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Relativistic Diffusion in Gödel's Universe

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Abstract: K. Gödel [G1] published his exact solution to Einstein's field equations in 1949. On the other hand, a general Lorentz invariant operator, associated to the so-called "relativistic diffusion", and making sense in any Lorentz manifold, was introduced recently by Franchi-Le Jan in [F-LJ]. Here is proposed a study of the relativistic diffusion in the framework of Gödel's universe, which contains matter. Such study is related to the determination of a boundary for this non-causal universe.

1. Introduction

K. Gödel [G1] published his exact solution to Einstein equations in 1949. The most striking feature of this cosmological model is that it is non-causal (though locally and geodesically causal), containing closed timelike curves. For this reason, it is generally considered as rather unphysical. Possessing a series of interesting properties, it aroused however a great interest among physicists. For example, it contains rotating matter, but no singularity. Moreover, the explicit exact solutions to Einstein equations are not so many.

W. Kundt [Ku] and S. Chandrasekhar-J.P. Wright [C-W] studied its geodesics, and S. Hawking-G. Ellis ([H-E], Sect. 5.7) emphasise on coordinates (defined by Gödel himself) showing up its rotational symmetry (about any point), to draw a nice picture of its dynamics. D. Malament ([M1,M2]) calculated the minimal energy of a closed timelike curve. K. Gödel [G2] discussed other rotating universes, which are spatially homogeneous, finite, and expanding, and he showed in particular that there are many examples of such strongly causal cosmological models. A.V. Levichev [L] takes advantage of a group structure on Gödel's universe G, to study all left-invariant Lorentz metrics on G (including Gödel's).

The relativistic diffusion was introduced by J. Franchi and Y. Le Jan in [F-LJ], in the framework of general relativity, on an arbitrary Lorentz manifold, as the only diffusion which is covariant under Lorentz isometries. In this sense, it is the Lorentzian analogue of the Brownian motion on a Riemannian manifold. It can be seen as a random perturbation of the timelike geodesic flow. It lives on the pseudo-unit tangent bundle of the considered Lorentz manifold, and is roughly the development of the integrated Brownian motion of the unit pseudo-sphere (in a fixed tangent space). Its precise construction, by means of stochastic differential geometry on the frame bundle, is the purpose of ([F-LJ], Sect. 3). Note that another type of diffusion has been introduced on a Lorentz manifold in [De], maybe physically more significant, but without Lorentz covariance.

As for the Brownian motion of a Riemannian manifold ([A, A-S, Ki, S]), there are natural questions about the relativistic diffusion of a Lorentz manifold such, as: What could be said about its long-proper time behaviour? Which knowledge about the manifold could it yield? It is reasonable to hope that the importance of heat kernels in Riemannian geometry could have an analogue in Lorentzian geometry. This seems to justify studies about the relativistic diffusion.

As for Brownian motion, some answers could depend heavily on the base manifold. Anyway, it seems now hard to formulate and establish general results concerning a generic relativistic diffusion. To appreciate this difficulty, and why it is larger as in the Riemannian-Brownian case, recall that, beyond the non-positivity of the underlying metric, the relativistic diffusion does not live on the base manifold, but only on the pseudo-unit tangent bundle, implying in particular that it is basically seven-dimensional (the Lorentz manifolds of main interest, for obvious physical reasons, have four dimensions); and there is no general reason that it must contain lower-dimensional subdiffusions. Even in the highly symmetric Schwarzschild case (studied in [F-LJ]), the most reduced significant subdiffusion is three-dimensional, which is to be compared with the constantly curved Riemannian case, for which a crucial one-dimensional (radial) subdiffusion fortunately exists.

Till now, the relativistic diffusion has been studied only in two space-times (namely, Minkowski and Schwarzschild-Kruskal-Szekeres ones), which are empty (having vanishing Ricci tensor) and satisfy all causality conditions considered in [H-E]. The simplest Lorentz manifold is Minkowski space, which is flat. The relativistic diffusion in Minkowski space, and especially its long-time behaviour, was first studied by Dudley [Du]. Very recently, Bailleul [B] performed the non-trivial determination of the Poisson boundary of Minkowski space (endowed with the relativistic operator), which is equivalent to the determination of the invariant σ-field of the natural filtration of the relativistic diffusion, and gave a geometric description of this boundary. The long-time (or long-proper time) behaviour of the relativistic diffusion in Schwarzschild-Kruskal-Szekeres space-time was studied in [F-LJ], however without reaching the full determination of the Poisson boundary, which appears complicated, even in this empty and highly symmetric space-time.

Here a study of the relativistic diffusion on Gödel's universe G is proposed, which presents the interest of also having a lot of symmetries (G admits a group structure with respect to which its metric is left-invariant), but to have a non-vanishing Ricci tensor and to be non-causal, two significant differences with the two preceding examples. The leading purpose of the present work is to study, in the framework of G, the behaviour of the relativistic diffusion, with an emphasis on the asymptotic behaviour, which as mentioned above, is intended to open on the Poisson boundary of G. The aim of this work is thus two-fold: on one hand, to show that, despite the non-existence of a causal boundary (but trivial), there is for a space-time like G a non-trivial intrinsic notion of boundary, which admits some geometric description (in terms of beams, i.e. classes of light rays, which can be seen as cylinders); on the other hand, to reinforce with this third example a guess that should hold generally: relativistic diffusions should asymptotically behave as light rays.

This article begins with a detailed study of timelike and lightlike geodesics, taking a different view from Kundt [Ku], Chandrasekhar-Wright [C-W], and going into more detail. For example, a detailed proof of (piece-wise) geodesic transitivity of G is given (Proposition 2). As a conclusion of the study of lightlike geodesics, a definition (Definition 2) of a beam (or boundary point, as an equivalence class of lightlike geodesics, without use of causality) and of convergence to a beam is given, which on one hand appears to be rather natural in this non-causal universe, and on the other hand becomes then justified by the determination of the Poisson boundary of G, even if incomplete. This notion of convergence is reinforced to a certain extent, in the last Sect. 3.8. Thus, the set $\mathcal{B}$ of beams has a natural structure of geometric 3-dimensional boundary, on which the isometry group of Gödel's universe operates. To have a maybe more geometrical intuition or a picture of beams, it is possible to see them as oriented cylinders in G, as explained below, in Remark 4 and just after Definition 2.

Then the relativistic diffusion of G is introduced. In order to study such a 7-dimensional diffusion, some sub-diffusions are considered, of dimensions 1, 2, and 4. A leading concern is here to bring out all asymptotic variables of the relativistic diffusion, or in other words, the invariant σ-field of its natural filtration, which is in turn closely related to the Poisson boundary of G (endowed with its invariant relativistic operator). The clue in this direction is the guess that convergence to a beam should eventually occur. The following theorem, progressively established in Sect. 3 below, shows that this general guess stands out as reinforced, and that some space of beams could generically yield a good notion of Lorentzian boundary, independent of global causality.

The main results of the present article are summarised in the following (see precisely Theorem 1 in Sect. 3.7):

Theorem. (i) *The relativistic diffusion is irreducible (on its 7-dimensional phase space).*

(ii) *Almost surely, the relativistic diffusion path possesses a 3-dimensional asymptotic random variable, and converges to a beam (in the sense of Definition 2 and Sect. 3.8).*

(iii) *The support of possible beams the relativistic diffusion can converge to, is the whole 3-dimensional boundary space of beams.*

Note that Property (i) distinguishes strongly the relativistic diffusion of G from its analogues of Minkowski and Schwarzschild space-times, for which the absolute time component increases strictly with proper time. See Sect. 3.6 below.

As a consequence of this theorem, and on the basis of some secondary results and considerations, the following conjecture appears to hold likely: by the determination of the 3-dimensional asymptotic random variable evoked in (ii) above, the whole invariant σ-field of the relativistic diffusion of Gödel's universe G has been brought out, and then its whole Poisson boundary, which would thus identify with the geometric boundary $\mathcal{B}$.

The only relativistic case in which the Poisson boundary has been determined, up to now, is Minkowski space, by two different methods: Doob's conditioning and then couplings, using an explicit expression of the laws of already found asymptotic variables, in [B]; or alternatively: study of the random walk associated with a lifted relativistic diffusion on Poincaré group, in [B-R]. The use of either method does not seem to be easy in the present curved case (likely as in any other curved case), since neither are the laws of the asymptotic variables explicit, nor is there any Poincaré group symmetry.

The group structure of G (with left-invariant metric, see Sect. 2.1 below) does not seem to lift to some Poincaré-like group structure on the tangent bundle T^1G (or on the frame bundle), as it should to prove efficient in the study of the relativistic diffusion (which lives in T^1G), see Remark 5 below. Finally the Poisson boundary can be determined using time-reversing, in some Riemannian frameworks, see [A-T-U]. But this method does not seem to work here, where a (from infinity on) proper time-reversed relativistic diffusion appears out of reach.

2. Gödel's Pseudo-Metric

Definition 1. *Gödel's universe G is the manifold* $\mathbb{R}^4$*, endowed with coordinates* $\xi := (t, x, y, z)$*, and with the pseudo-metric (having signature* $(+, -, -, -)$*) defined by:*

$$ds^2 := dt^2 - dx^2 + \tfrac{1}{2}e^{2\sqrt{2}\omega x}\, dy^2 + 2e^{\sqrt{2}\omega x}\, dt\, dy - dz^2,$$

for some strictly positive constant ω.

The inverse matrix of this pseudo-metric $((g_{ij}))$ is as follows:

$$((g^{ij})) = \begin{pmatrix} -1 & 0 & 2\,e^{-\sqrt{2}\,\omega\,x} & 0 \\ 0 & -1 & 0 & 0 \\ 2\,e^{-\sqrt{2}\,\omega\,x} & 0 & -2\,e^{-2\sqrt{2}\,\omega\,x} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Recall that a vector $(\dot t, \dot x, \dot y, \dot z)$ above (t, x, y, z) is *timelike* if it belongs to the light cone based at (t, x, y, z), i.e. if and only if $\dot t^2 - \dot x^2 + \frac{1}{2}\,e^{2\sqrt{2}\,\omega\,x}\,\dot y^2 + 2\,e^{\sqrt{2}\,\omega\,x}\,\dot t\,\dot y - \dot z^2 > 0$, or equivalently if and only if $\frac{1}{2}\left[e^{\sqrt{2}\,\omega\,x}\,\dot y + 2\,\dot t\right]^2 > \dot t^2 + \dot x^2 + \dot z^2$. This same vector will be said to be *future-directed* if moreover $e^{\sqrt{2}\,\omega\,x}\,\dot y + 2\,\dot t > 0$. This prescribes continuously a half of the light cone as indicating a preferred direction, seen as that of future.

Accordingly, a piece-wise C^1 path is said to be *timelike* if its tangent vector is everywhere timelike, and *future-directed* if its tangent vector is moreover everywhere future-directed. Along any timelike curve (t_s, x_s, y_s, z_s), the unit pseudo-norm relation, defining proper time s, is:

$$1 + \dot t_s^2 + \dot x_s^2 + \dot z_s^2 = \tfrac{1}{2}\left[e^{\sqrt{2}\,\omega\,x_s}\,\dot y_s + 2\,\dot t_s\right]^2. \tag{0}$$

The isometry group of Gödel's universe is the five-dimensional Lie group generated by:

1) the translations $(t, x, y, z) \mapsto (t + t_0, x, y + y_0, z + z_0)$ of the linear (t, y, z) 3-subspace;
2) the hyperbolic dilatations $(t, x, y, z) \mapsto (t, x + x_0, y\, e^{-\sqrt{2}\,\omega\,x_0}, z)$;
3) the rotational symmetries $(u, r, \phi, z) \mapsto (u, r, \phi + \phi_0, z)$, in the new coordinates system $(u, r, \phi, z) \in \mathbb{R} \times \mathbb{R}_+ \times (\mathbb{R}/\frac{2\pi}{\omega}\mathbb{Z}) \times \mathbb{R}$ defined by $|t - u| < \pi/\omega$ and:

$$e^{\sqrt{2}\omega x} = \cosh(2r) + \sinh(2r)\cos(\omega\phi);\; e^{\sqrt{2}\omega x}\,\omega y = \sinh(2r)\sin(\omega\phi);$$
$$\mathrm{tg}[\tfrac{\omega}{2}(\phi + t - u)] = e^{-2r}\,\mathrm{tg}[\tfrac{\omega\phi}{2}];$$

we have indeed: $ds^2 = [du + 2\sinh^2 r\, d\phi]^2 - 2\omega^{-2}dr^2 - \frac{1}{2}\sinh^2(2r)d\phi^2 - dz^2$.

Gödel ([G1], Sect. 4) proved that these three types of isometries generate indeed the full isometry group. As the action of this group is clearly transitive on $\mathbb{R}^4$, Gödel's universe is an homogeneous space-time.

Letting ω go to 0, we recover Minkowski space-time as limit of Gödel's universe.

Some generalisations of Gödel's pseudo-metric have been proposed since, which include a second parameter, see in particular [R-T]. Not to add computational complexity, we restrict here to the genuine Gödel pseudo-metric of Definition 1.

2.1. A group structure. Gödel's universe can be viewed as a matrix group, hence a Lie group, by means of the following identification:

$$G \ni \xi = (t, x, y, z) \equiv \begin{pmatrix} e^{-\sqrt{2}\,\omega x} & 0 & 0 & y \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ so that } \xi^{-1} \equiv \begin{pmatrix} e^{\sqrt{2}\,\omega x} & 0 & 0 & -e^{\sqrt{2}\,\omega x} y \\ 0 & 1 & 0 & -z \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Lie algebra $\mathcal{G}$ of G is generated by

$$X := -\sqrt{2}\omega \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, Y := \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, Z := \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, T := \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}.$$

The only non-trivial commutation relation is $[Y, X] = \sqrt{2}\omega\, Y$. For $(t, x, y, z) \in \mathbb{R}^4$ we have:

$$\exp[tT + xX + yY + zZ] = \Big(t, x, \big(\tfrac{1-e^{-\sqrt{2}\,\omega x}}{\sqrt{2}\,\omega x}\big)y, z\Big).$$

The left-invariant vector fields $\Big(\mathcal{L}_A f(\xi) := \frac{d_o}{ds} f(\xi\, e^{sA}),\ \forall \xi \in G,\ A \in \mathcal{G},\ f \in C^1(G)\Big)$ are given by:

$$(\mathcal{L}_T, \mathcal{L}_X, \mathcal{L}_Y, \mathcal{L}_Z) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, e^{-\sqrt{2}\,\omega x}\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Considering the Lorentz metric g^0 on $\mathcal{G}$, given in the basis (T, X, Y, Z) by:

$$((g^0_{ij})) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

we see that Gödel's metric g happens to be the left-invariant metric generated on G by the Lorentz metric g^0 on $\mathcal{G}$: $\langle \mathcal{L}_A, \mathcal{L}_A \rangle_g = \langle A, A \rangle_{g^0}$ for any $A \in \mathcal{G}$.

But g is not bi-invariant: the right-invariant vector field associated with the Lie derivative $\frac{d_o}{ds} f(e^{sX}\xi)$ is $\mathcal{L}'_X = \frac{\partial}{\partial x} - \sqrt{2}\,\omega\,\frac{\partial}{\partial y}$, so that $\langle \mathcal{L}'_X, \mathcal{L}'_X \rangle_g = \omega^2 e^{2\sqrt{2}\,\omega x} - 1$ is not constant on G.

All possible left-invariant metrics on G are considered in [L]. Among them, only one kind, of which Gödel's metric is typical, happens to satisfy both important conditions: it is complete and it satisfies the weak energy condition (see Sect. 2.4 below).

