

CENTRAL LIMIT THEOREM FOR THE GEODESIC FLOW ASSOCIATED WITH A KLEINIAN GROUP, CASE $\delta > d/2$

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ABSTRACT. – Let Γ be a geometrically finite Kleinian group, relative to the hyperbolic space $\mathbb{H} = \mathbb{H}^{d+1}$, and let δ denote the Hausdorff dimension of its limit set, that we suppose here strictly larger than $d/2$. We prove a central limit theorem for the geodesic flow on the manifold $\mathcal{M} := \Gamma \backslash \mathbb{H}$, with respect to the Patterson–Sullivan measure. The argument uses the ground-state diffusion and its canonical lift to the frame bundle, for which the existence of a potential operator is proved. © 2001 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

Consider the hyperbolic space $\mathbb{H} = \mathbb{H}^{d+1}$, endowed with some geometrically finite Kleinian group Γ . The Hausdorff dimension $\delta \in [0, d]$ of its limit set (see [13,17] or [18]) plays a fundamental role. When δ is larger than $d/2$, $\delta(\delta - d)$ is the highest eigenvalue of the Laplacian on a fundamental domain. The associated eigenstate Φ plays an important role in the study of the quotient $\mathcal{M} = \Gamma \backslash \mathbb{H}$ and of its geodesic flow. The corresponding ground-state diffusion Z_t^Φ , which we call “ Φ -diffusion”, is then also a natural object and tool in this framework: see [17, 2–4].

As in [3], where the problem of the asymptotic law of windings in cusps of hyperbolic surfaces has been treated, we approximate the Patterson–Sullivan measure m by the images under the geodesic flow θ_t of a quasi-invariant measure ν , which is stationary for the lift of the ground-state diffusion Z_t^Φ to $T^1\mathcal{M}$. An important step, which was not needed in [3], but which extends the proof given in [9] for the finite volume case, is the existence of a potential operator V for the lift of the Φ -diffusion.

Denote by \mathcal{L}_0 the Lie derivative along the geodesic flow, and by $\mathcal{L}_1, \dots, \mathcal{L}_d$ the Lie derivatives along the stable horocycle flows (which are no longer defined on the tangent

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bundle $T^1\mathcal{M}$, but only on the frame bundle $O\mathcal{M}$). Let f_1, \dots, f_d denote the conjugate functions of any given function f having bounded derivatives on $O\mathcal{M}$; they are defined by: $f_j := -\int_0^\infty e^{-s} \mathcal{L}_j f(\cdot \theta_s) ds$. Set $f_0 := f$ for convenience. Let us also introduce the canonical projection π_2 from $O\mathcal{M}$ onto \mathcal{M} , and the divergence operator K :

$$Kf := \frac{1}{2} \sum_{j=0}^d \mathcal{L}_j f_j + \sum_{j=0}^d (\mathcal{L}_j \log \Phi \circ \pi_2) f_j - \frac{d}{2} f.$$

Our main result, assuming that $\delta > d/2$, is the following central limit theorem for the geodesic flow on $T^1\mathcal{M}$, relating to the Patterson–Sullivan measure m :

THEOREM. – *Let us fix a real function f on $T^1\mathcal{M}$, such that $\int f dm = 0$, and of class C^2 with bounded and Hölderian derivatives. Then for all $a \in \mathbb{R}$ we have:*

$$\lim_{t \rightarrow \infty} \int_{T^1\mathcal{M}} \exp\left(\frac{a\sqrt{-1}}{\sqrt{t}} \int_0^t f(\xi \theta_s) ds\right) dm(\xi) = m(T^1\mathcal{M}) \times \exp\left(-\frac{a^2}{2} \mathcal{V}(f)\right),$$

where $\mathcal{V}(f) := \sum_{j=0}^d \int (f_j + \mathcal{L}_j \mathcal{V} Kf)^2 dv$ vanishes if and only if f is a \mathcal{L}_0 -derivative.

“ F Hölderian on $O\mathcal{M}$ ” precisely means: there exists some $r > 0$ such that $\text{dist}(\xi, \xi')^{-r} |F(\xi) - F(\xi')|$ is bounded on $\{(\xi, \xi') \in O\mathcal{M}^2 \mid 0 < \text{dist}(\xi, \xi') < 1\}$.

