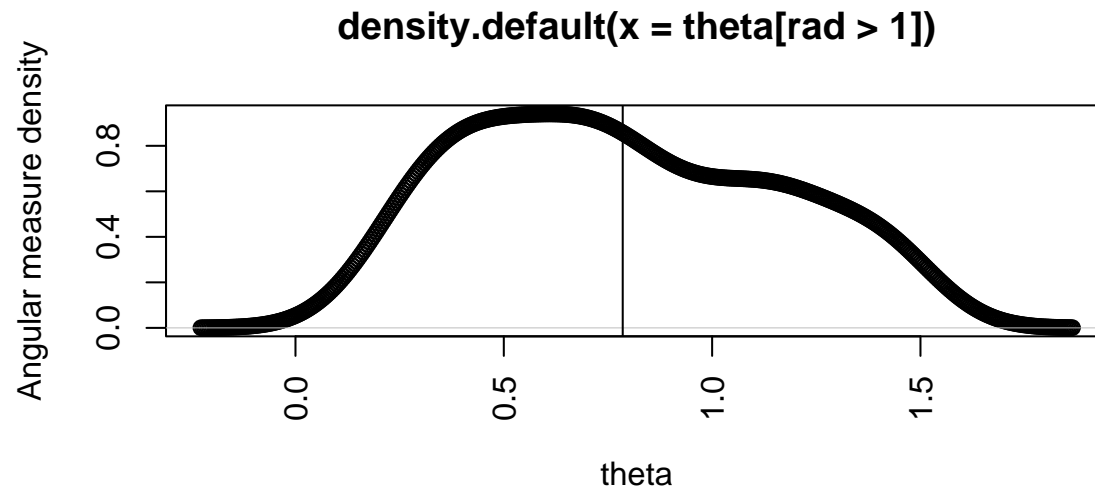
In case of **Asymptotic Independence :**

S is concentrated on the axes ; via another parametrization (using L^2 -norm and usual polar decomposition), we should see a distribution **concentrated on 0 and $\frac{1}{4}=2$.**

Next page : Estimation of the density function and the distribution function of the angular measure of a gaussian sample first, and of the wave data second.



number of upper order statistics used 100



number of upper order statistics used 100

2. Modeling under asymptotic independence

- Bivariate case? **Sibuya** (1960) :

(X_1, X_2) has *asymptotically independent* components iff

$$\text{(AI)} : \lim_{u \downarrow 1} P \{X_2 > F_{X_2}^{i-1}(u) \mid X_1 > F_{X_1}^{i-1}(u)\} = \chi = 0.$$

- Multivariate case?

Theorem : (**Berman**, 1961) Let $\{\mathbf{X}_n; n \geq 1\}$ be i.i.d. from F , with common univariate margins F_1 s.t. $F_1^n(a_n x + b_n) \rightarrow G_1(x)$.

The following assertions are equivalent :

(i) $F^n(a_n \mathbf{x} + b_n \mathbf{1}) = P\left(\bigvee_{i=1}^n \mathbf{X}_i \leq a_n \mathbf{x} + b_n \mathbf{1}\right) \rightarrow \prod_{j=1}^d G_1(x_j):$

(ii) For all $1 \leq k < \ell \leq d$,

$$P\left(\bigvee_{i=1}^n X_{i,k} \leq a_n x_k + b_n ; \bigvee_{i=1}^n X_{i,\ell} \leq a_n x_\ell + b_n\right) \rightarrow G_1(x_k) G_1(x_\ell):$$

(iii) For all $1 \leq k < \ell \leq d$,

$$\lim_{t \rightarrow x^*} P(X_{1,k} > t \mid X_{1,\ell} > t) = 0:$$

2.1 Alternative models for joint tails

Ledford & Tawn (1996, 1997)

Main model (for unit Fréchet margins) :

$$P(Z_1 > z_1 ; Z_2 > z_2) \sim \frac{\mathcal{L}(z_1; z_2)}{z_1^{c_1} z_2^{c_2}}; \quad (2)$$

when $z_1; z_2 \rightarrow \infty$, where $c_1; c_2 > 0$ and $c_1 + c_2 \geq 1$, and $\mathcal{L} : \mathbf{x} \in \mathbb{R}^2 \mapsto \mathcal{L}(\mathbf{x}) > 0$ is a bivariate slowly varying function, ie :

$$\lim_{t \rightarrow \infty} \frac{\mathcal{L}(t\mathbf{x})}{\mathcal{L}(t\mathbf{1})} = \varphi(\mathbf{x});$$

for some positive function φ satisfying $\varphi(t\mathbf{x}) = \varphi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$, $t > 0$. (See **Bingham, Goldie & Teugels**, 1989).

Under this model

$$P(Z_1 > r, Z_2 > r) \sim \frac{\mathcal{L}(r, r)}{r^{c_1+c_2}}$$

Coefficient of tail dependence : $\eta = \frac{1}{c_1 + c_2} \in (0, 1]$

Asymptotic dependence ! $\eta = 1$ and $\mathcal{L}(r, r) \rightarrow 0$

Asymptotic independence ! $\eta < 1$.

Remark : $\bar{\chi} = 2\eta - 1$

More formalization : See De Haan & Ferreira (2006)

Suppose $(X; Y)$ has distribution function F , with continuous marginal distribution functions denoted by F_1 and F_2 .

Second-order model : Assume the existence and positivity of

$$H(x; y) = \lim_{t \rightarrow 0} \frac{P\{1 - F_1(X) < tx; 1 - F_2(Y) < ty\}}{P\{1 - F_1(X) < t; 1 - F_2(Y) < t\}} :$$

Then $P\{1 - F_1(X) < t; 1 - F_2(Y) < t\}$ is a regularly varying function with index $1-\eta$:

$$H(tx; ty) = t^{1/\eta} H(x; y):$$

Estimation of η and inference in submodels of (2) :

Ledford & Tawn (1996, 1997) — Bruun & Tawn (1998)

de Haan & de Ronde (1998) — Peng (1999)

Draisma et al. (2004) Beirlant et al. (2004).

Related models : **Hidden regular variation**

Resnick (2002), **Maulik & Resnick** (2002, 2004),
Heffernan & Resnick (2005)...

2.2 Alternative models for conditional excesses

Modeling and estimation procedures :

Heffernan & Tawn (05) - **Heffernan & Resnick** (06)

Balkema & Embrechts (07) - **Fougères & Soulier** (08)