We have for any $\xi_0 = (t_0, x_0, y_0, z_0)$, $\xi = (t, x, y, z) \in G$:

$$\xi_0 \times \xi = (t + t_0, x + x_0, y\, e^{-\sqrt{2}\,\omega\, x_0} + y_0, z + z_0).$$

Note that the two first types of isometries listed above are thus merely left translations on G. The third type is however not that simple. The map $\xi \mapsto (e^{-\sqrt{2}\,\omega\, x}, y, t, z)$ is a Lie group isomorphism from G onto $\mathrm{Aff}^{+}(\mathbb{R}) \times \mathbb{R}^2$, where $\mathrm{Aff}^{+}(\mathbb{R})$ denotes the group of increasing affine maps of $\mathbb{R}$. G is two-step solvable, but not nilpotent. It has left Haar measure $d\mu(\xi) = e^{\sqrt{2}\,\omega\, x} dt dx dy dz$. As $\int f(\xi\xi_0) d\mu(\xi) = e^{-\sqrt{2}\,\omega\, x_0} \int f d\mu$, G is not unimodular. It is not semi-simple: the only non-zero scalar product (in the basis (T, X, Y, Z)), with respect to its Killing metric $g^K(\cdot,\cdot) := \mathrm{Trace}(\mathrm{ad}(\cdot) \circ \mathrm{ad}(\cdot))$, is: $g^K(X, X) = 2\omega^2$.

2.2. Timelike geodesics. Geodesics are associated with the Lagrangian $L(\dot\xi, \xi)$, given by:

$$2\, L(\dot\xi_s, \xi_s) = \dot t_s^2 - \dot x_s^2 + \tfrac{1}{2} e^{2\sqrt{2}\,\omega\, x_s}\, \dot y_s^2 + 2\, e^{\sqrt{2}\,\omega\, x_s}\, \dot t_s\, \dot y_s - \dot z_s^2.$$

The equation of geodesics $\dfrac{\partial}{\partial s}\left(\dfrac{\partial L(\dot\xi_s, \xi_s)}{\partial \dot\xi_s^j}\right) = \dfrac{\partial L(\dot\xi_s, \xi_s)}{\partial \xi_s^j}$ reads here:

$$\dot t_s + e^{\sqrt{2}\,\omega\, x_s}\, \dot y_s = a; \tag{1}$$
$$e^{2\sqrt{2}\,\omega\, x_s}\, \dot y_s + 2\, e^{\sqrt{2}\,\omega\, x_s}\, \dot t_s = b; \tag{2}$$
$$\dot z_s = c; \tag{3}$$
$$\ddot x_s + (\omega/\sqrt{2})\, e^{2\sqrt{2}\,\omega\, x_s}\, \dot y_s^2 + \sqrt{2}\,\omega\, e^{\sqrt{2}\,\omega\, x_s}\, \dot t_s\, \dot y_s = 0; \tag{4}$$

for constant a, b, c.

Equations (1) and (2) jointly are equivalent to:

$$\dot t_s = b\, e^{-\sqrt{2}\,\omega\, x_s} - a; \tag{1'}$$

$$\dot y_s = 2\, a\, e^{-\sqrt{2}\,\omega\, x_s} - b\, e^{-2\sqrt{2}\,\omega\, x_s}; \tag{2'}$$

and then using Eqs. (1′), (2′), (3), we see that Eq. (0) is equivalent to:

$$1 + \Big[b\, e^{-\sqrt{2}\,\omega\, x_s} - a\Big]^2 + \dot x_s^2 + c^2 = \tfrac{1}{2}\, b^2\, e^{-2\sqrt{2}\,\omega\, x_s},$$

or equivalently:

$$\dot x_s^2 + \tfrac{1}{2}\Big[2\, a - b\, e^{-\sqrt{2}\,\omega\, x_s}\Big]^2 = a^2 - c^2 - 1. \tag{0'}$$

Note that necessarily $a^2 \geq 1 + c^2$, and also $ab > 0$ (since by (0′), $ab \leq 0$ would imply $a^2 - 1 \geq a^2 - c^2 - 1 - \dot x_s^2 = \frac{1}{2}[2a - be^{-\sqrt{2}\,\omega\, x_s}]^2 \geq 2a^2$, which is clearly impossible).

Then, owing to Eq. (2), Eq. (4) is equivalent to:

$$\frac{\sqrt{2}}{\omega\, b}\; \dot{x}_s + y_s = Y, \qquad \text{for some constant } Y. \tag{4'}$$

Set

$$k := \sqrt{\tfrac{1}{2}[1-(1+c^2)\, a^{-2}]} \;\in [0, \tfrac{1}{\sqrt{2}}[.$$

By (0′), we must have:

$$\dot{x}_s = \sqrt{2}\, a\, k\, \cos(\omega\, \varphi_s), \quad b\, e^{-\sqrt{2}\,\omega\, x_s} = 2\, a - 2\, a\, k\, \sin(\omega\, \varphi_s), \tag{5}$$

for some angular component φ_s; whence after derivation:

$$\dot{\varphi}_s = b\, e^{-\sqrt{2}\,\omega\, x_s} = 2\, a\, [1 - k\, \sin(\omega\, \varphi_s)], \tag{5'}$$

and then:

$$2\, a\, \omega\, (s - s_0) = \int^{\varphi_s} \frac{\omega\, d\varphi}{1 - k\, \sin(\omega\, \varphi)} = \frac{2}{\sqrt{1-k^2}}\, \text{Arctg}\left[\frac{\text{tg}\,(\omega\, \varphi_s/2) - k}{\sqrt{1-k^2}}\right].$$

Therefore

$$\text{tg}\,(\omega\, \varphi_s/2) = \sqrt{1-k^2}\; \text{tg}\left[a\sqrt{1-k^2}\, \omega\, (s - s_0)\right] + k. \tag{6}$$

Moreover by (1′), (1) we have:

$$\dot{t}_s = a - 2\, a\, k\, \sin(\omega\, \varphi_s) = \dot{\varphi}_s - a, \quad e^{\sqrt{2}\,\omega\, x_s}\, \dot{y}_s + 2\dot{t}_s = \dot{\varphi}_s,$$

so that we deduce now directly the following:

Proposition 1. *The timelike geodesics of Gödel's universe G are determined by the following equations:*

$$e^{-\sqrt{2}\,\omega\, x_s} = \tfrac{2a}{b}\, [1 - k\, \sin(\omega\, \varphi_s)], \quad y_s = Y - \tfrac{2\, a}{\omega\, b} k \cos(\omega\, \varphi_s),$$
$$t_s = T_0 - a\, s + \varphi_s, \quad z_s = z_0 + c\, s,$$

the smooth angle φ_s being determined by:

$$\text{tg}\,(\omega\, \varphi_s/2) = \sqrt{1-k^2}\text{tg}\left[a\sqrt{1-k^2}\, \omega\, (s - s_0)\right] + k. \tag{6}$$

The real parameters $a, b, c, Y, T_0, z_0, s_0$ are constant along each geodesic, and such that
$a^2 \geq 1 + c^2$, $ab > 0$, without any other constraint, and $k = \sqrt{\tfrac{1}{2}[1-(1+c^2)\, a^{-2}]} \in [0, \tfrac{1}{\sqrt{2}}[$.
The projection in the (x, y)-plane is bounded and periodic, and satisfies:

$$\left[\tfrac{b}{2\, a}\, e^{-\sqrt{2}\,\omega\, x_s} - 1\right]^2 + \left[\tfrac{\omega\, b}{2\, a}\, (y_s - Y)\right]^2 = k^2. \tag{7}$$

Such timelike geodesic is future-directed if and only if the non-vanishing speed

$$e^{\sqrt{2}\,\omega\, x_s}\, \dot{y}_s + 2\, \dot{t}_s = \dot{\varphi}_s = 2\, a\, [1 - k\, \sin(\omega\, \varphi_s)]$$

is positive, hence if and only if $a > 0$.

Remark 1. We deduce from Proposition 1 the following formula:

$$t_s = T_0' - a\,s + \tfrac{2}{\omega}\,\text{Arctg}\Big(\sqrt{1-k^2}\;\text{tg}\Big[a\sqrt{1-k^2}\,\omega\,(s-s_0)\Big] + k\Big), \tag{8}$$

in which the successive determinations of Arctg, at the successive proper time values $s \in s_0 + \frac{\pi}{a\sqrt{1-k^2}\,\omega}(\frac{1}{2}+\mathbb{Z})$, are understood to be chosen conveniently, in order that the absolute time coordinate (t_s) be continuous, as it must be. Observe that (t_s) is strictly monotonic if and only if $k \le \frac{1}{2}$, or equivalently, if and only if $(1+c^2) \le a^2 \le 2(1+c^2)$. Note that this last observation is related to the global non-causality of Gödel's space-time: there are future-directed geodesic arcs along which absolute time (t_s) decreases.

For example, taking $a > 0$ and $k = 1/\sqrt{3}$, and running the proper time interval $\Big[s_0, s_0 + \frac{\sqrt{3}\text{Arctg}\sqrt{2}}{\sqrt{2}\,a\,\omega}\Big]$, we see tg $(\frac{\omega\varphi_s}{2})$ running the interval $\Big[\frac{1}{\sqrt{3}}, \sqrt{3}\Big]$ and then φ_s running an interval $\big[\frac{\pi}{3\omega} + \frac{2n\pi}{\omega}, \frac{2\pi}{3\omega} + \frac{2n\pi}{\omega}\big]$, so that $t_{s_0+\frac{\sqrt{3}\,\text{Arctg}\sqrt{2}}{\sqrt{2}\,a\,\omega}} - t_{s_0} = \frac{1}{\omega}\big[\frac{\pi}{3} - \frac{\sqrt{3}\,\text{Arctg}\sqrt{2}}{\sqrt{2}}\big] < 0$.

Remark 2. The case $k=0$ is particular. It implies (using Eqs. (0′), (1′), (2′)): $\dot{t}^2 = 1 + \dot{z}^2$ and $\dot{x} = \dot{y} = 0$, and then:

$$(x_s, y_s) \text{ constant and } t_s = t_0 + as,\; z_s = z_0 + c\,s, \text{ with } a^2 = 1 + c^2.$$

Reciprocally, if $\dot{x}_0 = \dot{y}_0 = 0$, then (by Eqs. (2′), (0′)) the corresponding geodesic must satisfy also $k=0$, and then be included in the phase subspace defined by:

$$\mathcal{E}_0 := \{\dot{x} = \dot{y} = 0\} = \Big\{\dot{t}^2 = 1 + \dot{z}^2;\; \dot{x} = \dot{y} = 0\Big\}.$$

Therefore, the case $k=0$ corresponds to the geodesically stable phase subspace $\mathcal{E}_0$.

Remark 3. Every timelike geodesic is defined for all proper times s, unbounded and causal. Moreover, it never accumulates near its past.

Proof. This is clear for timelike geodesics such that $k=0$ by Remark 2, so that we can restrict to a timelike geodesic such that $k>0$. Then, if for proper times $s < s'$ we had $(x_{s'}, y_{s'}, t_{s'}) = (x_s, y_s, t_s)$, then by Proposition 1 we should have:

$$\text{tg}\Big[a\sqrt{1-k^2}\,\omega(s'-s_0)\Big] = \text{tg}\Big[a\sqrt{1-k^2}\,\omega(s-s_0)\Big], \quad \text{then } s' = s + \tfrac{\pi\,n}{|a|\omega\sqrt{1-k^2}}$$
$$\text{with } n \in \mathbb{N}^*,$$

whence by Eq. (8) : $t_{s'} - t_s = \text{sign}(a)\Big(\frac{2\pi\,n}{\omega} - \frac{\pi\,n}{\omega\sqrt{1-k^2}}\Big)$, and then the contradiction:

$$0 = \tfrac{n\pi}{\omega}\,(2 - (1-k^2)^{-1/2}) \ge \tfrac{\pi}{\omega}.$$

Similarly, if we had proper times $s < s'$ with difference $s'-s$ bounded away from 0, such that $(x_{s'}, y_{s'}, t_{s'})$ be arbitrarily close to (x_s, y_s, t_s), then by Proposition 1 we should have tg $\Big[a\sqrt{1-k^2}\,\omega(s'-s_0)\Big]$ arbitrarily close to tg $\Big[a\sqrt{1-k^2}\,\omega(s-s_0)\Big]$, whence $s'-s$ arbitrarily close to $\frac{\pi}{|a|\omega\sqrt{1-k^2}}\mathbb{N}^*$, whence by Eq. (8), 0 arbitrarily close to $\Big(\frac{2\pi}{\omega} - \frac{\pi}{\omega\sqrt{1-k^2}}\Big)\mathbb{Z}^*$, which yields the same contradiction as above. □

The following statement, which will be used later to ensure the irreducibility of the relativistic diffusion, shows up the non-causal structure of Gödel's universe, despite the preceding Remark 3: the causal past of any point of Gödel's universe is the whole Gödel's universe. In particular, the causal boundary, in the sense of Penrose (or Geroch-Kronheimer-Penrose, see ([H-E], Sect. 6.8)), reduces to a single point.

Proposition 2. *Gödel's universe G is piece-wise geodesically transitive: any two points of it can be linked by a piece-wise geodesic future-directed timelike continuous path.*

Proof. This result derives from the two following references:

- Property (6) of [G1] shows up future-directed timelike curves, whose spacial projection draws a circle and along which absolute time decreases, which entails in fact (though this is not explained in [G1]) the transitivity of space-time under future-directed timelike curves;
- [P], according to which a point p of any given space-time can be linked to another point p' by a future-directed timelike curve if and only if p can be linked to p' by a piece-wise geodesic future-directed timelike path.

However, for the sake of completeness, let us deduce the claim directly from the above.

Using first Remark 2, by means of a single geodesic arc, with $a = \sqrt{2}$ and $c = \pm 1$, we reduce the proof to the case of points p, p' (to be linked) having the same coordinate z. Fixing from now on $c = 0$, we can forget the coordinate z. By homogeneity of the space-time, we can then consider that $p = (t, 0, 0, 0)$ and $p' = (0, x, y, 0)$, for arbitrary given $(t, x, y) \in \mathbb{R}^3$.

Let us use Proposition 1, considering the three following examples of future-directed geodesic moves:

1) Using Remark 1, i.e. taking $k = \frac{1}{\sqrt{3}}, a = \sqrt{3}, \varphi_s = \frac{\pi}{3\omega}, \varphi_{s'} = \frac{2\pi}{3\omega},$ we get:

$$x_{s'} = x_s, \quad y_{s'} - y_s = \frac{2}{\omega\, b\sqrt{3}}, \quad t_{s'} - t_s = \frac{1}{\omega}\left[\frac{\pi}{3} - \frac{\sqrt{3}\,\mathrm{Arctg}\sqrt{2}}{\sqrt{2}}\right] < 0.$$

Hence, moving so several times, we can decrease the absolute time t arbitrarily largely, without changing the x coordinate, and, choosing b large, almost without changing the y coordinate. We can thus suppose now that $p = (t, 0, y_0, 0)$, with $t < -\frac{6|x|}{\log 3}\left[1 - \frac{1}{\sqrt{3}}\right] - \frac{\pi}{\omega}$ and $|y_0| < 1$.

2) Taking $k = \frac{1}{2}, a = \sqrt{2}$, and $(\varphi_s, \varphi_{s'}) = (\frac{\pi}{2\omega}, \frac{3\pi}{2\omega})$ or $(\frac{-\pi}{2\omega}, \frac{\pi}{2\omega})$, we get:

$$x_{s'} - x_s = \pm\frac{\log 3}{\omega\sqrt{2}}, \quad y_{s'} = y_s, \quad t_{s'} - t_s = \frac{\pi}{\omega}\left[1 - \frac{1}{\sqrt{3}}\right].$$

Hence, moving so approximately $\frac{|x|}{\log 3}\omega\sqrt{2}$ times, we can suppose now that $p = (t', x', y_0, 0)$, with $t' < -\frac{|x|}{\log 3}\left[1 - \frac{1}{\sqrt{3}}\right] - \frac{\pi}{\omega}$ and $0 < x - x' < \frac{\log 3}{\omega\sqrt{2}}$. Moving once more in a similar way, i.e. taking $k = \frac{1}{2}, \varphi_s = \frac{\pi}{2\omega}$, a large b, and a convenient $\varphi_{s'} \in]\frac{\pi}{2\omega}, \frac{3\pi}{2\omega}[$, we link p to $p_1 = (t_1, x, y_1, 0)$, with $t_1 < -\frac{\pi}{\omega}$ and $|y_1| < 2$.

3) Taking $k = \frac{1}{2}, a = \sqrt{2}$, and $(\varphi_s, \varphi_{s'}) = (0, \frac{\pi}{\omega})$ or $(\frac{\pi}{\omega}, \frac{2\pi}{\omega})$, we get:

$$x_{s'} = x_s, \quad y_{s'} - y_s = \pm\frac{2\sqrt{2}}{\omega b}, \quad t_{s'} - t_s = \frac{\pi}{\omega}\left[1 - \frac{2}{3\sqrt{3}}\right].$$

Hence, moving so once, and choosing b conveniently, we link p_1 to $p_2 = (t_2, x, y, 0)$, with $t_2 < 0$.