Equivalently, our result reads (with $c(\delta)$ given in Corollary 1): *for all $a \in \mathbb{R}$ we have:*

$$\lim_{t \rightarrow \infty} m \left[\xi \in T^1\mathcal{M} \mid \int_0^t f(\xi \theta_s) ds \leq a\sqrt{t\mathcal{V}(f)} \right] = \|\Phi\|_2^2 c(\delta)^{-1} \times (2\pi)^{-1/2} \int_{-\infty}^a \exp(-s^2/2) ds.$$

The particular case of \mathcal{M} being convex-cocompact, that is to say without cusp, or associated with a group Γ without parabolic element, can be handled by the coding method of [15]. See also [1,8] and [19].

The particular case of \mathcal{M} having finite volume (corresponding to $\delta = d$) was handled in [9], the Patterson–Sullivan measure being in this case just the Liouville measure. Our result concerns the more general case $d/2 < \delta \leq d$, in which \mathcal{M} may have both infinite volume and cusps.

For dealing with this new case, we use here globally the same strategy as in [9], which consists roughly in comparing the geodesics with the paths of a diffusion on $T^1\mathcal{M}$, for which the existence of potentials has to be exhibited. But the infinite volume case is much more involved, since m is not invariant under the horocycle flows and is distinct of ν , which is only quasi-invariant under the geodesic and stable horocycle flows.

Finally note that the remaining case $\delta \leq d/2$ cannot be handled in the same way, since in that case there does not exist any fundamental diffusion Z_t^Φ associated with δ on the base manifold \mathcal{M} .

2. Notations and basic data

Let \mathbb{H} denote the hyperbolic space \mathbb{H}^{d+1} , with boundary $\partial\mathbb{H}$, unitary tangent bundle $T^1\mathbb{H}$, and orthonormal frame bundle $O\mathbb{H}$.

Let us identify \mathbb{H} with its Poincaré half-space model $\mathbb{R}^d \times \mathbb{R}_+^*$, and the current point $z \in \mathbb{H}$ with its canonical coordinates $(x, y) = (x_1, \dots, x_d, y) \in \mathbb{R}^d \times \mathbb{R}_+^*$. Set $e_0 := (0, 1)$. Recall that

the metric of \mathbb{H} is given by: $|dz|^2 = (|dx|^2 + dy^2)/y^2$, that its Riemannian volume measure is given by: $d\tilde{V} := y^{-d-1} dx dy$, and that its (hyperbolic) Laplacian is given by:

$$\Delta = y^2 \times \left(\frac{\partial^2}{\partial y^2} + \frac{1-d}{y} \times \frac{\partial}{\partial y} + \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} \right).$$

Let us denote by G the Möbius group of orientation-preserving isometries of \mathbb{H} , which is generated by the following elements:

- the translations $\theta_x^+ = \theta_{x_1}^1 \dots \theta_{x_d}^d$, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$;
 - the homotheties θ_t (= the linear dilatation by e^t), for $t \in \mathbb{R}$;
 - the Euclidian rotations $R_t^{i,j}$ (= the rotation by t in the plane (x_i, x_j));
 - the hyperbolic rotations $R_t^{0,j}$ (defined by: $R_t^{0,j}(x_1, \dots, x_j, \dots, x_d, y) = (x_1, \dots, x'_j, \dots, x_d, y')$,
- with $x'_j + y'\sqrt{-1} = [(x_j + y\sqrt{-1})\cos(t/2) + \sin(t/2)]/[\cos(t/2) - (x_j + y\sqrt{-1})\sin(t/2)]$.

Let us set, for any $z = (x, y) \in \mathbb{H}$:

$$T_z := \theta_x^+ \theta_{\text{Log } y}.$$

Observe the following important classical relation:

$$(1) \quad T_{(x,y)} T_{z'} = T_{(x,0)+yz'}.$$

This means in particular that the set $\{T_z \mid z \in \mathbb{R}^d \times \mathbb{R}_+^*\}$ constitutes a group, isomorphic to a subgroup of the affine group of \mathbb{R}^d .

Observe that the Euclidian rotations above generate the subgroup SO_d of G , and that SO_d and the hyperbolic rotations together, generate a subgroup of G isomorphic to SO_{d+1} , which we identify with SO_{d+1} . So SO_{d+1} is the subgroup of those $g \in G$ which fix e_0 , and SO_d is the subgroup of those $g \in SO_{d+1}$ whose differential at e_0 fixes $\frac{\partial}{\partial y}$. This identifies $T^1\mathbb{H}$ with $O\mathbb{H}/SO_d$, and \mathbb{H} with $O\mathbb{H}/SO_{d+1}$.