Finally, observe from Remark 2 that (taking $k = c = 0$) there are future-directed timelike geodesics increasing at will the coordinate t, without changing any other coordinate: this allows to link p_2 to p'. □

2.3. Lightlike geodesics and boundary of G. Equations (1), (2), (3), (4) remain the same, while the pseudo-norm Eq. (0) is replaced by:

$$\dot{t}_s^2 + \dot{x}_s^2 + \dot{z}_s^2 = \tfrac{1}{2}\left[e^{\sqrt{2}\,\omega\, x_s}\,\dot{y}_s + 2\,\dot{t}_s\right]^2. \qquad (0'')$$

As previously, knowing Eqs. (1) and (2), Eq. (4) is equivalent to

$$\frac{\sqrt{2}}{\omega\, b}\,\dot{x} + y = Y, \quad \text{for some constant } Y, \qquad (4')$$

and is implied by Eqs. (0″), (1), (2), (3). Thus lightlike geodesics are the solutions to the system:

$$\dot{t}_s = b\, e^{-\sqrt{2}\,\omega\, x_s} - a; \qquad (1')$$

$$\dot{y}_s = 2\,a\, e^{-\sqrt{2}\,\omega\, x_s} - b\, e^{-2\sqrt{2}\,\omega\, x_s}; \qquad (2')$$

$$\dot{z}_s = c; \qquad (3)$$

and

$$\dot{x}_s^2 + \tfrac{1}{2}\left[2\,a - b\, e^{-\sqrt{2}\,\omega\, x_s}\right]^2 = a^2 - c^2. \qquad (0''')$$

Hence we must have again $a^2 \geq c^2$, and $ab > 0$ (if not, the trajectory must be constant), and, setting now

$$\kappa := \sqrt{\tfrac{1}{2}(1 - c^2/a^2)} \in [0, \tfrac{1}{\sqrt{2}}],$$

we get the same equations for the geodesic motion as in Proposition 1, merely with κ replacing k.

As the parameter s cannot here any longer have a meaning like proper time, but stands only for an affine parameter, determined up to a change $s \mapsto us+v$, the constant (a, b, c) is now irrelevant. Note that on the contrary, the constant Y of Eq. (4′) is relevant. The only meaningful a priori parameter for a lightlike geodesic is the "impact" parameter:

$$B = (\ell, \varrho, Y) := \left(\frac{c}{a}, \frac{b}{a}, Y\right) \in \mathcal{B} := [-1, 1] \times \mathbb{R}_+^* \times \mathbb{R}. \qquad (9)$$

Eliminating s, we see indeed that a lightlike geodesic having impact parameter B solves:

$$\ell\, dt = (\varrho\, e^{-\sqrt{2}\,\omega\, x} - 1)\, dz; \quad (2 - \varrho\, e^{-\sqrt{2}\,\omega\, x})\, e^{-\sqrt{2}\,\omega\, x}\, dt = (\varrho\, e^{-\sqrt{2}\,\omega\, x} - 1)\, dy;$$

$$(\varrho\, e^{-\sqrt{2}\,\omega\, x} - 1)\, dx = \pm\sqrt{1 - \ell^2 - \tfrac{1}{2}(2 - \varrho\, e^{-\sqrt{2}\,\omega\, x})^2}\; dt.$$

Let us sum up the description of lightlike geodesics in the following statement.

Proposition 3. *Any lightlike geodesic $(x_\tau, y_\tau, z_\tau, t_\tau)$ having impact parameter $B = (\ell, \varrho, Y) \in \mathcal{B}$ satisfies, for an additional parameter $(Z_0, T_0) \in \mathbb{R}^2$ and for any real τ:*

$$e^{-\sqrt{2}\,\omega\, x_\tau} = \frac{2}{\varrho} \times \left(1 - \frac{2\sqrt{1-\ell^2}\left(\sqrt{1+\ell^2}\ \mathrm{tg}\,\tau + \sqrt{1-\ell^2}\right)}{2 + \left(\sqrt{1+\ell^2}\ \mathrm{tg}\,\tau + \sqrt{1-\ell^2}\right)^2}\right);$$

$$y_\tau = Y + \frac{\sqrt{2(1-\ell^2)}}{\omega\varrho}\left(1 - \frac{4}{2 + \left(\sqrt{1+\ell^2}\ \mathrm{tg}\,\tau + \sqrt{1-\ell^2}\right)^2}\right);$$

$$z_\tau = Z_0 + \frac{\ell\,\tau}{\omega\sqrt{(1+\ell^2)/2}};$$

$$t_\tau = T_0 - \frac{\tau}{\omega\sqrt{(1+\ell^2)/2}} + \frac{2}{\omega}\,\mathrm{Arctg}\left(\sqrt{(1+\ell^2)/2}\ \mathrm{tg}\,\tau + \sqrt{(1-\ell^2)/2}\right);$$

$$\mathcal{C}_B:\quad \left[\frac{\varrho}{2}\, e^{-\sqrt{2}\,\omega\, x_\tau} - 1\right]^2 + \left[\frac{\omega\,\varrho}{2}\,(y_\tau - Y)\right]^2 = \frac{1-\ell^2}{2}.$$

Remark 4. The last equation in Proposition 3 shows that to any given lightlike geodesic is associated a cylinder $\mathcal{C}_B$, parallel to the (t, z)-coordinate plane. Reciprocally, by Proposition 3 again, any lightlike geodesic which is drawn on the cylinder $\mathcal{C}_B$ has a prescribed projection on the (x, y)-coordinate plane (up to changing affine parameter τ). The equations displayed in Proposition 3 define a lightlike geodesic associated to any given $B \in \mathcal{B}$.

The successive determinations of Arctg in the above expression of t_τ are understood as in Remark 1, to be consistent with the continuity of t.

Considering then any continuous angular parameter $\varphi = \varphi_\tau$ (determined modulo $2\pi/\omega$) such that

$$\mathrm{tg}\,(\omega\,\varphi_\tau/2) = \sqrt{(1+\ell^2)/2}\ \mathrm{tg}\,\tau + \sqrt{(1-\ell^2)/2},$$

by Proposition 3 we have:

$$\varrho\, e^{-\sqrt{2}\,\omega\, x_\tau} = 2 - \sqrt{2\,(1-\ell^2)}\,\sin(\omega\,\varphi_\tau) - \sqrt{2\,(1-\ell^2)}\,\cos(\omega\,\varphi_\tau), \quad \text{and} \quad \omega\,\varrho\, y_\tau = \omega\,\varrho\, Y$$

together with:

$$t_\tau = T_0 - \frac{\tau}{\omega\sqrt{(1+\ell^2)/2}} + \varphi_\tau, \quad z_\tau + \ell\, t_\tau = Z_0 + \ell\, T_0 + \ell\,\varphi_\tau.$$

Since the function $\tau \mapsto \omega\varphi_\tau - 2\tau$ is π-periodic, the functions

$$\tau \mapsto \omega\, t_\tau - 2\,(1 - [2(1+\ell^2)]^{-1/2})\,\tau \quad \text{and} \quad \tau \mapsto z_\tau - \frac{\ell\, t_\tau}{\sqrt{2(1+\ell^2)} - 1} = Z_0 - \frac{\ell(T_0 + \varphi_\tau - 2\tau/\omega)}{\sqrt{2(1+\ell^2)} - 1}$$

are π-periodic too. This implies that t_τ wanders out to infinity, nearly linearly, and that the projection on the (t, z)-coordinate plane of any lightlike geodesic has an asymptotic direction:

$$\lim_{\tau\to\pm\infty} \frac{z_\tau}{t_\tau} = \frac{\ell}{\sqrt{2(1+\ell^2)}\,-1}.$$

This prescribes geometrically the sign of parameter ℓ, which was not determined by the cylinder $\mathcal{C}_B$ alone, which however determines $(|\ell|, \varrho, Y)$. Note that the impact parameter $B = (\ell, \varrho, Y) \in \mathcal{B}$ has thus indeed a clear geometrical meaning or picture: it can be identified with the oriented cylinder $(\mathcal{C}_B = \mathcal{C}_B(|\ell|, \varrho, Y), \mathrm{sign}(\ell))$.

Note finally that the additional parameter (Z_0, T_0) depends on a translation on the parameter (τ, φ_τ), and then, contrary to $B = (\ell, \varrho, Y)$, is geometrically irrelevant.

Recall that in a strongly causal space-time, it seems natural to use the causal boundary, in the sense of Penrose, to classify lightlike geodesics by gathering in an equivalence class, called a beam, all geodesics which converge to a given causal boundary point (having asymptotically the same past, see ([H-E], Sect. 6.8)). On the contrary, in the present setting (recall Proposition 2) such classification is totally inoperative. It seems that no alternative classification has been proposed so far, which is relevant in a non-causal setting.

Now, owing to the above Remark 4, we are led to adopt here the following alternative classification of lightlike geodesics into beams, and then also, to see the 3-dimensional space of beams as an alternative notion of (non-causal) boundary, as follows:

Definition 2. *Let us call* **beam**, *or* **boundary point**, *of Gödel's universe, any equivalence class of lightlike geodesics, identifying those which have the same impact parameter* $B = (\ell, \varrho, Y) \in \mathcal{B}$. *Thus* $\mathcal{B} = [-1, 1] \times \mathbb{R}_+^* \times \mathbb{R}$ *is the* **boundary** *of Gödel's universe.*

Let us say that a curve $s \mapsto \xi_s = (t_s, x_s, y_s, z_s)$ *of class* C^1 *in Gödel's universe* **converges to the beam** $B = (\ell, \varrho, Y)$ *if, setting:*

$$a_s := \dot t_s + e^{\sqrt{2}\,\omega\, x_s}\,\dot y_s, \quad b_s := e^{\sqrt{2}\,\omega\, x_s}\,(2\,\dot t_s + e^{\sqrt{2}\,\omega\, x_s}\,\dot y_s), \quad \textit{and} \quad Y_s := \frac{\sqrt{2}\,\dot x_s}{\omega\, b_s} + y_s,$$

the following convergences hold, as $s \to +\infty$*:*

$$\frac{\dot z_s}{a_s} \longrightarrow \ell, \quad \frac{b_s}{a_s} \longrightarrow \varrho, \quad Y_s \longrightarrow Y, \quad \Big[\frac{\varrho}{2}\,e^{-\sqrt{2}\,\omega\, x_s} - 1\Big]^2 + \Big[\frac{\omega\,\varrho}{2}\,(y_s - Y)\Big]^2 \longrightarrow \frac{1-\ell^2}{2}.$$

Recall from Remark 4 the following picture of beams (or impact parameters): any beam $B = (\ell, \varrho, Y) \in \mathcal{B}$ can be identified with the oriented cylinder $(\mathcal{C}_B = \mathcal{C}_B(|\ell|, \varrho, Y), \mathrm{sign}(\ell))$.

The notion of convergence to the boundary $\mathcal{B}$ can be reinforced to a certain extent: see Corollary 7 and Remark 13, in the last Sect. 3.8. We saw above that any lightlike geodesic belonging to a beam B converges to it. On the contrary, a timelike geodesic does not converge to any beam: by Proposition 1, we get indeed $B = (\frac{c}{a}, \frac{b}{a}, Y) \in \mathcal{B}$, but the cylinder $\mathcal{C}_B$ has too small a "radius", since we have $k^2 = [1 - \ell^2 - a^{-2}]/2 < [1 - \ell^2]/2$.

Proposition 4. *The isometry group of Gödel's universe (recall Sect. 2) operates on the boundary $\mathcal{B}$, so that the above Definition 2 is consistent. It acts precisely as follows:*

(1) *The translation* $(t, x, y, z) \mapsto (t + t_0, x, y + y_0, z + z_0)$ *changes* (ℓ, ϱ, Y) *into* $(\ell, \varrho, Y + y_0)$.

(2) *The hyperbolic dilatation* $(t, x, y, z) \mapsto (t, x + x_0, y\, e^{-\sqrt{2}\,\omega\, x_0}, z)$ *changes* (ℓ, ϱ, Y) *into*

$$(\ell, \varrho\, e^{\sqrt{2}\,\omega\, x_0}, Y e^{-\sqrt{2}\,\omega\, x_0}).$$

(3) *The rotational symmetry* $(u, r, \phi, z) \mapsto (u, r, \phi + \phi_0, z)$ *changes* (ℓ, ϱ, Y) *into*

$$\bigg(\ell,\ \alpha + [\varrho - \alpha]\cos(\omega\,\phi_0) - \omega\,\varrho\, Y\, \sin(\omega\,\phi_0),\ \frac{\omega\,\varrho\, \cos(\omega\,\phi_0) + [\varrho - \alpha]\sin(\omega\,\phi_0)}{\omega\,[\alpha + [\varrho - \alpha]\cos(\omega\,\phi_0) - \omega\,\varrho\, \sin(\omega\,\phi_0)]}\bigg),$$

where $\alpha := \frac{2a\cosh(2r) - \sinh^2(2r)\,\dot\phi}{\dot u + 2\sinh^2 r\,\dot\phi}$ *is constant under* $\phi \mapsto \phi + \phi_0$*, and on each geodesic. We have indeed:* $\alpha = \frac{\varrho}{2}(1+\omega^2 Y^2) + \frac{1+\ell^2}{\varrho}$*, or equivalently:* $\varrho - \alpha = \frac{\varrho}{2}(1-\omega^2 Y^2) - \frac{1+\ell^2}{\varrho}$.

Proof. The two first items are straightforward. On the other hand, the action of the rotational symmetry $(u, r, \phi, z) \mapsto (u, r, \phi + \phi_0, z)$ is not so obvious. However, a computation shows that we have in coordinates (u, r, ϕ, z): $a = \dot u + 2(\sinh r)^2\,\dot\phi$, and:

$$b = A + \Psi\, \cos(\omega\,\phi) - 2\omega^{-1}\dot r\, \sin(\omega\,\phi), \quad Z := \omega\, b\, Y = \Psi\, \sin(\omega\,\phi) + 2\omega^{-1}\dot r\, \cos(\omega\,\phi),$$

with

$$A := 2a\cosh(2r) - \sinh^2(2r)\,\dot\phi \quad \text{and} \quad \Psi := [2a - \cosh(2r)\,\dot\phi]\sinh(2r).$$

Note that $A = 2a + 4[a - \cosh^2 r\,\dot\phi]\sinh^2 r = 2a + 2\,\frac{\partial L}{\partial\dot\phi}$ is seen to be constant on each geodesic, by looking at the expression of the Lagrangian L in coordinates (u, r, ϕ, z). Alternatively, a computation yields: $2A = b + \omega^2 b\,(2Yy - y^2) + (4a - b\, e^{-\sqrt{2}\,\omega\, x})\, e^{-\sqrt{2}\,\omega\, x}$, whence by using Proposition 3:

$$\begin{aligned}
2\alpha - \varrho = \tfrac{2A}{a} - \varrho &= \omega^2\varrho\left[Y^2 - \tfrac{2(1-\ell)^2}{\omega^2\varrho^2}\left[1 - \frac{4}{2+\mathrm{tg}^2(\frac{\omega\varphi_\tau}{2})}\right]^2\right] \\
&\quad + \frac{1}{\varrho}\left[4 - \left[\frac{4\sqrt{1-\ell^2}\,\mathrm{tg}^2(\frac{\omega\varphi_\tau}{2})}{2+\mathrm{tg}^2(\frac{\omega\varphi_\tau}{2})}\right]^2\right] \\
&= \omega^2\varrho\, Y^2 + \frac{4}{\varrho} - 2\,\frac{(1-\ell^2)}{\varrho}, \quad \text{whence} \quad \alpha = \frac{\varrho}{2}(1+\omega^2 Y^2) + \frac{1+\ell^2}{\varrho}.
\end{aligned}$$