As usual, we identify any $g \in G$ with $(g, \frac{dg}{|dg|})$ operating on $T^1\mathbb{H}$ or $O\mathbb{H}$. As there exists a unique $g \in G$ mapping a given $\xi \in O\mathbb{H}$ to another given frame of $O\mathbb{H}$, we may and do identify G with $O\mathbb{H}$, by identifying any $g \in G$ with the image it gives of the canonical frame at e_0 .

We have a unique decomposition of any $g \in G$: $g = T_z R R'$, with $z \in \mathbb{H}$, R a hyperbolic rotation in the plane $(\frac{\partial}{\partial y}, dg_{e_0}(\frac{\partial}{\partial y}))$, and $R' \in SO_d$. In particular, we have an identification between $O\mathbb{H}$ and $T^1\mathbb{H} \times SO_d$.

Note that these identifications have two important consequences:

- Firstly, the base-point of the frame ξ is $\pi_2(\xi) = \xi(e_0)$, π_2 being the canonical projection from $O\mathbb{H}$ onto \mathbb{H} . And $\pi_2(\xi T_z) = \xi(z)$, for any $z \in \mathbb{H}$ and any $\xi \in O\mathbb{H}$.
- Secondly, the right multiplication by θ_t moves any frame ξ by the geodesic flow of algebraic length t , in the direction of the vector $dg_{e_0}(\frac{\partial}{\partial y})$. And similarly, the right multiplication by θ_t^j moves the frame ξ by the horocycle flow of algebraic length t , in the direction of the vector $dg_{e_0}(\frac{\partial}{\partial x_j})$.

Indeed, this is clear when ξ is the unit element of G , that is to say the canonical frame at e_0 , and this remains clear for the other frames by invariance of π_2 and of the flows under isometries.

Note that since θ_t commutes with the Euclidian rotations, the geodesic flow still makes sense on $T^1\mathbb{H}$. On the contrary, the horocycle flow makes sense only on $O\mathbb{H}$.

Given (z, z', u) in $\mathbb{H} \times \mathbb{H} \times \partial\mathbb{H}$, denote by $\log[B_u(z, z')]$ the Busemann function, that is to say the algebraic hyperbolic distance, on any geodesic ending at u , from the stable horocycle $H(z, u)$ determined by z to the stable horocycle $H(z', u)$.

In the Poincaré half-space model, we have $B_u(z, z') = p(z', u)/p(z, u)$, $p(z, u)$ denoting the Poisson kernel: $p(z, u) = y \times |z - u|^{-2}$ if $u \neq \infty$ and $p(z, \infty) = y$.

We have the cocycle property: $B_u(z, z'') = B_u(z, z') \times B_u(z', z'')$.

We shall use on the unitary tangent bundle $T^1\mathbb{H}$ the two following systems of coordinates:

- firstly, $(z, u) \in \mathbb{H} \times \partial\mathbb{H}$, the geodesic running from z to u determining the unitary tangent vector at the base point z ; this identifies $T^1\mathbb{H}$ with $\mathbb{H} \times \partial\mathbb{H}$;
- secondly, given a reference point $z_0 \in \mathbb{H}$, the point (z, u) of $T^1\mathbb{H}$ can be represented by the triple $(u, v, s) \in \partial\mathbb{H} \times \partial\mathbb{H} \times \mathbb{R}$, where
 - v is the starting point of the geodesic ending at u and running through z ;
 - s is the algebraic hyperbolic distance from z to the orthogonal projection z_1 of z_0 onto the geodesic \overrightarrow{vu} .

Note that the first coordinates above extend to the following global coordinates on $O\mathbb{H}$: $(z, u, r) \in \mathbb{H} \times \partial\mathbb{H} \times SO_d$, by means of the identification between $O\mathbb{H}$ and $T^1\mathbb{H} \times SO_d$.

Denote by $\text{dist}(\zeta, uv)$ the hyperbolic distance from $\zeta \in \mathbb{H}$ to the geodesic \overrightarrow{vu} .

The following well-known identity is valid for any ζ in \mathbb{H} , any distinct u, v in $\partial\mathbb{H}$, and any z on the geodesic \overrightarrow{vu} running from v to u .

$$(*) \quad ch^2(\text{dist}(\zeta, uv)) = B_u(\zeta, z)B_v(\zeta, z).$$

(Indeed, since this is an intrinsic formula, we may consider the half-space model with $u = \infty$ and $v = 0$. Denoting then by (X, Y) the Euclidian coordinates of ζ in this model, and by $(0, y)$ those of z , it is elementary that $B_u(\zeta, z) = y/Y$, $B_v(\zeta, z) = (|X|^2 + Y^2)/(yY)$, and, using the classical formula for the distance (see [13]), that $ch^2(\text{dist}(\zeta, uv)) = ch^2\text{dist}(\zeta, (0, |z|)) = (|X|^2 + Y^2)^2/(Y|z|)^2 = 1 + |X|^2/Y^2 = B_u(\zeta, z)B_v(\zeta, z)$.)

Let Γ be a discrete torsion-free (non-elementary) subgroup of G , that we suppose geometrically finite. Let $\Lambda = \Lambda(\Gamma)$ denote its limit set, with Hausdorff dimension say δ . Recall that δ is also the critical convergence exponent of the Poincaré series relative to Γ (see for example ([18], Theorem 1)). Obviously $\delta \leq d$.

We make here the assumption that $\delta > d/2$.

Let $\{\mu_z \mid z \in \mathbb{H}\}$ denote the family of Patterson (finite) measures on Λ associated with Γ . It can be defined, up to a multiplicative constant (that we definitively fix), as the only family of measures on Λ satisfying the following geometric “conformal density” property:

$$d\mu_{z'}(u) = B_u^\delta(z, z') d\mu_z(u) \quad \text{for any } z, z' \text{ in } \mathbb{H}$$

together with the invariance property by the group Γ , in the sense that:

$$\gamma^* \mu_z = \mu_{\gamma z} \quad \text{for any } \gamma \text{ in } \Gamma \text{ and } z \text{ in } \mathbb{H},$$

with the convention $\gamma^* \mu := \mu \circ \gamma^{-1}$.

See for example ([13], Lecture 2), [18], or ([11], Sections 3.4 and 4.7).

Set

$$\Phi(z) := \int d\mu_z = \mu_z(\partial\mathbb{H}) = \mu_z(\Lambda), \quad \text{and} \quad \lambda_0 := \delta(\delta - d)/2.$$

It is a Γ -invariant function on \mathbb{H} which verifies $\Delta\Phi = 2\lambda_0\Phi$. See ([13], Theorem 1, p. 301).

Define the Patterson–Sullivan measure \tilde{m} on $T^1\mathbb{H}$ by

$$d\tilde{m}(u, v, s) := ch^{2\delta}(\text{dist}(z_0, uv)) d\mu_{z_0}(u) d\mu_{z_0}(v) ds.$$

Note that by the geometric property for (μ_z) and by the identity (*) above, \tilde{m} does not depend on the choice of the reference point z_0 . Hence it is intrinsic (it depends only on the subgroup Γ), and then it is Γ -invariant. Moreover it is plainly invariant with respect to the geodesic flow. It is also called Bowen–Margulis measure.

The Liouville measure $\tilde{\lambda}$ on $T^1\mathbb{H}$ can be expressed for any reference point z_0 , by:

$$d\tilde{\lambda}(u, v, s) = ch^{2d}(\text{dist}(z_0, uv)) d\mu_{z_0}^h(u) d\mu_{z_0}^h(v) ds,$$

where μ_z^h denotes the harmonic measure at z . Recall that we have in the half-space model: $d\mu_z^h(u) = p^d(z, u) du$.

Note that the above geometric property holds for harmonic measures, by changing δ into d : $d\mu_{z'}^h(u) = B_u^d(z, z') d\mu_z^h(u)$ for any z, z' in \mathbb{H} .

This and the identity (*) show the irrelevance of the reference point z_0 in the expression of the Liouville measure $\tilde{\lambda}$ above. As can be verified by a direct elementary computation, the expression of $\tilde{\lambda}$ in the (z, u) coordinates is: $d\tilde{\lambda}(z, u) = d\mu_z^h(u) d\tilde{V}(z)$.

$\tilde{\lambda}$ is naturally lifted to the Liouville measure λ' on $O\mathbb{H}$, by taking λ' uniform on each fibre SO_d . This Liouville measure λ' is the Haar measure on G , and thus it is invariant by the horocycle and geodesic flows.

We are interested in the quotient manifold $\mathcal{M} := \Gamma \backslash \mathbb{H}$.

It is known that when $\delta > d/2$, Φ is square-integrable with respect to the Riemannian volume measure dV of \mathcal{M} , and is the fundamental eigenstate on \mathcal{M} . See ([13], Theorem 1, p. 301).

Note that as a consequence, the volume of \mathcal{M} is finite if and only if $\delta = d$.