Now we have at once: under $\phi \mapsto \phi + \phi_0$, (a, b, Z) is changed into

$$(a, A + [b - A]\cos(\omega\,\phi_0) - Z\sin(\omega\,\phi_0),\ Z\cos(\omega\,\phi_0) + [b - A]\sin(\omega\,\phi_0)),$$

so that (ℓ, ϱ, Y) is changed into (recall from the above that $\alpha = A/a$):

$$\Bigg(\ell,\ \tfrac{A}{a} + [\varrho - \tfrac{A}{a}]\cos(\omega\,\phi_0) - \omega\,\varrho\,Y\ \sin(\omega\,\phi_0),\qquad \frac{\omega\,\varrho\ \cos(\omega\,\phi_0) + [\varrho - \frac{A}{a}]\sin(\omega\,\phi_0)}{\omega\,[\frac{A}{a} + [\varrho - \frac{A}{a}]\cos(\omega\,\phi_0) - \omega\,\varrho\ \sin(\omega\,\phi_0)]}\Bigg).$$

□

2.4. Ricci curvature and energy tensor. Recall that the Christoffel symbols are computed by:

$$\Gamma^k_{ij} = \tfrac{1}{2}\,g^{k\ell}\left(\frac{\partial g_{\ell j}}{\partial \xi^i} + \frac{\partial g_{i\ell}}{\partial \xi^j} - \frac{\partial g_{ij}}{\partial \xi^\ell}\right),$$

or by the fact that geodesics solve $\ddot{\xi}^k + \Gamma^k_{ij}\dot{\xi}^i\dot{\xi}^j = 0$, and that the Ricci tensor (R_{ij}) is computed by:

$$R_{ij} = \frac{\partial \Gamma^k_{ij}}{\partial \xi^k} - \frac{\partial \Gamma^k_{ik}}{\partial \xi^j} + \Gamma^k_{k\ell}\Gamma^\ell_{ij} - \Gamma^k_{i\ell}\Gamma^\ell_{jk}.$$

From Eqs. (1′), (2′), (3), we find all non-vanishing Christoffel coefficients:

$$\Gamma^t_{xy} = \Gamma^x_{ty} = (\omega/\sqrt{2}\,)\,e^{\sqrt{2}\,\omega\,x},\ \ \Gamma^t_{tx} = \sqrt{2}\,\omega,\ \ \Gamma^x_{yy} = (\omega/\sqrt{2}\,)\,e^{2\sqrt{2}\,\omega\,x},$$
$$\Gamma^y_{tx} = -\sqrt{2}\,\omega\,e^{-\sqrt{2}\,\omega\,x}.$$

Therefore we get all non-vanishing Ricci coefficients:

$$R_{tt} = 2\omega^2;\ \ R_{ty} = R_{yt} = 2\omega^2\,e^{\sqrt{2}\,\omega\,x};\ \ R_{yy} = 2\omega^2\,e^{2\sqrt{2}\,\omega\,x}.$$

Hence, the scalar curvature is $\quad R = g^{ij}\,R_{ij} = 2\omega^2$.
Einstein equations

$$R_{ij} - \tfrac{1}{2}\,R\,g_{ij} + \Lambda\,g_{ij} = T_{ij}$$

are satisfied, with cosmological constant $\Lambda = \omega^2$ representing a positive pressure, and energy tensor $(T_{ij}) = (R_{ij}) = (u_i u_j)$, where $u := (\sqrt{2}\,\omega, 0, \sqrt{2}\,\omega\,e^{\sqrt{2}\,\omega\,x}, 0)$ represents the four-velocity of matter, which rotates with constant velocity ω. The energy is thus:

$$E(\xi,\dot{\xi}) := T_{ij}(\xi)\,\dot{\xi}^i\,\dot{\xi}^j = 2\omega^2\left[\dot{t} + e^{\sqrt{2}\,\omega\,x}\,\dot{y}\right]^2 = 2\omega^2\,a(\xi,\dot{\xi})^2.$$

In particular, the *weak energy condition* of [H-E] holds: $E(\xi,\dot{\xi}) \geq 0$ for any timelike vector $\dot{\xi} = (\dot{\xi}^i)$. The *dominant energy condition* of [H-E] means that moreover the vector $(T^{ij}(\xi)\dot{\xi}_j)$ must be non-spacelike, which is equivalent to $g^{k\ell}(\xi)T_{ki}(\xi)T_{\ell j}(\xi)\dot{\xi}^i\dot{\xi}^j \geq 0$. It holds here too, since we get: $g^{k\ell}(\xi)T_{ki}(\xi)T_{\ell j}(\xi)\dot{\xi}^i\dot{\xi}^j = g^{k\ell}(\xi)u_k u_\ell \times (u_i\dot{\xi}^i)^2 = 4\omega^4\,a(\xi,\dot{\xi})^2$.

Similarly, the *strong energy condition* of [H-E]: $E(\xi,\dot{\xi}) \geq \frac{1}{2}\,g^{k\ell}(\xi)T_{k\ell}(\xi)g_{ij}(\xi)\dot{\xi}^i\dot{\xi}^j$, is satisfied too, since it is here equivalent to: $2\,a(\xi,\dot{\xi})^2 \geq \dot{t}^2 - \dot{x}^2 + \frac{1}{2}\,e^{2\sqrt{2}\,\omega\,x}\,\dot{y}^2 + 2\,e^{\sqrt{2}\,\omega\,x}\,\dot{t}\dot{y} - \dot{z}^2$, or also to: $\dot{t}^2 + \frac{7}{2}\,e^{2\sqrt{2}\,\omega\,x}\,\dot{y}^2 + 2\,e^{\sqrt{2}\,\omega\,x}\,\dot{t}\dot{y} + \dot{x}^2 + \dot{z}^2 \geq 0$, which holds clearly for any timelike $\dot{\xi} \in T_\xi G$.

3. Relativistic Diffusion $(\xi_s, \dot{\xi}_s)$ on G

Recall from [F-LJ] that the general expression of the relativistic operator $\mathcal{L}$ is:

$$\mathcal{L} = \dot{\xi}^k \frac{\partial}{\partial \xi^k} + \left(\tfrac{3\sigma^2}{2}\,\dot{\xi}^k - \dot{\xi}^i \dot{\xi}^j\, \Gamma^k_{ij}(\xi)\right) \frac{\partial}{\partial \dot{\xi}^k} + \tfrac{\sigma^2}{2}\,(\dot{\xi}^k \dot{\xi}^l - g^{kl}(\xi))\, \frac{\partial^2}{\partial \dot{\xi}^k\, \partial \dot{\xi}^l},$$

σ being an arbitrary fixed strictly positive (speed or heat) parameter.

Equivalently in the present setting, the relativistic diffusion $(\xi_s, \dot{\xi}_s)$, in coordinates $\xi = (t, x, y, z)$, solves the following system of stochastic differential equations:

$$\begin{aligned}
&dt_s = \dot{t}_s\, ds; \quad dx_s = \dot{x}_s\, ds; \quad dy_s = \dot{y}_s\, ds; \quad dz_s = \dot{z}_s\, ds;\\
&d\dot{t}_s = -2\sqrt{2}\,\omega\, \dot{t}_s\, \dot{x}_s\, ds - \sqrt{2}\,\omega\, e^{\sqrt{2}\,\omega\, x_s}\, \dot{x}_s\, \dot{y}_s\, ds + \tfrac{3\sigma^2}{2}\, \dot{t}_s\, ds + \sigma\, dM_s^t;\\
&d\dot{x}_s = -\sqrt{2}\,\omega\, e^{\sqrt{2}\,\omega\, x_s}\, \dot{t}_s\, \dot{y}_s\, ds - (\omega/\sqrt{2})\, e^{2\sqrt{2}\,\omega\, x_s}\, \dot{y}_s^2\, ds + \tfrac{3\sigma^2}{2}\, \dot{x}_s\, ds + \sigma\, dM_s^x;\\
&d\dot{y}_s = 2\sqrt{2}\,\omega\, e^{-\sqrt{2}\,\omega\, x_s}\, \dot{t}_s\, \dot{x}_s\, ds + \tfrac{3\sigma^2}{2}\, \dot{y}_s\, ds + \sigma\, dM_s^y;\\
&d\dot{z}_s = \tfrac{3\sigma^2}{2}\, \dot{z}_s\, ds + \sigma\, dM_s^z;
\end{aligned}$$

where the $\mathbb{R}^4$-valued martingale $M_s := (M_s^t, M_s^x, M_s^y, M_s^z)$ has (rank 3) quadratic covariant matrix:

$$((K_s^{ij})) := \frac{\langle dM_s^i, dM_s^j \rangle}{ds} = \begin{pmatrix} \dot{t}_s^2 + 1 & \dot{t}_s\, \dot{x}_s & \dot{t}_s\, \dot{y}_s - 2\, e^{-\sqrt{2}\,\omega\, x_s} & \dot{t}_s\, \dot{z}_s \\ \dot{t}_s\, \dot{x}_s & \dot{x}_s^2 + 1 & \dot{x}_s\, \dot{y}_s & \dot{x}_s\, \dot{z}_s \\ \dot{t}_s\, \dot{y}_s - 2\, e^{-\sqrt{2}\,\omega\, x_s} & \dot{x}_s\, \dot{y}_s & \dot{y}_s^2 + 2\, e^{-2\sqrt{2}\,\omega\, x_s} & \dot{y}_s\, \dot{z}_s \\ \dot{t}_s\, \dot{z}_s & \dot{x}_s\, \dot{z}_s & \dot{y}_s\, \dot{z}_s & \dot{z}_s^2 + 1 \end{pmatrix}.$$

Recall that the unit pseudo-norm relation reads:

$$1 + \dot{t}_s^2 + \dot{x}_s^2 + \dot{z}_s^2 = \tfrac{1}{2}\left[e^{\sqrt{2}\,\omega\, x_s}\, \dot{y}_s + 2\, \dot{t}_s\right]^2. \tag{0}$$

Thus, the relativistic diffusion $(\xi_s, \dot{\xi}_s)$ is 7-dimensional, having phase space:

$$\mathcal{E} := \left\{(t, x, y, z, \dot{t}, \dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^8 \,\middle|\, 1 + \dot{t}^2 + \dot{x}^2 + \dot{z}^2 = \tfrac{1}{2}\left[e^{\sqrt{2}\,\omega\, x}\, \dot{y} + 2\, \dot{t}\right]^2\right\},$$

or equivalently:

$$\mathcal{E} = \left\{(t, x, y, z, \dot{t}, \dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^8 \,\middle|\, 1 + \dot{x}^2 + \dot{z}^2 + \tfrac{1}{2}\, e^{2\sqrt{2}\,\omega\, x}\, \dot{y}^2 = \left[\dot{t} + e^{\sqrt{2}\,\omega\, x}\, \dot{y}\right]^2\right\}.$$

Note that the particular phase subspace distinguished in Remark 2:

$$\mathcal{E}_0 := \mathcal{E} \cap \{\dot{x} = \dot{y} = 0\} = \mathcal{E} \cap \left\{\dot{t}^2 = 1 + \dot{z}^2\,;\ \dot{x} = \dot{y} = 0\right\}$$

is clearly not stable under the relativistic diffusion $(\xi_s, \dot{\xi}_s)$, contrary to the geodesic flow, and even instantly unstable: starting from any point in $\mathcal{E}_0$, its exit time from $\mathcal{E}_0$ is null.

Remark 5. On one hand, it is convenient in the Riemannian case (see for example [N-R-W]) to get the Brownian motion on a symmetric space by means of a left Brownian motion on a covering Lie group. On the other hand, Theorem 3.2(ii) of [F-LJ] states that the relativistic diffusion on a Lorentz manifold is got by development (from a fixed tangent space) of its Minkowskian analogue, the Minkowskian relativistic diffusion being essentially an integrated hyperbolic Brownian motion (living however on the tangent bundle).

Thus, in the case of G, Sect. 2.1 could let it be thought that developing such an integrated hyperbolic Brownian motion would yield a more invariant and tractable presentation of the relativistic diffusion.

However, on one hand Gödel's metric is not the Killing one (which is degenerate), and solving explicitly the equation of parallel transport is not easy here. And on the other hand, the group structure of G does not seem to lift to a natural group structure on or over the pseudo-unit tangent bundle T^1G, and therefore does not seem to allow a more invariant presentation of the relativistic diffusion (this latter exists only at the level of the tangent bundle). Even geodesics of G through the unit are not simply deduced from geodesics of $\mathcal{G}$ (which are straight lines), despite the Gavrilov equation $\dot{\xi}_s = \mathrm{Jac}_{\mathbf{1}} L_{\xi_s} \times (g^0)^{-1} \times \mathrm{Jac}_{\mathbf{1}}\mathrm{Ad}(\xi_s) \times g^0 \times \dot{\xi}_0$ appearing in ([L], (6)) (which does not simplify the solution in Sect. 2.2 above).

3.1. Reduction of the dimension. The study of geodesics induces consideration of the following quantities (which, as $\dot{z}_s$, are constant along each geodesic), setting (as in Definition 2):

$$a_s := \dot{t}_s + e^{\sqrt{2}\,\omega\, x_s}\, \dot{y}_s \qquad \text{and} \qquad b_s := e^{\sqrt{2}\,\omega\, x_s}\,(2\,\dot{t}_s + e^{\sqrt{2}\,\omega\, x_s}\, \dot{y}_s). \tag{10}$$

Recall from Sect. 2.4 that a_s^2 is an energy. Then we have:

$$da_s = \tfrac{3\sigma^2}{2}\, a_s\, ds + \sigma\, dM_s^a = \tfrac{3\sigma^2}{2}\, a_s\, ds + \sigma\,(dM_s^t + e^{\sqrt{2}\,\omega\, x_s}\, dM_s^y);$$

and

$$db_s = \tfrac{3\sigma^2}{2}\, b_s\, ds + \sigma\, dM_s^b = \tfrac{3\sigma^2}{2}\, b_s\, ds + \sigma\, e^{\sqrt{2}\,\omega\, x_s}\,(2\, dM_s^t + e^{\sqrt{2}\,\omega\, x_s}\, dM_s^y).$$

Moreover we have:

$$d\dot{x}_s = (\omega/\sqrt{2}\,)\, e^{-2\sqrt{2}\,\omega\, x_s}\, b_s^2\, ds - \sqrt{2}\,\omega\, e^{-\sqrt{2}\,\omega\, x_s}\, a_s\, b_s\, ds + \tfrac{3\sigma^2}{2}\, \dot{x}_s\, ds + \sigma\, dM_s^x\,,$$

and the $\mathbb{R}^4$-valued martingale $\tilde{M}_s := (M_s^a, M_s^b, M_s^x, M_s^z)$ has (rank 3) quadratic covariation matrix:

$$((\tilde{K}_s^{ij})) = \begin{pmatrix} a_s^2 - 1 & a_s\, b_s - 2\, e^{\sqrt{2}\,\omega\, x_s} & a_s\, \dot{x}_s & a_s\, \dot{z}_s \\ a_s\, b_s - 2\, e^{\sqrt{2}\,\omega\, x_s} & b_s^2 - 2\, e^{2\sqrt{2}\,\omega\, x_s} & b_s\, \dot{x}_s & b_s\, \dot{z}_s \\ a_s\, \dot{x}_s & b_s\, \dot{x}_s & \dot{x}_s^2 + 1 & \dot{x}_s\, \dot{z}_s \\ a_s\, \dot{z}_s & b_s\, \dot{z}_s & \dot{x}_s\, \dot{z}_s & \dot{z}_s^2 + 1 \end{pmatrix}.$$

From this, we deduce the following, which will allow, in the following sections, the asymptotic study (as proper time s goes to infinity) of relativistic paths.

Corollary 1. *The (7-dimensional) relativistic diffusion* $(\xi_s, \dot{\xi}_s)$ *admits the following subdiffusions:* (a_s); $(\dot{z}_s)$; $(a_s, \dot{z}_s)$; $(x_s, \dot{x}_s, a_s, b_s)$.