Denote by π the canonical projection from the unitary tangent bundle $T^1\mathcal{M} = \Gamma \backslash T^1\mathbb{H}$ onto \mathcal{M} , by π_1 the canonical projection from the orthonormal frame bundle $O\mathcal{M} = \Gamma \backslash O\mathbb{H}$ onto $T^1\mathcal{M}$, and by $\pi_2 = \pi \circ \pi_1$ the canonical projection from $O\mathcal{M}$ onto \mathcal{M} . By taking the left quotient by Γ , we see that the identifications at the level of \mathbb{H} give at the level of \mathcal{M} : $T^1\mathcal{M} \equiv O\mathcal{M}/SO_d$ and $\mathcal{M} \equiv O\mathcal{M}/SO_{d+1}$, and

$$(3) \quad \pi_2(\xi T_z) = \xi(z) \quad \text{for any } \xi \in O\mathcal{M} \text{ and } z \in \mathbb{H}.$$

In particular, we identify from now on the functions on $T^1\mathcal{M}$ with the SO_d -invariant functions on $O\mathcal{M}$.

Recall that any Γ -invariant measure \tilde{n} on $T^1\mathbb{H}$ induces a measure n on $T^1\mathcal{M}$.

In particular, denote by λ the Liouville measure induced by $\tilde{\lambda}$, and by m the Patterson–Sullivan measure induced by \tilde{m} . Similarly, denote by $dV = \pi^*\lambda$ the volume measure on \mathcal{M} , induced by $d\tilde{V}$.

Observe that since the flows act on the right-hand side, while Γ acts on the left-hand side, these two actions commute. Thus the geodesic flow makes sense on $T^1\mathcal{M}$ and on $O\mathcal{M}$, and the horocycle flow makes sense on $O\mathcal{M}$.

Let us introduce then the Lie derivatives: for any smooth function F on $O\mathcal{M}$, any ξ in $O\mathcal{M}$, $0 \leq i < j \leq d$ we set:

$$(4) \quad \mathcal{L}_0 F(\xi) := \frac{d_0}{dt} F(\xi \theta_t), \quad \mathcal{L}_j F(\xi) := \frac{d_0}{dt} F(\xi \theta_t^j), \quad \mathcal{L}_{i,j} F(\xi) := \frac{d_0}{dt} F(\xi R_t^{i,j}).$$

d_0/dt means and will mean the derivative at $t = 0$ with respect to t .

We get from the definitions that for $1 \leq i, j \leq d$ and $0 \leq k \leq d$:

$$(5) \quad \begin{aligned} [\mathcal{L}_0, \mathcal{L}_j] &= \mathcal{L}_j, & [\mathcal{L}_i, \mathcal{L}_j] &= 0, & [\mathcal{L}_k, \mathcal{L}_{i,j}] &= 1_{\{i=k\}}\mathcal{L}_j - 1_{\{j=k\}}\mathcal{L}_i, \\ [\mathcal{L}_0, \mathcal{L}_{0,j}] &= \mathcal{L}_j - \mathcal{L}_{0,j}, & [\mathcal{L}_i, \mathcal{L}_{0,j}] &= 1_{\{i \neq j\}}\mathcal{L}_{j,i} - 1_{\{i=j\}}\mathcal{L}_0. \end{aligned}$$

It also follows immediately from (3) that:

$$(6) \quad \mathcal{L}_0 F(\xi T_{(x,y)}) = y \frac{\partial}{\partial y} F(\xi T_{(x,y)}), \quad \mathcal{L}_j F(\xi T_{(x,y)}) = y \frac{\partial}{\partial x_j} F(\xi T_{(x,y)}).$$

3. An intrinsic measure on $T^1\mathcal{M}$

We introduce an intrinsic measure ν on $T^1\mathcal{M}$, which was already used in [2–4]. Its interest is to be smooth along the stable leaves and quasi-invariant under the geodesic and stable horocycle flows, and to be an invariant measure for two dual diffusions on $O\mathcal{M}$, which are both projected by π_2 onto the Φ -diffusion.

DEFINITION 1. – Let $\tilde{\nu}$ be the Γ -invariant measure on $T^1\mathbb{H}$ defined by:

$$d\tilde{\nu}(z, u) = \Phi(z) d\mu_z(u) d\tilde{V}(z).$$

Denote by ν the measure it induces on $T^1\mathcal{M}$.

Denote by ν' the unique measure on $O\mathcal{M}$ which has marginal ν on $T^1\mathcal{M}$ and whose conditional laws on the fibres are the normalized Haar measure on $SO_d \equiv O\mathcal{M}/T^1\mathcal{M}$.

Set also $dV^\Phi(z) := \Phi^2(z) dV(z)$.

Remark 1. – Observe that by definition of Φ we have $\pi_2^* \nu' = \pi^* \nu = V^\Phi$, and then that ν and ν' are finite (since $\delta > d/2$), with mass $\|\Phi\|_2^2$.

Observe also that, due to the geometric property, the Patterson measure μ_z , seen as a measure on $T_z^1\mathbb{H}$ by means of the coordinate system (z, u) , makes sense on $T_z^1\mathcal{M}$ as well. So in the preceding definition the second coordinate u can be seen as a unit tangent vector based at z , and the expression for $\tilde{\nu}$ (with \tilde{V} replaced by V) can then be understood directly as the expression for ν .

Remark 2. – In the finite volume case, we have $\delta = d$, Φ constant, $d\mu(u)$ is proportional to the uniform measure du , and then our measure ν is proportional to the Liouville measure λ (and to the Patterson–Sullivan measure m).

PROPOSITION 1. – The measure ν' is quasi-invariant under the geodesic and positive horocycle flows:

$$\frac{d(T_z^* \nu')}{d\nu'}(\xi) = y^{(d-\delta)} \times \frac{\Phi \circ \pi_2(\xi T_z^{-1})}{\Phi \circ \pi_2(\xi)} \quad \text{for any } \xi \in O\mathcal{M} \text{ and } z = (x, y) \in \mathbb{R}^d \times \mathbb{R}_+^*.$$

Note that this quasi-invariance property is what remains from the invariance of the Liouville measure λ' under the flows, in the finite volume case. The proof was already given in [4] (and in [3] in the case of surfaces).

The following theorem appeared already in [3] for $d = 1$. The same proof is valid, with only obvious modifications.

THEOREM 1. – *The measure ν can be expressed by a convolution of m along the stable horocycles of $T^1\mathcal{M}$. Precisely, we have:*

$$\nu = \int_{\mathbb{R}^d} (\theta_x^+)^* m(1 + |x|^2)^{-\delta} dx.$$

Note that the right-hand side above is well defined as a measure on $T^1\mathcal{M}$, although θ_x^+ does not act on $T^1\mathcal{M}$, since one integrates with respect to a SO_d -invariant measure. This right-hand side can also be thought with m replaced by its lift m' to $O\mathcal{M}$, uniform on each fibre SO_d .

Using Remark 1, we immediately deduce from Theorem 1:

COROLLARY 1. – *Let us set $c(\delta) := \int_{\mathbb{R}^d} (1 + |x|^2)^{-\delta} dx$.*

Then (for $\delta > d/2$) the mass of the measure m on $T^1\mathcal{M}$ equals $\|\Phi\|_2^2/c(\delta)$.

We also deduce, as in [3], the following approximation result on the measure m :

COROLLARY 2. – *The measure $\theta_S^* \nu$ converges as $S \rightarrow +\infty$, in the sense of the evaluation on each bounded function on $T^1\mathcal{M}$ which is continuous along the horocycles, towards the normalized Patterson–Sullivan measure $c(\delta)m$.*

4. Diffusions on \mathcal{M} and on $O\mathcal{M}$

This section is essentially taken from [4].

4.1. The diffusions Z_t^δ, ξ_t^δ and Z_t^Φ

Let (w_t, W_t) denote a Brownian motion on $\mathbb{R} \times \mathbb{R}^d$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Set

$$y_t := \exp[w_t + (\delta - d/2)t], \quad x_t := \int_0^t y_s dW_s, \quad Z_t^\delta := (x_t, y_t) \in \mathbb{H}.$$

For all δ , Z_t^δ is the diffusion on \mathbb{H} starting from $e_0 = (0, 1)$, with invariant measure $y^{2\delta-2d} dx dy$, and generator:

$$\frac{1}{2} \Delta^\delta := \frac{1}{2} \Delta + \delta y \frac{\partial}{\partial y} = \frac{y^2}{2} \left(\frac{\partial^2}{\partial y^2} + \frac{2\delta + 1 - d}{y} \times \frac{\partial}{\partial y} + \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} \right).$$

Similarly, denote by $Z_t^b = (x_t^b, y_t^b)$ the analogous process with δ replaced by $b \in [0, d]$. In particular, Z_t^0 is the Brownian motion on \mathbb{H} .