The unit pseudo-norm relation can be written:

$$1 + \dot{x}_s^2 + \dot{z}_s^2 + (a_s - e^{-\sqrt{2}\,\omega\, x_s}\, b_s)^2 = \tfrac{1}{2}\, e^{-2\sqrt{2}\,\omega\, x_s}\, b_s^2, \tag{00}$$

or equivalently:

$$1 + \dot{x}_s^2 + \dot{z}_s^2 + \tfrac{1}{2}\,(2\, a_s - e^{-\sqrt{2}\,\omega\, x_s}\, b_s)^2 = a_s^2. \tag{00'}$$

Hence the phase space $\mathcal{E}$ of the relativistic diffusion $(\xi_s, \dot{\xi}_s)$ can be written equivalently:

$$\mathcal{E} = \left\{ (t, x, y, z, a, b, \dot{x}, \dot{z}) \in \mathbb{R}^8 \,\middle|\, 1 + \dot{x}^2 + \dot{z}^2 + \tfrac{1}{2}\,(2\, a - e^{-\sqrt{2}\,\omega\, x}\, b)^2 = a^2 \right\}.$$

And the particular phase subspace $\mathcal{E}_0$ distinguished in Remark 2 can be written:

$$\mathcal{E}_0 = \mathcal{E} \cap \left\{ a^2 = 1 + \dot{z}^2;\; 2\, a = e^{-\sqrt{2}\,\omega\, x}\, b;\; \dot{x} = 0 \right\} = \mathcal{E} \cap \left\{ a^2 = 1 + \dot{z}^2 \right\}.$$

Remark 6. We see in particular that $a_s^2 \geq 1$ and that $b_s^2 \geq 2\, e^{2\sqrt{2}\,\omega\, x_s}$, for any proper time $s \geq 0$. Therefore, (a_s) and (b_s) almost surely never vanish. Moreover, they must have the same sign, since (00′) implies $\left| e^{-\sqrt{2}\,\omega\, x_s} \dfrac{b_s}{a_s} - 2 \right| \leq \sqrt{2}$ and then $e^{-\sqrt{2}\,\omega\, x_s} \dfrac{b_s}{a_s} \geq 2 - \sqrt{2}$. This implies also $\left| e^{\sqrt{2}\,\omega\, x_s} \dfrac{a_s}{b_s} - 1 \right| \leq 1/\sqrt{2}$.

Remark 7. The phase space $\mathcal{E}$ splits into two connected components: $\mathcal{E} = \mathcal{E}^+ \sqcup \mathcal{E}^-$, with $\mathcal{E}^+ := \mathcal{E} \cap \{a \geq 1,\ b > 0\}$ and $\mathcal{E}^- := \mathcal{E} \cap \{a \leq -1, b < 0\}$. Similarly, $\mathcal{E}_0 = \mathcal{E}_0^+ \sqcup \mathcal{E}_0^-$, with $\mathcal{E}_0^+ := \mathcal{E}_0 \cap \mathcal{E}^+$ and $\mathcal{E}_0^- := \mathcal{E}_0 \cap \mathcal{E}^-$. Note that since $2\dot{t}_s + e^{\sqrt{2}\,\omega\, x_s}\, \dot{y}_s = e^{-\sqrt{2}\,\omega\, x_s}\, b_s$, the paths in $\mathcal{E}^+$ are always future-directed. Since the symmetry $(a, b) \mapsto (-a, -b)$ exchanges $(\mathcal{E}^+, \mathcal{E}_0^+)$ and $(\mathcal{E}^-, \mathcal{E}_0^-)$, from now on, we can restrict the phase space of the relativistic diffusion $(\xi_s, \dot{\xi}_s)$ to $\mathcal{E}^+$ (its behaviour on $\mathcal{E}^-$ being trivially related).

3.2. Sub-processes (λ_s) and (φ_s). The unit pseudo-norm relation (00′) and the determination of timelike geodesics (recall Sect. 2.2) induce to consider two real sub-processes (λ_s) and (φ_s), defined by:

$$\lambda_s := \operatorname{argch}\left[\sqrt{a_s^2 - \dot{z}_s^2} \right], \quad \dot{x}_s = \sinh(\lambda_s) \cos(\omega \varphi_s),$$
$$e^{-\sqrt{2}\,\omega\, x_s}\, b_s = 2\, a_s - \sqrt{2}\, \sinh(\lambda_s) \sin(\omega \varphi_s).$$

Proposition 5. *The sub-processes (λ_s) and (φ_s) satisfy the following equations:*

$$d\lambda_s = \sigma\, d\beta_s^\lambda + \sigma^2 \coth(2\lambda_s) ds, \quad d\varphi_s = \sigma\, dM_s^\varphi + e^{-\sqrt{2}\,\omega\, x_s}\, b_s\, ds,$$

where $\beta_s^\lambda := \displaystyle\int_0^s \frac{2}{\sinh(2\lambda_\tau)} [a_\tau\, dM_\tau^a - \dot{z}_\tau\, dM_\tau^{\dot{z}}]$ *is a Brownian motion, and the martingale* (M_s^φ) *defined by* $dM_s^\varphi := \frac{\sinh(\lambda_s) \cos(\omega\varphi_s)\, d\beta_s^\lambda - dM_s^{\dot{x}}}{\omega \sinh(\lambda_s) \sin(\omega\varphi_s)}$ *satisfies :* $\langle dM_s^\varphi \rangle = \frac{ds}{\omega^2 \sinh^2(\lambda_s)}$, $\langle dM_s^\varphi, d\beta_s^\lambda \rangle = 0$.

In particular, (λ_s) is a non-negative sub-diffusion (of the relativistic diffusion).

Proof. We have first (using Sect. 3.1):

$$\sinh(2\lambda_s)d\lambda_s + \cosh(2\lambda_s)\langle d\lambda_s\rangle = d[\cosh^2\lambda_s] = d[a_s^2 - \dot{z}_s^2]$$
$$= 2\sigma[a_s dM_s^a - \dot{z}_s dM_s^z] + 2\sigma^2\cosh(2\lambda_s)ds,$$

whence

$$d\lambda_s + \coth(2\lambda_s)\langle d\lambda_s\rangle = \sigma\, d\beta_s^\lambda + 2\sigma^2\coth(2\lambda_s)ds,$$

and it is easily verified that $\langle d\beta_s^\lambda\rangle = ds$, whence the first formula. The two other quadratic covariations displayed in the statement are also directly computed. Then,

$$d\dot{x}_s = d[\sinh(\lambda_s)\cos(\omega\varphi_s)] = \sigma\cosh(\lambda_s)\cos(\omega\varphi_s)d\beta_s^\lambda - \omega\sinh(\lambda_s)\sin(\omega\varphi_s)d\varphi_s$$

$$+\tfrac{\sigma^2}{2}\left[\frac{\cos(\omega\varphi_s)}{\sinh(\lambda_s)} + 3\,\dot{x}_s\right]ds - \tfrac{\omega^2}{2}\,\dot{x}_s\langle d\varphi_s\rangle - \tfrac{\omega}{2}\,\dot{x}_s\cosh(\lambda_s)\sin(\omega\varphi_s)\langle d\lambda_s, d\varphi_s\rangle.$$

Using the definition of (M_s^φ), and Sect. 3.1 again, we deduce the equation giving $d\varphi_s$. □

Remark 8. The phase subspace $\mathcal{E}_0$ of Remarks 2 and 7 is precisely: $\mathcal{E}_0 = \mathcal{E} \cap \{\lambda = 0\}$. The equation satisfied by (λ_s) can be precisely solved as follows, provided $\lambda_0 > 0$, using some real Brownian motion β, started from $\beta_0 = \frac{1}{2}\log[\coth\lambda_0]$:

$$\lambda_s = \tfrac{1}{2}\log\left[\coth\left(\beta\left[\inf\left\{u \,\Big|\, \int_0^u \coth^2(2\beta_v)\,dv = \sigma^2 s\right\}\right]\right)\right].$$

This implies that we have almost surely: $\lambda_s > 0$ for any $s > 0$: the state subspace $\mathcal{E}_0$ of Remark 2 is polar for the relativistic diffusion. It is also instantly unstable. Hence we can finally restrict the state space $\mathcal{E}^+$ (recall Remark 7) of the relativistic diffusion to $\mathcal{E}^+ \setminus \mathcal{E}_0^+$.

3.3. Study of the one-dimensional sub-diffusions (a_s), $(\dot{z}_s)$, (λ_s). These three one-dimensional sub-diffusions are easily handled.

Lemma 1. *There exist three standard real Brownian motions* (w_s), (w'_s), $(\tilde{w}_s)$, *and three almost surely converging processes* (η_s), (η'_s), $(\tilde{\eta}_s)$, *such that we have:*

$$a_s = \exp\left[\sigma^2 s + \sigma\, w_s + \eta_s\right] \quad \textit{for any proper time } s \geq 0,$$

$$|\dot{z}_s| = \exp\left[\sigma^2 s + \sigma\, w'_s + \eta'_s\right] \quad \textit{for any sufficiently large proper time } s,$$

and

$$\lambda_s = \sigma^2 s + \sigma\,\tilde{w}_s + \tilde{\eta}_s \quad \textit{for any proper time } s \geq 0.$$

Proof. The stochastic differential equations satisfied by (a_s) and $(\dot{z}_s)$ are respectively:

$$da_s = \tfrac{3\sigma^2}{2}\, a_s\, ds + \sigma\, \sqrt{a_s^2 - 1}\, dw_s, \quad \text{and} \quad d\dot{z}_s = \tfrac{3\sigma^2}{2}\, \dot{z}_s\, ds + \sigma\, \sqrt{\dot{z}_s^2 + 1}\, dw'_s,$$

for two standard real Brownian motions (w_s) and (w'_s). These equations are solved as follows; we have real Brownian motions (W_u) and (W'_u) such that:

$$a_s = F\left(W\left[\inf\left\{u \,\Big|\, \int_0^u (W_v^2 - 1)^{-2}\, dv > \sigma\, s\right\}\right]\right),$$

and

$$\dot{z}_s = G\left(W'\left[\inf\left\{u \,\Big|\, \int_0^u (1 - |W'_v|^2)^{-2}\, dv > \sigma\, s\right\}\right]\right),$$

with $F(W) := \frac{W}{\sqrt{W^2-1}}$ and $G(W') := \frac{W'}{\sqrt{1-|W'|^2}}$.

Clearly, as u increases to the hitting time of 1 by W, then $\int_0^u (W_v^2 - 1)^{-2}\, dv$ increases to infinity, W_u goes to 1, and $F(W_u)$ goes to infinity, showing that a_s goes almost surely to infinity with s. The same reasoning holds for $(\dot{z}_s)$ (except that $|W'_0|$ must be smaller than 1, while W_0 must be larger than 1 (recall Remark 7)), so that, in the same way, $|\dot{z}_s|$ goes almost surely to infinity with s.

Then we have almost surely, for any sufficiently large proper time s:

$$d \log a_s = (1 + \tfrac{1}{2\,a_s^2})\sigma^2 ds + \sigma\sqrt{1 - a_s^{-2}}\, dw_s,$$
$$d \log |\dot{z}_s| = (1 - \tfrac{1}{2\,\dot{z}_s^2})\sigma^2 ds + \sigma\sqrt{1 + \dot{z}_s^{-2}}\, dw'_s.$$

Whence, for real Brownian motions $\tilde{w}$, $\tilde{w}'$ and for sufficiently large proper times s_0, s:

$$\log a_s = \log a_0 + \sigma^2 s + \sigma\, w_s + \int_0^s \frac{\sigma^2 du}{2\,a_u^2} - \tilde{w}\left[\int_0^s \frac{\sigma^2\, a_u^{-4}\, du}{\left(1 + \sqrt{1 - \frac{1}{a_u^2}}\right)^2}\right]$$
$$= \sigma^2 s + o(s) > \sigma^2 s/2,$$

and similarly:

$$\log \frac{|\dot{z}_s|}{|\dot{z}_{s_0}|} = \sigma^2(s - s_0) + \sigma(w'_s - w'_{s_0}) - \int_{s_0}^s \frac{\sigma^2 du}{2\,\dot{z}_u^2} + \tilde{w}'\left[\int_{s_0}^s \frac{\sigma^2\, \dot{z}_u^{-4}\, du}{\left(1 + \sqrt{1 + \frac{1}{\dot{z}_u^2}}\right)^2}\right]$$
$$= \sigma^2 s + o(s) > \sigma^2 s/2.$$

The convergence of the integrals in the above formulas follows, implying the two first claims. Then, since $\coth(2\lambda_s) > 1$, the comparison theorem and Proposition 5 ensure that we have almost surely:

$$\lambda_s \geq \lambda_0 + \sigma\, \tilde{w}_s + \sigma^2\, s \longrightarrow +\infty.$$

Moreover, we have almost surely $\lambda_s \geq \sigma^2 s/2$, for large enough s_0 and for $s \geq s_0$. We deduce that $\tilde{\eta}_s := \lambda_s - \sigma^2 s - \sigma\,\tilde{w}_s = \tilde{\eta}_{s_0} + 2\sigma^2 \displaystyle\int_{s_0}^{s} \frac{du}{e^{4\lambda_u}-1}$ converges almost surely. $\square$

Lemma 1 and Proposition 5 imply immediately the following.

Corollary 2. *There exists an almost surely converging process* $(\check{\eta}_s)$ *such that:*

$$\varphi_s = \int_0^s e^{-\sqrt{2}\,\omega\,x_u}\,b_u\,du + \check{\eta}_s,\ \ \textit{for any}\ \ s \geq 0. \quad \textit{And}\quad |z_s| = e^{\sigma^2 s + o(s^{5/9})}\ \ \textit{as}\ \ s \to \infty.$$

We have also the following lower control, which we shall use later.

Lemma 2. *For any* $A > \sqrt{3}$, *we have* $\mathbb{P}\Big[(\exists\, s>0)\ |\dot{z}_s| \leq A\, e^{\sigma^2 s/2} \Big/ |\dot{z}_0| \geq A^2\Big] < 1/\sqrt{A}$, *and* $\mathbb{P}\Big[(\exists\, s>0)\ a_s \leq A\, e^{\sigma^2 s/2} \Big/ a_0 \geq A^2\Big] \leq 1/A$.

Proof. Fix $A > \sqrt{3}$ and $|\dot{z}_0| \geq A^2$. The stochastic differential equation satisfied by $(\log|\dot{z}_s|)$, already written in the proof of Lemma 1, is equivalent to:

$$d\log\left|\dot{z}_s\, e^{-\sigma^2 s/2}\right| = \tfrac{\sigma^2}{2}\,(1-\dot{z}_s^{-2})\,ds + \sigma\sqrt{1+\dot{z}_s^{-2}}\,dw'_s.$$

Let us apply the comparison theorem (see for example ([I-W], Theorem 4.1)): setting $T^z_A := \inf\Big\{s \,\Big|\, |\dot{z}_s| = A\,e^{\sigma^2 s/2}\Big\}$ and $\log r_s := \log A^2 + \frac{\sigma^2}{2}\,(1-A^{-2})\,s + \sigma\sqrt{1+A^{-2}}\,w'_s$, we have: $\inf\limits_{0\leq s\leq T^z_A}\left|\dot{z}_s\,e^{-\sigma^2 s/2}\right| \geq \inf\limits_{0\leq s\leq T^z_A} r_s$, whence

$$\begin{aligned}\mathbb{P}[T^z_A < \infty] &\leq \mathbb{P}[\ \log r_s\ \text{hits}\ \log A]\\ &= \mathbb{P}[\ w'_s - \tfrac{1-A^{-2}}{1+A^{-2}}\,s/2\ \text{hits}\ \log A] = A^{-\frac{1-A^{-2}}{1+A^{-2}}} < 1/\sqrt{A}.\end{aligned}$$

Similarly, for $a_0 \geq A^2$ and $T^a_A := \inf\{u \,|\, a_u = A\,e^{\sigma^2 s/2}\}$: since

$$d\left(a_s\,e^{-\sigma^2 s/2}\right) = \tfrac{\sigma^2}{2}\,a_s^{-2}\,ds + \sigma\sqrt{1-a_s^{-2}}\,dw_s,$$

we get:

$$\mathbb{P}[T^a_A < \infty] \leq \mathbb{P}[\ w_s - s/2\ \text{hits}\ \log A] = 1/A.$$

$\square$

3.4. Study of the two-dimensional sub-diffusion $(a_s, \dot{z}_s)$. We get easily an asymptotic variable of the sub-diffusion $(a_s, \dot{z}_s)$.

Proposition 6. *The process* $(\dot{z}_s/a_s)$ *converges almost surely, toward some random limit* ℓ *such that* $0 < |\ell| \leq 1$.

Proof. Using Itô’s Formula and Sect. 3.1, we get:

$$d\left[\frac{\dot z_s}{a_s}\right]=\frac{d\dot z_s}{a_s}-\frac{\dot z_s\,da_s}{a_s^2}+\frac{\dot z_s\,\langle da_s\rangle}{a_s^3}-\frac{\langle da_s,d\dot z_s\rangle}{a_s^2}=\frac{\sigma}{a_s}\left[dM_s^z-\frac{\dot z_s}{a_s}\,dM_s^a\right]-\frac{\sigma^2\,\dot z_s}{a_s^3}\,ds,$$

with

$$\left\langle a_s^{-1}\left[dM_s^z-\frac{\dot z_s}{a_s}\,dM_s^a\right]\right\rangle=\frac{1-|\dot z_s/a_s|^2}{a_s^2}\,ds.$$

Hence, we have some real Brownian motion $\check W$ such that almost surely, for any $s\ge 0$:

$$\frac{\dot z_s}{a_s}=\frac{\dot z_0}{a_0}-\sigma^2\int_0^s\frac{\dot z_u}{a_u^3}\,du+\sigma\,\check W\left[\int_0^s\frac{1-|\dot z_u/a_u|^2}{a_u^2}\,du\right],$$

which almost surely converges toward some random limit $\ell\in\mathbb{R}$, by Lemma 1. The unit pseudo-norm relation ($00'$) implies $a_s^2-1\ge\dot z_s^2$, and then $\ell^2\le 1$.

Similarly, using Lemma 1 again, for any large enough proper time s we have:

$$d\left[\frac{a_s}{\dot z_s}\right]=\frac{da_s}{\dot z_s}-\frac{a_s\,d\dot z_s}{\dot z_s^2}+\frac{a_s\,\langle d\dot z_s\rangle}{\dot z_s^3}-\frac{\langle da_s,d\dot z_s\rangle}{\dot z_s^2}=\frac{\sigma}{\dot z_s}\left[dM_s^a-\frac{a_s}{\dot z_s}\,dM_s^z\right]+\frac{\sigma^2\,a_s}{\dot z_s^3}\,ds,$$

with

$$\left\langle \dot z_s^{-1}\left[dM_s^a-\frac{a_s}{\dot z_s}\,dM_s^z\right]\right\rangle=\frac{(a_s/\dot z_s)^2-1}{\dot z_s^2}\,ds,$$

whence the almost sure convergence of $(a_s/\dot z_s)$, which proves that $\ell\neq 0$ almost surely. □

Remark 9. It can be shown that the process $(\dot z_s-\ell\,a_s)$ converges in law, toward the law of $\left[(1-\ell^2)\int_0^\infty e^{-2u-2w_u}\,du\right]^{1/2}\times N$, N denoting a $\mathcal{N}(0,1)$ Gaussian variable, independent from (ℓ,w). It can also be shown that it does not converge in probability, letting one think that the asymptotic σ-algebra of $(a_s,\dot z_s)$ is generated by the only variable ℓ.

The following statement ensures that the range of possible limits ℓ in Proposition 6, is the whole $[-1,0[\cup]0,1]$. This provides a continuum of non-trivial bounded harmonic functions for the relativistic operator $\mathcal{L}$ on T^1G.

Proposition 7. *For any real* ℓ_0 *such that* $0<|\ell_0|\le 1$, *and for any* $\varepsilon>0$, *we have* $\mathbb{P}\left[\ell_0-\varepsilon<\ell=\lim_{s\to\infty}\frac{\dot z_s}{a_s}<\ell_0+\varepsilon\right]>1-\varepsilon$, *provided* $\dot z_0/a_0$ *is close enough from* ℓ_0 *and* a_0 *is large enough.*

Proof. Fix $A>9$, $a_0>A^2$ and $|\dot z_0|\ge A^2$, such that $\dot z_0/a_0$ is close to ℓ_0 (precisely, we demand $|\log(\frac{\dot z_0}{a_0\ell_0})|<A^{-2}$), and consider the event:

$$\mathcal{A}:=\left\{a_s^2>1+A^2e^{\sigma^2 s}\ \text{ and }\ \dot z_s^2>A^2e^{\sigma^2 s}\quad\text{for all } s\ge 0\right\}.$$

By Lemma 2 we have: $\mathbb{P}(\mathcal{A}) > 1 - 2/\sqrt{A}$. Now, on $\mathcal{A}$ we have:

$$\int_0^\infty \frac{du}{a_u^2 - 1} + \int_0^\infty \frac{du}{\dot{z}_u^2} \le 2\sigma^{-2} A^{-2} \quad \text{and} \quad \int_0^\infty \frac{du}{\dot{z}_u^2} \le \sigma^{-2} A^{-2}.$$

Hence, we see from the expression giving $\log\left[\frac{|\dot{z}_s|}{\sqrt{a_s^2 - 1}}\right]$, displayed in the proof of Proposition 6, that we have on $\mathcal{A}$:

$$|\log(\ell/\ell_0)| \le 2\,A^{-2} + \sigma\,\max\{|\check{W}_s|\,|\,0 \le s \le \sigma^{-2} A^{-2}\}.$$

Finally, as

$$\begin{aligned}\mathbb{P}[\sigma\,\max\{|\check{W}_s|\,|\,0 \le s \le \sigma^{-2}A^{-2}\} > A^{-1/2}] &\le 2\,\mathbb{P}[\max\{\check{W}_s\,|\,0 \le s \le A^{-2}\} > A^{-1/2}] \\ &= 2\,\mathbb{P}[\,|\check{W}_{A^{-2}}| > A^{-1/2}] = 4\,\mathbb{P}[\check{W}_1 > \sqrt{A}\,] < e^{-A/2},\end{aligned}$$

we obtain: $\mathbb{P}[\,|\log(\ell/\ell_0)| \le 2\,A^{-2} + A^{-1/2}] > 1 - 2/\sqrt{A} - e^{-A/2}$. □

We shall need to know that in fact $|\ell| < 1$ almost surely.

Lemma 3. *The random limit $\ell = \lim_{s\to\infty}(\dot{z}_s/a_s)$ of Proposition 6 satisfies almost surely:*

$$0 < |\ell| < 1.$$

Proof. Let us consider $A_s := \sqrt{1 - (1 + \dot{z}_s^2)a_s^{-2}} = a_s^{-1}\sinh\lambda_s$ (recall Sect. 3.2), which goes almost surely to $\sqrt{1 - \ell^2}$, by Lemma 1 and Proposition 6. On the other hand, using Itô's Formula and Proposition 5, we get:

$$\begin{aligned} d(\log A_s) &= d(\log[\sinh\lambda_s]) - d(\log a_s) \\ &= \coth\lambda_s\,d\lambda_s - \frac{d\langle\lambda_s\rangle}{2\sinh^2\lambda_s} - \sigma^2(1 + \tfrac{1}{2}\,a_s^{-2})ds - \frac{\sigma}{a_s}\,dM_s^a \\ &= \sigma\left[\left(\frac{a_s}{\sinh^2\lambda_s} - \frac{1}{a_s}\right)dM_s^a - \frac{\dot{z}_s}{\sinh^2\lambda_s}\,dM_s^z\right] \\ &\quad + \frac{\sigma^2}{2}\left[2\coth(\lambda_s)\coth(2\lambda_s) - \frac{1}{\sinh^2\lambda_s} - 2 - \frac{1}{a_s^2}\right]ds \\ &= \frac{\sigma}{a_s\sinh^2\lambda_s}\left[(\dot{z}_s^2 + 1)\,dM_s^a - a_s\,\dot{z}_s\,dM_s^z\right] - \frac{\sigma^2}{2\,a_s^2}ds. \end{aligned}$$

Hence, we have for some real Brownian motion $\check{B}$:

$$\log(1 - \ell^2) = 2\,\log A_0 + 2\,\sigma\,\check{B}\left[\int_0^\infty \frac{\dot{z}_s^2 + 1}{a_s^2\sinh^2\lambda_s}\,ds\right] - \sigma^2\int_0^\infty \frac{ds}{a_s^2},$$

which converges (in $\mathbb{R}$) almost surely, by Lemma 1 and Proposition 6, showing that indeed $\ell^2 < 1$ almost surely. □

We have furthermore the following.

Proposition 8. *The law of the random limit $\ell = \lim_{s\to\infty}(\dot{z}_s/a_s)$ has no atom.*

Proof. Fix any $\ell_0 \in]-1,1[$, and set $\delta_s := \dot{z}_s - \ell_0\, a_s$. The stochastic differential equation satisfied by (δ_s) is easily seen to be:

$$d\delta_s = \tfrac{3\sigma^2}{2}\,\delta_s\,ds + \sigma\sqrt{\delta_s^2 + 1 - \ell_0^2}\;d\beta_s,$$

for some standard real Brownian motion (β_s). This diffusion equation can be solved as follows: we have a real Brownian motion (W_u) (started from $W_0 \in]\frac{-1}{1-\ell_0^2}, \frac{1}{1-\ell_0^2}[$) such that:

$$\delta_s = F\left(W\left[\inf\left\{u \,\middle|\, \int_0^u \frac{(1-\ell_0^2)\,dv}{1-(1-\ell_0^2)^2\,W_v^2} > \sigma\, s\right\}\right]\right),$$
$$\text{with}\;\; F(W_u) := \frac{(1-\ell_0^2)^{3/2}\,W_u}{\sqrt{1-(1-\ell_0^2)^2\,W_u^2}}.$$

As u increases to the hitting time of $\pm(1-\ell_0^2)^{-1}$ by W, then $\int_0^u \frac{(1-\ell_0^2)\,dv}{1-(1-\ell_0^2)^2\,W_v^2}$ increases to infinity, W_u goes to $\pm(1-\ell_0^2)^{-1}$, and $F(W_u)$ goes to $\pm\infty$, showing that $|\delta_s|$ goes almost surely to infinity with s. (The invariant measure of the diffusion (δ_s) is $\sqrt{\delta^2 + 1 - \ell_0^2}\;d\delta$.) Then we have almost surely, for any sufficiently large proper time s:

$$d\log|\delta_s| = \big(1 - \tfrac{1-\ell_0^2}{2\,\delta_s^2}\big)\,\sigma^2 ds \pm \sigma\sqrt{1 + \tfrac{1-\ell_0^2}{\delta_s^2}}\;d\beta_s.$$

Whence, for real Brownian motions $w, \tilde{w}$ and for sufficiently large proper times s_0, s:

$$\begin{aligned}\log\frac{|\delta_s|}{|\delta_{s_0}|} &= \sigma^2(s-s_0) + \sigma\,(w_s - w_{s_0}) - \frac{\sigma^2(1-\ell_0^2)}{2}\int_{s_0}^s \frac{du}{\delta_u^2} + \sigma\sqrt{1-\ell_0^2}\;\tilde{w}\left[\int_{s_0}^s \frac{du}{\delta_u^2}\right]\\ &= \sigma^2 s + o(s) > \sigma^2 s/2.\end{aligned}$$

This implies the convergence of the integrals in the above formula, and then the existence of a standard real Brownian motion $(w_s^{\ell_0})$ and of an almost surely converging process $(\eta_s^{\ell_0})$, such that almost surely, for any sufficiently large proper time s we have:

$$|\delta_s| = |\dot{z}_s - \ell_0\,a_s| = \exp\left[\sigma^2 s + \sigma\, w_s^{\ell_0} + \eta_s^{\ell_0}\right] = \exp\left[\sigma^2 s + o(s^{5/9})\right].$$

For the same ℓ_0 and (δ_s) as above, we get as in the proof of Proposition 6:

$$\log\frac{|\delta_s/a_s|}{|\delta_{s_0}/a_{s_0}|} = -\sigma^2\int_{s_0}^s \frac{\dot{z}_u\,du}{a_u^2\,\delta_u} - \frac{\sigma^2}{2}\int_{s_0}^s \frac{1-(\dot{z}_u/a_u)^2\,du}{\delta_u^2} + \sigma\,\check{W}\left[\int_{s_0}^s \frac{1-(\dot{z}_u/a_u)^2\,du}{\delta_u^2}\right],$$

almost surely for any sufficiently large s_0, s. Using the above, this shows the almost sure convergence of $\log\left|\frac{\dot{z}_s}{a_s} - \ell_0\right|$, hence by Proposition 6, that indeed $\mathbb{P}[\ell = \ell_0] = 0$. □

3.5. Study of the four-dimensional sub-diffusion $(x_s, \dot{x}_s, a_s, b_s)$. The process (b_s) alone is easily handled, in a way similar to Lemma 2.

Lemma 4. *There exist a real standard, real Brownian motion* (w''_s)*, and an almost surely converging process* (η''_s)*, such that we have:*

$$b_s = \exp\left[\sigma^2 s + \sigma\, w''_s + \eta''_s\right] \quad \textit{for any proper time } s.$$

Proof. By Remark 7 and Sect. 3.1, there exists a real standard, real Brownian motion (w''_s) such that for any proper time s we have:

$$d \log b_s = \sigma^2 ds + \sigma^2\, e^{2\sqrt{2}\,\omega\, x_s}\, b_s^{-2}\, ds + \sigma \sqrt{1 - 2\, e^{2\sqrt{2}\,\omega\, x_s}\, b_s^{-2}}\, dw''_s,$$

so that there exists another real Brownian motion $\tilde{w}''$ such that:

$$\log b_s = \log b_0 + \sigma^2 s + \sigma^2 \int_0^s e^{2\sqrt{2}\,\omega\, x_u}\, \frac{du}{b_u^2}$$
$$+\sigma\, w''_s + \sigma\, \tilde{w}''\left(\int_0^s \left[\frac{2\, e^{2\sqrt{2}\,\omega\, x_u}\, b_u^{-1}}{b_u + \sqrt{b_u^2 - 2\, e^{2\sqrt{2}\,\omega\, x_u}}}\right]^2 du\right).$$

Now using Remark 6 and Lemma 1 we get:

$$\int_0^\infty e^{2\sqrt{2}\,\omega\, x_u}\, \frac{du}{b_u^2} = \int_0^\infty \left[e^{\sqrt{2}\,\omega\, x_u}\, \frac{a_u}{b_u}\right]^2 a_u^{-2}\, du < \infty,$$

which shows that $\eta''_s := \log b_s - \sigma^2 s - \sigma\, w''_s$ almost surely converges, as $s \to \infty$. $\square$

Then we get easily a new asymptotic variable of the relativistic diffusion.

Lemma 5. *The process* $\log(b_s/a_s)$ *converges almost surely, as* $s \to \infty$.

Proof. Recalling from Remark 6 that $b_s/a_s > 0$, we have for any proper time $s \geq 0$:

$$d \log\left[\frac{b_s}{a_s}\right] = \sigma^2\, e^{2\sqrt{2}\,\omega\, x_s}\, b_s^{-2}\, ds - \tfrac{1}{2}\,\sigma^2\, a_s^{-2}\, ds + \sigma\,(b_s^{-1} dM_s^b - a_s^{-1} dM_s^a),$$

or equivalently, for some real Brownian motion W:

$$\log\left[\frac{b_s}{a_s}\right] - \log\left[\frac{b_0}{a_0}\right] = \sigma^2 \int_0^s e^{2\sqrt{2}\,\omega\, x_u}\, \frac{du}{b_u^2} - \sigma^2 \int_0^s \frac{du}{2\, a_u^2}$$
$$+\sigma\, W\left(\int_0^s \left[4\, \frac{e^{\sqrt{2}\,\omega\, x_u}}{a_u\, b_u} - \frac{e^{2\sqrt{2}\,\omega\, x_u}}{b_u^2} - \frac{1}{a_u^2}\right] du\right).$$

Now, as already noticed in the proof of Lemma 4 for $\int_0^s e^{2\sqrt{2}\,\omega\, x_u}\, \frac{du}{b_u^2}$, by Remark 6 and Lemma 1 we get the following, which guarantees the almost sure convergence of $\log(b_s/a_s)$:

$$\int_0^\infty \frac{e^{\sqrt{2}\,\omega\, x_u}}{a_u\, b_u}\, du = \int_0^\infty \left[e^{\sqrt{2}\,\omega\, x_u}\, \frac{a_u}{b_u}\right] a_u^{-2}\, du < \infty.$$

$\square$

By Remark 6 again, we deduce at once the following:

Corollary 3. *The process (x_s) is almost surely bounded. Setting $\varrho := \lim\limits_{s\to\infty} \dfrac{b_s}{a_s}$, we have:*

$$0 < \left(1 - \tfrac{1}{\sqrt{2}}\right) \varrho \le \liminf_{s\to\infty} e^{\sqrt{2}\,\omega\, x_s} \le \limsup_{s\to\infty} e^{\sqrt{2}\,\omega\, x_s} \le \left(1 + \tfrac{1}{\sqrt{2}}\right) \varrho < \infty.$$

Remark 10. It is possible to complete Corollary 3, as Remark 9 completed Proposition 6: it can be shown that the process $(b_s - \varrho\, a_s)$ converges in law, but not in probability.