By formula (2), we see that $\pi_2(\xi T_{Z_t^0}) = \xi(Z_t^0)$ is a Brownian motion on \mathcal{M} , started from $\pi_2(\xi)$, for any $\xi \in O\mathcal{M}$. As a consequence, denoting by P_t the Brownian semi-group on \mathcal{M} , we have $\mathbb{E}(f \circ \pi_2(\xi T_{Z_t^0})) = P_t f(\pi_2(\xi))$.

Observe then that $T_{Z_t^0}$ is a right Brownian motion on a subgroup of the affine group of \mathbb{R}^d . Indeed, for any $b \in [0, d]$,

$$T_{Z_t^b}^{-1} T_{Z_{t+s}^b} = T_{\left(\frac{x_{t+s}^b - x_t^b}{y_t^b}, \frac{y_{t+s}^b}{y_t^b} \right)} = T_{Z_s^b} \circ \Theta_t$$

is independent of the sub- σ -field \mathcal{F}_t generated by the coordinates until time t .

DEFINITION 2. – For any $\xi \in \mathcal{OM}$, set $\xi_t^\delta := \xi T_{Z_t^\delta}$.

Set $\Delta^\Phi := \Phi^{-1} \Delta \circ \Phi - 2\lambda_0 = \Delta + 2(\nabla \log \Phi) \cdot \nabla$, and $P_t^\Phi := \exp(\frac{t}{2} \Delta^\Phi)$.

Denote by Z_t^Φ and call “ Φ -diffusion” the diffusion on \mathcal{M} with generator $\frac{1}{2} \Delta^\Phi$.

By the preceding observation, ξ_t^δ is a diffusion on \mathcal{OM} , starting from ξ .

From (5) we get $\Delta^\delta[F(\xi T_z)] = (D^\delta F)(\xi T_z)$, where

$$(7) \quad D^\delta := \sum_{j=0}^d \mathcal{L}_j^2 + (2\delta - d)\mathcal{L}_0 = D^0 + 2\delta\mathcal{L}_0.$$

Then the generator of the diffusion ξ_t^δ is $\frac{1}{2} D^\delta$.

Moreover, note that the Φ -diffusion is symmetrical and has invariant measure V^Φ and semi-group P_t^Φ . In the finite volume case $\delta = d$, this is just the Brownian motion.

Remark 3. – We have for any test-function F on \mathcal{OM} :

$$D^0(F \circ \pi_2)(\xi T.) = \Delta[F \circ \pi_2(\xi T.)] = \Delta(F \circ \xi) = (\Delta F) \circ \xi = (\Delta F) \circ \pi_2(\xi T.),$$

whence $D^0(F \circ \pi_2) = (\Delta F) \circ \pi_2$.

4.2. ν' as an invariant measure

We deduce now from the quasi-invariance property of ν' an adjonction property for ν' , and thus its invariance with respect to two dual diffusions on \mathcal{OM} .

The following results were already in [4]:

PROPOSITION 2. – We have for all δ and all test functions F, G on \mathcal{OM} :

$$\int (D^\delta F)G \, d\nu' = \int F(D^\Phi G) \, d\nu',$$

with $D^\Phi := \sum_{j=0}^d \mathcal{L}_j^2 - d\mathcal{L}_0 + 2 \sum_{j=0}^d (\mathcal{L}_j \log \Phi \circ \pi_2) \mathcal{L}_j = (\Phi \circ \pi_2)^{-1} D^0 \circ (\Phi \circ \pi_2) - 2\lambda_0$.

COROLLARY 3. – For $\xi \in \mathcal{OM}$ and each $\delta > d/2$ we have:

- (i) under \mathbb{P} , the diffusion ξ_t^δ admits the invariant measure ν' ;
- (ii) under $\nu' \otimes \mathbb{P}$, ξ_t^δ extends to a stationary diffusion defined for all real t , and ξ_{-t}^δ is the stationary diffusion associated with the infinitesimal generator $\frac{1}{2} D^\Phi$, say ξ_t^Φ .

Remark 4. – (i) We see from Remark 3 and from the h -process form of D^Φ in Proposition 2 above that we have for any test-function F on \mathcal{OM} : $D^\Phi(F \circ \pi_2) = (\Delta^\Phi F) \circ \pi_2$.