The following statement, analogous to Proposition 7, ensures that the range of possible limits ϱ in Corollary 3 (and Lemma 5), is the whole $]0, \infty[$. This provides another continuum of non-trivial bounded harmonic functions for the relativistic operator $\mathcal{L}$.

Proposition 9. *For any real ϱ_0 such that $0 < \varrho_0 < \infty$, and for any $\varepsilon > 0$, we have $\mathbb{P}\left[\varrho_0 - \varepsilon < \varrho = \lim\limits_{s\to\infty} \dfrac{b_s}{a_s} < \varrho_0 + \varepsilon\right] > 1 - \varepsilon$, provided b_0/a_0 is close enough from ϱ_0 and a_0 is large enough.*

Proof. Fix $A > \sqrt{3}$, $a_0 > A^2$, and use the expression displayed for $\log(b_s/a_s)$ in the proof of Lemma 5, Remark 6, and Lemma 2, to get an event of probability $\ge 1 - 1/A$:

$$\begin{aligned}\left|\log \varrho - \log\left[\frac{b_0}{a_0}\right]\right| &\le \sigma^2 (1 + \tfrac{1}{\sqrt{2}})^2 \int_0^\infty \frac{du}{a_u^2} \\ &\quad + \sigma \max\left\{|W_u| \,\middle|\, 0 \le u \le 4(1 + \tfrac{1}{\sqrt{2}}) \int_0^\infty \frac{du}{a_u^2}\right\} \\ &\le 3\, A^{-2} + \sigma \max\left\{|W_u| \,\middle|\, 0 \le u \le 7\, \sigma^{-2} A^{-2}\right\},\end{aligned}$$

so that $\mathbb{P}\left(|\log \varrho - \log \varrho_0| \le \left|\log\left[\dfrac{b_0}{a_0\, \varrho_0}\right]\right| + 3\, A^{-2} + A^{-1/2}\right) > 1 - 1/A - e^{-A/14}.$ □

3.6. Irreducibility. We establish here mainly the irreducibility property (i) of the main result (Theorem 1 below, also stated in the Introduction). In particular, the relativistic diffusion of G can perform any absolute time-stemming, in accordance with the non-causality of G. Note that this distinguishes it strongly from its analogues of Minkowski and Schwarzschild space-times, for which the absolute time component increases strictly with proper time.

Proposition 10. (i) *The relativistic diffusion is irreducible: from any starting point, it hits any non-empty open subset of the phase space $\mathcal{E}^+ \setminus \mathcal{E}_0^+$ with strictly positive probability.*
(ii) *For any starting point (in $\mathcal{E}^+$), the law of the asymptotic variable (ℓ, ϱ) charges any non-empty open subset of the range $(]-1, 0[\cup]0, 1[) \times]0, \infty[$.*

Proof. (i) We know from Proposition 2 (in Sect. 2.2) that there are piece-wise geodesic future-directed timelike continuous paths, and then trajectories in the support of the relativistic diffusion $(\xi_\cdot, \dot{\xi}_\cdot)$, moving at will the coordinates (t, x, y, z).

Owing to the quadratic covariation (rank 3) matrix of the $\mathbb{R}^3$-valued martingale (M_s^a, M_s^b, M_s^z) (recall Sect. 3.1) and to the unit pseudo-norm relation (00), we can find three independent standard real Brownian motions (w^1, w^2, w^3) such that:

$$d\dot z_s = \tfrac{3\sigma^2}{2}\,\dot z_s\,ds + \sigma\sqrt{\dot z_s^2+1}\;dw_s^1;$$

$$da_s = \tfrac{3\sigma^2}{2}\,a_s\,ds + \sigma\,\frac{a_s\,\dot z_s}{\sqrt{\dot z_s^2+1}}\,dw_s^1 + \sigma\sqrt{\frac{a_s^2-\dot z_s^2-1}{\dot z_s^2+1}}\;dw_s^2;$$

$$db_s = \tfrac{3\sigma^2}{2}\,b_s\,ds + \sigma\,\frac{b_s\,\dot z_s}{\sqrt{\dot z_s^2+1}}\,dw_s^1 + \sigma\,\frac{a_s\,b_s - 2\,e^{\sqrt2\,\omega\,x_u}(\dot z_s^2+1)}{\sqrt{(\dot z_s^2+1)(a_s^2-\dot z_s^2-1)}}\,dw_s^2$$
$$+\sigma\,\frac{\sqrt2\,e^{\sqrt2\,\omega\,x_u}\,\dot x_s}{\sqrt{a_s^2-\dot z_s^2-1}}\,dw_s^3.$$

Let us use the support theorem of Stroock and Varadhan (see for example ([I-W], Theorem VI.8.1)). We see thus from the above stochastic differential system, that the following trajectories belong to the support of $(\xi_., \dot\xi_.) \equiv (t_., x_., y_., z_., \dot z_., a_., b_., \dot x_.)$:

- trajectories moving at will the coordinate $\dot z$, without changing the coordinates (t, x, y, z);
- trajectories moving at will the coordinate a, without changing the coordinates $(t, x, y, z, \dot z)$;
- trajectories moving at will the coordinate b, provided $\dot x \neq 0$, without changing the coordinates $(t, x, y, z, \dot x, a)$.

So far, it has become clear that it is possible, within the support of the relativistic diffusion, to move any point of the phase space $\mathcal{E}^+\backslash\mathcal{E}_0^+$ having given first coordinates $(t, x, y, z, \dot z, a) \in \mathbb{R}^6$, onto some point of the phase space $\mathcal{E}^+\backslash\mathcal{E}_0^+$ having prescribed first coordinates $(t', x', y', z', \dot z', a') \in \mathbb{R}^6$.

It remains only to consider the last two coordinates $(b, \dot x)$. They are of course constrained by the unit pseudo-norm relation ($00'$), which tells precisely that they run some ellipse of this plane of coordinates, which is centred on the axis $\{\dot x = 0\}$. The last type of trajectory mentioned above allows now to move $(b, \dot x)$ arbitrarily on the upper half and on the lower half this ellipse, without changing the other coordinates, within the support of the relativistic diffusion.

Hence, we have shown that the support of the relativistic diffusion connects any couple of points belonging to the same connected component of $(\mathcal{E}^+ \setminus \mathcal{E}_0^+) \cap \{\dot x \neq 0\}$.

Now, by Sect. 3.1 again, we can find three independent standard real Brownian motions $(\bar w^1, \bar w^2, \bar w^3)$ such that:

$$d\dot x_s = (\omega/\sqrt2)\,e^{-2\sqrt2\,\omega\,x_s}\,b_s^2\,ds - \sqrt2\,\omega\,e^{-\sqrt2\,\omega\,x_s}\,a_s\,b_s\,ds$$
$$+\tfrac{3\sigma^2}{2}\,\dot x_s\,ds + \sigma\sqrt{\dot x_s^2+1}\;d\bar w_s^1;$$

$$da_s = \tfrac{3\sigma^2}{2}\,a_s\,ds + \sigma\,\frac{a_s\,\dot x_s}{\sqrt{\dot x_s^2+1}}\,d\bar w_s^1 + \sigma\sqrt{\frac{a_s^2-\dot x_s^2-1}{\dot x_s^2+1}}\;d\bar w_s^2;$$

$$db_s = \tfrac{3\sigma^2}{2}\, b_s\, ds + \sigma\, \frac{b_s\, \dot{x}_s}{\sqrt{\dot{x}_s^2+1}}\, d\bar{w}_s^1 + \sigma\, \frac{a_s\, b_s - 2\, e^{\sqrt{2}\,\omega\, x_u}(\dot{x}_s^2+1)}{\sqrt{(\dot{x}_s^2+1)(a_s^2-\dot{x}_s^2-1)}}\, d\bar{w}_s^2$$
$$+\sigma\, \frac{\sqrt{2}\, e^{\sqrt{2}\,\omega\, x_u}\, \dot{z}_s}{\sqrt{a_s^2-\dot{x}_s^2-1}}\, d\bar{w}_s^3.$$

The same argument as above, applied to this new decomposition, shows similarly that the support of the relativistic diffusion connects any couple of points belonging to the same connected component of $(\mathcal{E}^+ \setminus \mathcal{E}_0^+) \cap \{\dot{z} \neq 0\}$, hence of $(\mathcal{E}^+ \setminus \mathcal{E}_0^+) \cap \{\dot{x}^2 + \dot{z}^2 \neq 0\}$. This ends actually the proof of irreducibility, since the latter set is a connected and dense open subset of the whole phase space $\mathcal{E}^+ \setminus \mathcal{E}_0^+$.

(ii) This is a direct consequence of (i) above and of Propositions 7 and 9: by (i), it is indeed enough to start the relativistic diffusion so that $\dot{z}_0/a_0$ be close to a given $\ell_0 \in (]-1,0[\,\cup\,]0,1[)$, b_0/a_0 be close to a given $\varrho_0 > 0$, and a_0 be large enough. □

3.7. Convergence to a beam. Recall that, owing to Definition 2, we are looking for a limiting beam $B = (\ell, \varrho, Y)$. Let us exhibit now the third asymptotic random variable Y, for the relativistic diffusion.

Proposition 11. *The process* $Y_s := y_s + \frac{\sqrt{2}}{\omega}\, \frac{\dot{x}_s}{b_s}$ *converges almost surely, as* $s \to \infty$, *toward some real random variable* Y.

Proof. Recall from Formulas (10) that we have $\dot{y}_s = e^{-\sqrt{2}\,\omega\, x_s}(2\, a_s - e^{-\sqrt{2}\,\omega\, x_s} b_s)$. We have then:

$$d\left[\frac{\dot{x}_s}{b_s}\right] = \frac{d\dot{x}_s}{b_s} - \frac{\dot{x}_s\, db_s}{b_s^2} + \frac{\dot{x}_s\, \langle db_s\rangle}{b_s^3} - \frac{\langle db_s, d\dot{x}_s\rangle}{b_s^2}$$
$$= \tfrac{\omega}{\sqrt{2}}\, e^{-2\sqrt{2}\,\omega\, x_s}\, b_s\, ds - \sqrt{2}\,\omega\, e^{-\sqrt{2}\,\omega\, x_s}\, a_s\, ds - 2\sigma^2\, e^{2\sqrt{2}\,\omega\, x_s}\, \frac{\dot{x}_s}{b_s^3}\, ds$$
$$+\frac{\sigma}{b_s}\, dM_s^x - \sigma\, \frac{\dot{x}_s}{b_s^2}\, dM_s^b,$$

whence

$$\tfrac{\omega}{\sqrt{2}}\, dY_s = -2\sigma^2\, e^{2\sqrt{2}\,\omega\, x_s}\, \frac{\dot{x}_s}{b_s^3}\, ds + \frac{\sigma}{b_s}\, dM_s^x - \sigma\, \frac{\dot{x}_s}{b_s^2}\, dM_s^b,$$

and for some Brownian motion W':

$$\tfrac{\omega}{\sqrt{2}}\, Y_s = \tfrac{\omega}{\sqrt{2}}\, Y_0 - 2\sigma^2 \int_0^s e^{2\sqrt{2}\,\omega\, x_u}\, \frac{\dot{x}_u}{b_u^3}\, du + \sigma\, W'\left[\int_0^s \left[1 - 2\, e^{2\sqrt{2}\,\omega\, x_u} \left|\frac{\dot{x}_u}{b_u}\right|^2\right] \frac{du}{b_u^2}\right].$$

By Corollary 3, Lemma 4, and Proposition 6 (which implies, according to Sect. 3.2, that $\dot{x}_s/b_s$ is bounded), the two above integrals, and then Y_s, converge almost surely. □

Corollary 4. *We have almost surely, as* $s \to \infty$:

$$\left[\frac{\varrho}{2}\, e^{-\sqrt{2}\,\omega\, x_s} - 1\right]^2 + \left[\frac{\omega\,\varrho}{2}\,(y_s - Y)\right]^2 \longrightarrow \tfrac{1}{2}(1-\ell^2).$$

Proof. By definition of Y_s (in Proposition 11) and Sect. 3.2, the left-hand side equals:

$$\left[\frac{\varrho\, a_s}{b_s} - 1 - \frac{\varrho \sinh(\lambda_s)}{\sqrt{2}\, b_s}\, \sin(\omega\, \varphi_s)\right]^2 + \left[\frac{\omega\, \varrho}{2}\, (Y_s - Y) - \frac{\varrho \sinh(\lambda_s)}{\sqrt{2}\, b_s}\, \cos(\omega\, \varphi_s)\right]^2$$
$$= \left[\frac{\varrho \sinh \lambda_s}{\sqrt{2}\, b_s}\right]^2 + \left[\frac{\varrho a_s}{b_s} - 1\right]\left[\tfrac{\varrho a_s}{b_s} - 1 - \tfrac{\sqrt{2}\varrho \sinh \lambda_s}{b_s}\, \sin(\omega\varphi_s)\right]$$
$$+ \left[\tfrac{\omega\varrho}{2}(Y_s - Y)\right]\left[\tfrac{\omega\varrho}{2}(Y_s - Y) - \tfrac{\sqrt{2}\varrho \sinh \lambda_s}{b_s}\, \cos(\omega\varphi_s)\right],$$

which goes to $\frac{1-\ell^2}{2}$, by Proposition 6, Corollary 3, Proposition 11, and since (by Lemma 1, Corollary 3, and Proposition 6): $\dfrac{\varrho \sinh \lambda_s}{b_s} = \varrho\, \dfrac{a_s}{b_s} \times \sqrt{1 - \left|\dfrac{\dot{z}_s}{a_s}\right|^2 - a_s^{-2}} \longrightarrow \sqrt{1-\ell^2}$. □

The following statement, analogous to Propositions 7 and 9, ensures that the range of possible limits Y in Proposition 11 is the whole $\mathbb{R}$. This provides again another continuum of non-trivial bounded harmonic functions for the relativistic operator $\mathcal{L}$ acting on $T^1 G$.

Proposition 12. *For any real y and any $\varepsilon > 0$, we have $\mathbb{P}[y - \varepsilon < Y < y + \varepsilon] > 1 - \varepsilon$, provided Y_0 is close enough from y and a_0 is large enough.*

Proof. Recall from Lemma 2 that the event $\mathcal{A} := \left\{a_s \geq \sqrt{a_0}\, e^{\sigma^2 s/2} \text{ for any } s \geq 0\right\}$ has (for $a_0 > 3$) probability larger than $1 - a_0^{-1/2}$. The proof of Proposition 11 shows that

$$|Y - Y_0| = \mathcal{O}(1)\int_0^\infty \frac{du}{a_u^2} + \max\left\{|W_s| \,\Big|\, 0 \leq s \leq \mathcal{O}(1)\int_0^\infty \frac{du}{a_u^2}\right\}$$
$$= \mathcal{O}(\tfrac{1}{a_0}) + W^*[\mathcal{O}(\tfrac{1}{a_0})] \;\text{ on } \mathcal{A},$$

so that $\mathbb{P}\left(|Y - Y_0| \leq 2\, a_0^{-1/3}\right) > 1 - 2\, a_0^{-1/2}$, for large enough a_0. □

Proposition 12 improves Proposition (10,(ii)). We deduce indeed at once the following.

Corollary 5. *For any starting point (in $\mathcal{E}$), the law of the asymptotic variable (ℓ, ϱ, Y) charges any non-empty open subset of the range $(]-1, 0[\,\cup\,]0, 1[) \times]0, \infty[\times\mathbb{R}$.*

More precisely, if the starting point of the relativistic diffusion satisfies: $\dot{z}_0/a_0$ close enough to $\ell_0 \in]-1, 1[$, b_0/a_0 close enough to $\varrho_0 > 0$, Y_0 close enough to $y \in \mathbb{R}$, and a_0 large enough, then with arbitrary large probability, (ℓ, ϱ, Y) is arbitrary close to (ℓ_0, ϱ_0, y).

The theorem of the Introduction (Sect. 1) is now established. Indeed, gathering successively Remark 8 and Proposition 10, Propositions 6, Lemma 3, Corollary 3, Proposition 11 and Corollary 4, and Corollary 5, we get the following main result (for which $\sigma > 0$ is necessary, due to the observation made after Definition 2, in Sect. 2.3).

Theorem 1. (i) *The relativistic diffusion is irreducible, on its phase space* $\mathcal{E}^+\setminus\mathcal{E}_0^+$.
(ii) *Almost surely, the relativistic diffusion path possesses a 3-dimensional asymptotic random variable* $B = (\ell, \varrho, Y) \in \mathcal{B}$*, and converges to this beam* B*, in the sense of Definition 2. Indeed, we have almost surely, as proper time* s *goes to infinity:*

$$\dot{z}_s/a_s \longrightarrow \ell \in]-1, 0[\cup]0, 1[; \quad b_s/a_s \longrightarrow \varrho \in]0, \infty[; \quad Y_s \longrightarrow Y \in \mathbb{R};$$

$$\left[\frac{\varrho}{2}\, e^{-\sqrt{2}\omega\, x_s} - 1\right]^2 + \left[\frac{\omega\,\varrho}{2}\,(y_s - Y)\right]^2 \longrightarrow \tfrac{1}{2}(1-\ell^2).$$

(iii) *The asymptotic random variable* (ℓ, ϱ, Y) *can be arbitrary close to any given* $(\ell_0, \varrho_0, y) \in]-1, 1[\times]0, \infty[\times\mathbb{R}$*, with positive probability. Hence, the whole boundary (space of beams)* $\mathcal{B}$ *is the support of beams the relativistic diffusion can converge to.*

Remark 11. A rapid look at Remark 4 could let one think that there could be a fourth asymptotic random variable for the relativistic diffusion, namely a possible almost sure limit for $X_s := z_s + \ell\, t_s - \ell\, \varphi_s$. But as a matter of fact, it can be shown (in the same vein as Remarks 9 and 10) that there is no such limit, in accordance with the last sentence of Remark 4, on the geometric irrelevance of the additional parameter (Z_0, T_0).

Remark 12. From the proofs of Proposition 14, Proposition 11, and Proposition 6, we have the following representation of the asymptotic variable $B = (\ell, \varrho, Y)$:

$$\varrho = \frac{b_0}{a_0} + 2\sigma^2 \int_0^\infty e^{\sqrt{2}\,\omega\, x_u}\, \frac{du}{a_u^2} - \sigma^2 \int_0^\infty \frac{b_u\, du}{a_u^3} + \sigma \int_0^\infty a_u^{-1}\left[dM_u^b - \tfrac{b_u}{a_u}\, dM_u^a\right];$$

$$Y = Y_0 - \tfrac{2\sqrt{2}\,\sigma^2}{\omega} \int_0^\infty e^{2\sqrt{2}\,\omega\, x_u}\, \frac{\dot{x}_u}{b_u^3}\, du + \tfrac{\sqrt{2}\,\sigma}{\omega} \int_0^\infty b_u^{-1}\left[dM_u^x - \tfrac{\dot{x}_u}{b_u}\, dM_u^b\right];$$

$$\ell = \frac{\dot{z}_0}{a_0} - \sigma^2 \int_0^\infty \frac{\dot{z}_u}{a_u^3}\, du + \sigma \int_0^\infty a_u^{-1}\left[dM_u^z - \tfrac{\dot{z}_u}{a_u}\, dM_u^a\right].$$

By Proposition 8, the law of the asymptotic variable B has no atom, and by Theorem (1, (iii)), it is really three-dimensional. None of ℓ, ϱ, Y is a function of the other two.

Theorem 1 and Remarks 9, 10, 11, 12, incite to believe in the following, which, by classical methods, would imply that the Poisson boundary of G identifies with its geometric boundary $\mathcal{B}$:

Conjecture. The invariant σ-field of the relativistic diffusion in Gödel's universe is the σ-field generated by the asymptotic three-dimensional random variable $B = (\ell, \varrho, Y)$ of Theorem 1 (exhibited by Proposition 6, Corollary 3, and Proposition 11).

3.8. Improvement of convergence. We show here that the convergence of Theorem (1,(ii)), of the generic diffusion path $(\xi_s, \dot{\xi}_s)$, occurs in fact in some stronger sense: on one hand, it is exponentially fast, as stated in Corollary 7 below, and on the other hand, it holds partially in the sense of Skorohod topology, as explained in the concluding Remark 13.

Lemma 6. *We have almost surely, for any* $n \geq 1$ *and* $\varepsilon > 0$*:* $\lim_{s\to\infty} a_s^{n-\varepsilon} \int_s^\infty a_u^{-n}\, du = 0.$

Proof. Recall from the proof of Lemma 1 that we have for any $0 \le s \le u$:

$$\frac{a_s}{a_{s+u}} = \exp\left[-\sigma^2 u - \sigma(w_{s+u} - w_s) - \tfrac{\sigma^2}{2}\int_s^{s+u}\frac{dv}{a_v^2} + \sigma\int_s^{s+u}\frac{a_v^{-2}\,dw_v}{1+\sqrt{1-a_v^{-2}}}\right].$$

Hence, by Lemma 1, for $s \to \infty$:

$$\begin{aligned}\int_s^\infty a_u^{-n}\,du &\sim a_s^{-n}\int_0^\infty \exp\Big(-n\,\sigma^2 u - n\,\sigma\,(w_{s+u} - w_s)\Big)\,du \\ &= o\left(a_s^{\varepsilon-n}\right)\int_0^\infty \exp\Big(-n\,\sigma^2 u - n\,\sigma\,(w_{s+u} - w_s) - \tfrac{\varepsilon}{2}(\sigma^2 s + \sigma w_s)\Big)\,du \\ &= o\left(a_s^{\varepsilon-n}\right)\int_0^\infty \exp\Big(-\sigma^2\Big[(n-\tfrac{\varepsilon}{4})u + \Big(\tfrac{\varepsilon}{4} + \tfrac{n w_{s+u}}{\sigma(s+u)}\Big)(s+u) + \Big(\tfrac{\varepsilon}{4} - \tfrac{(n-\frac{\varepsilon}{2})w_s}{\sigma\, s}\Big)s\Big]\Big)du \\ &= o\left(a_s^{\varepsilon-n}\right).\end{aligned}$$

□

Proposition 13. *We have for any $\varepsilon > 0$, almost surely:* $\lim\limits_{s\to+\infty} (\dot z_s - \ell\, a_s)\, a_s^{-\varepsilon} = 0$.

Proof. Since $\left\langle dM_s^z - \frac{\dot z_s}{a_s}\, dM_s^a\right\rangle = \left(1 - \left|\frac{\dot z_s}{a_s}\right|^2\right) ds$, using the expression for $d\left[\frac{\dot z_s}{a_s}\right]$ displayed in the proof of Proposition 6, we have:

$$\dot z_s - \ell\, a_s = \sigma^2 a_s \int_s^\infty \frac{\dot z_u}{a_u^3}\,du - \sigma\, a_s\, \tilde W\left[\int_s^\infty \left[1 - \left|\frac{\dot z_u}{a_u}\right|^2\right]\frac{du}{a_u^2}\right],$$

for some Brownian Motion $\tilde W$. Now, by Proposition 6 and Lemma 6, we have almost surely:

$$\begin{aligned}\int_s^\infty \tfrac{\dot z_u}{a_u^3}du &= \int_s^\infty \mathcal{O}(a_u^{-2})du = o\left(a_s^{\varepsilon-2}\right), \quad \tilde W\left[\int_s^\infty\left[1 - \left|\tfrac{\dot z_u}{a_u}\right|^2\right]\tfrac{du}{a_u^2}\right] \\ &= o\left[\int_s^\infty \tfrac{du}{a_u^2}\right]^{\frac{1-\varepsilon}{2}} = o\left(a_s^{2\varepsilon-1}\right).\end{aligned}$$

Therefore we get finally: $\dot z_s - \ell\, a_s = o\left(a_s^{\varepsilon-1}\right) + o\left(a_s^{2\varepsilon}\right) = o\left(a_s^{2\varepsilon}\right)$. □

Lemma 3, Proposition 13 and Sect. 3.2 imply at once the following.

Corollary 6. *For any $\varepsilon > 0$, we have almost surely:* $\dfrac{\sinh \lambda_s}{a_s} = \sqrt{1-\ell^2} + o(a_s^{\varepsilon-1})$,

$$e^{-\sqrt{2}\,\omega\, x_s}\,\frac{b_s}{a_s} = 2 - \sqrt{2(1-\ell^2)}\,\sin(\omega\,\varphi_s) + o(a_s^{\varepsilon-1}) \quad \textit{and}$$
$$\dot x_s = \sqrt{1-\ell^2}\; a_s\,\cos(\omega\,\varphi_s) + o(a_s^{\varepsilon}).$$

In the same vein as Proposition 13, we have the following.

Proposition 14. *We have for any $\varepsilon > 0$, almost surely:* $\lim\limits_{s\to+\infty} (b_s - \varrho\, a_s)\, a_s^{-\varepsilon} = 0.$

Proof. We have: $d\left[\frac{b_s}{a_s}\right] = 2\sigma^2\,\frac{e^{\sqrt{2}\,\omega\,x_s}}{a_s^2}\,ds - \sigma^2\,\frac{b_s}{a_s^3}\,ds + \frac{\sigma}{a_s}\left[dM_s^b - \frac{b_s}{a_s}\,dM_s^a\right]$,

and $\left\langle dM_s^b - \frac{b_s}{a_s}\,dM_s^a\right\rangle = \left[4\,e^{\sqrt{2}\,\omega\,x_s}\,\frac{b_s}{a_s} - 2\,e^{2\sqrt{2}\,\omega\,x_s} - \frac{b_s^2}{a_s^2}\right]ds$,

so that there exists a standard real Brownian motion W such that:

$$\varrho - \frac{b_s}{a_s} = 2\sigma^2\int_s^\infty \frac{e^{\sqrt{2}\,\omega\,x_u}}{a_u^2}du - \sigma^2\int_s^\infty \frac{b_u}{a_u^3}\,du$$
$$+\sigma\,W\left[\int_s^\infty\left[4\,e^{\sqrt{2}\,\omega\,x_u}\,\frac{b_u}{a_u} - 2\,e^{2\sqrt{2}\,\omega\,x_u} - \frac{b_u^2}{a_u^2}\right]\frac{du}{a_u^2}\right].$$

Now, by Corollary 3 and Lemma 6, we have almost surely:

$$\int_s^\infty \frac{e^{\sqrt{2}\,\omega\,x_u}}{a_u^2}\,du + \int_s^\infty \frac{b_u}{a_u^3}du$$
$$= o\left(a_s^{\varepsilon-2}\right),\quad \int_s^\infty\left[4\,e^{\sqrt{2}\,\omega\,x_u}\,\frac{b_u}{a_u} - 2\,e^{2\sqrt{2}\,\omega\,x_u} - \frac{b_u^2}{a_u^2}\right]\frac{du}{a_u^2} = o\left(a_s^{\varepsilon-2}\right).$$

Hence, as in the proof of Proposition 13, we deduce $\varrho - b_s/a_s = o\left(a_s^{2\varepsilon-1}\right)$. □

Proposition 15. *We have* $Y - Y_s = o\left(a_s^{\varepsilon-1}\right)$ *almost surely, for any* $\varepsilon > 0$.

Proof. From the proof of Proposition 11, we have (for some Brownian motion W'):

$$Y_s - Y = Y_s - Y_\infty = \frac{2\sqrt{2}\,\sigma^2}{\omega}\int_s^\infty e^{2\sqrt{2}\,\omega\,x_u}\,\frac{\dot{x}_u}{b_u^3}\,du$$
$$+W'\left[\frac{2\,\sigma^2}{\omega^2}\int_s^\infty\left[1 - 2\,e^{2\sqrt{2}\,\omega\,x_u}\left|\frac{\dot{x}_u}{b_u}\right|^2\right]\frac{du}{b_u^2}\right]$$
$$= \mathcal{O}(1)\int_s^\infty a_s^{-2}\,ds + W'\left[\mathcal{O}(1)\int_s^\infty a_u^{-2}\,du\right] = o\left(a_s^{\varepsilon-1}\right),$$

as in the proof of Proposition 13, by Lemma 6. □

As Proposition 6, Corollary 3, and Proposition 11 implied Corollary 4 in Sect. 3.7, the following is easily implied by the above Propositions 13, 14, 15.

Corollary 7. *For any* $\varepsilon > 0$*, we have almost surely, as* $s \to \infty$*:*

$$\left[\frac{\varrho}{2}\,e^{-\sqrt{2}\,\omega\,x_s} - 1\right]^2 + \left[\frac{\omega\,\varrho}{2}\,(y_s - Y)\right]^2 = \tfrac{1}{2}(1-\ell^2) + o\left(a_s^{\varepsilon-1}\right).$$

In Theorem (1, (ii)*), the four converging processes are at distance* $o(e^{(\varepsilon-1)\sigma^2 s})$ *of their limits.*

Finally, another type of improvement of the convergence is noticed in the following.

Remark 13. By Corollary 6 and Propositions 14 and 15, we have on one hand:

$$e^{-\sqrt{2}\,\omega\, x_s} = \tfrac{2}{\varrho} - \tfrac{\sqrt{2(1-\ell^2)}}{\varrho}\,\sin(\omega\,\varphi_s) + o(e^{(\varepsilon-1)\sigma^2 s}) \quad \text{and}$$
$$y_s = Y - \tfrac{\sqrt{2(1-\ell^2)}}{\omega\,\varrho}\,\cos(\omega\,\varphi_s) + o(e^{(\varepsilon-1)\sigma^2 s}),$$

while by Remark 4, for any light ray $\overline{\xi} = (\overline{t}, \overline{x}, \overline{y}, \overline{z})$ belonging to the beam $B = (\ell, \varrho, Y)$, using the increasing diffeomorphism $\bar{\varphi} = (\tau \mapsto \bar{\varphi}_\tau)$, we have on the other hand:

$$\exp[-\sqrt{2}\,\omega\,\overline{x}_{\bar{\varphi}^{-1}(\varphi_s)}] = \tfrac{2}{\varrho} - \tfrac{\sqrt{2(1-\ell^2)}}{\varrho}\,\sin(\omega\,\varphi_s) \quad \text{and}$$
$$\overline{y}_{\bar{\varphi}^{-1}(\varphi_s)} = Y - \tfrac{\sqrt{2(1-\ell^2)}}{\omega\,\varrho}\,\cos(\omega\,\varphi_s).$$

Hence, we have in the (x, y)-plane a strong convergence, of the projection of the generic relativistic diffusion path to the projection of a lightlike geodesic, in the Skorohod topology:

$$\left|x_s - \overline{x}_{\bar{\varphi}^{-1}(\varphi_s)}\right| + \left|y_s - \overline{y}_{\bar{\varphi}^{-1}(\varphi_s)}\right| = o(e^{(\varepsilon-1)\sigma^2 s}).$$

Otherwise, by Corollary 6, Propositions 5, 13, and Remark 4, we have:

$$\begin{aligned} z_s + \ell\, t_s &= z_s + \ell t_0 + \ell \int_0^s (e^{-\sqrt{2}\,\omega\, x_u} b_u - a_u)du \\ &= \ell\,\varphi_s + \mathcal{O}(1) + \int_0^s (\dot{z}_u - \ell\, a_u)du = \ell\,\varphi_s + o(e^{\varepsilon\, s}) \\ &= \overline{z}_{\bar{\varphi}^{-1}(\varphi_s)} + \ell\,\overline{t}_{\bar{\varphi}^{-1}(\varphi_s)} + o(e^{\varepsilon\, s}) \\ &= \left(\overline{z}_{\bar{\varphi}^{-1}(\varphi_s)} + \ell\,\overline{t}_{\bar{\varphi}^{-1}(\varphi_s)}\right)\left[1 + o\left(e^{(\varepsilon-1)\sigma^2 s}\right)\right] \longrightarrow \infty. \end{aligned}$$

Hence, again in the Skorohod topology, and in the (z, t)-plane, the projection of the limiting light ray $\overline{\xi}$ stands for an asymptotic direction for the projection of the generic relativistic diffusion path, but there is no exactly asymptotic lightlike geodesic (a parabolic branch occurs).

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