(ii) The h -process form of D^Φ shows that ξ_t^Φ can be defined by the following formula, where $0 = t_0 < t_1 < \dots < t_n$, F_0, \dots, F_n are test-functions on \mathcal{OM} , and $\xi_t^0 = \xi T_{Z_t^0}$:

$$\int \int \prod_{j=0}^n F_j(\xi_{t_j}^\Phi) \, d\nu'(\xi) \, d\mathbb{P} = \int \int \frac{e^{-\lambda_0 t_n}}{\Phi \circ \pi_2(\xi)} \times \Phi \circ \pi_2(\xi_{t_n}^0) \times \prod_{j=0}^n F_j(\xi_{t_j}^0) \, d\nu'(\xi) \, d\mathbb{P}.$$

Letting F_0 go to $1_{\{\xi\}}$, we get the law of ξ_t^Φ started from ξ . Then taking $F_j = f_j \circ \pi_2$ for $j \geq 1$, and using that $\pi_2(\xi_t^0) = \xi(Z_t^0)$ is a Brownian motion starting from $\xi(e_0) = \pi_2(\xi)$, and the h -process form of Δ^Φ in Definition 2, we deduce the following:

PROPOSITION 3. – Under \mathbb{P} , the projection $\pi_2(\xi_t^\Phi)$ of the diffusion ξ_t^Φ starting from ξ on OM is the Φ -diffusion starting from $\pi_2(\xi)$ on \mathcal{M} .

5. Existence of potentials along the stable foliation

Here we establish the existence of a kind of spectral gap for the degenerate foliated diffusion ξ_t^Φ , in order to ensure that enough regular functions possess a potential along its paths. This will be crucial for proving our main result.

Precisely, the aim of this section is to establish the following:

THEOREM 2. – Let us denote by Q_t^Φ the semi-group of the diffusion ξ_t^Φ .

For any Borelian bounded function F on $T^1\mathcal{M}$ which is rotationally Hölderian (this means: there exists some $r > 0$ such that $|F(\xi g) - F(\xi)|/d(g, SO_d)^r$ is bounded independently from $g \in SO_{d+1} - SO_d$ and $\xi \in T^1\mathcal{M}$), and such that $\int F d\nu = 0$, there exists some $\varrho > 0$ such that we have: $\|Q_t^\Phi F\|_{L^2(\nu)} \leq \varrho^{-1} e^{-\varrho t}$ for all $t \geq 0$.

By means of the coordinates (z, u) , we have for each $z \in \mathbb{H}$ an identification between $\partial\mathbb{H}$ and $T_z^1\mathbb{H}$. This allows to consider the Patterson measure μ_z as a measure on $T_z^1\mathcal{M}$, and then to localize the measure ν' as follows:

DEFINITION 3. – For each $\xi \in OM$ and each Borelian bounded function F on OM , set

$$\overline{F}(\pi_2(\xi)) := \int_{SO_{d+1}} F(\xi g) d\nu_\xi(g) := \Phi(\pi_2(\xi))^{-1} \times \int_{T_{\pi_2(\xi)}^1\mathcal{M}} F \circ \pi_1 d\mu_{\pi_2(\xi)}.$$

This defines a probability measure ν_ξ on SO_{d+1} and a function \overline{F} on \mathcal{M} .

Remark 5. – (i) The measure ν_ξ depends only on $\pi_1(\xi)$, and then its restriction to any fibre of SO_{d+1}/SO_d is uniform.

(ii) We have, for F Borelian and non-negative on $T^1\mathcal{M}$:

$$\int_{\mathcal{M}} \overline{F} dV^\Phi = \int_{\mathcal{M}} \int_{T_z^1\mathcal{M}} F \circ \pi_1 d\mu_z \Phi(z) dV(z) = \int_{T^1\mathcal{M}} F \circ \pi_1 d\nu = \int_{OM} F d\nu'.$$

Hence if $F \in L^2(T^1\mathcal{M}, \nu)$ and $\int F d\nu = 0$, then $\overline{F} \in L^2(\mathcal{M}, V^\Phi)$ and $\int \overline{F} dV^\Phi = 0$.

We shall use the following commutation relation in G :

LEMMA 1. – For all $g \in SO_{d+1}$ and $z \in \mathbb{R}^d \times \mathbb{R}_+^*$, there exists a unique $g_z \in SO_{d+1}$ such that $gT_z = T_{g(z)}g_z$.

Denoting by $u = u(g) := g\theta_\infty \in \mathbb{R}^d$ the extremity of the half-geodesic $g\theta_{\mathbb{R}_+}$ started from g , and setting $u' = g_z\theta_\infty$ and $g(z) = (x', y') \in \mathbb{R}^d \times \mathbb{R}_+^*$, we have $u' = (u - x')/y'$.

Proof. – Each element of G can be written $T_{z'}g'$, and in particular gT_z . This implies that $z' = \pi_2(T_{z'}g') = \pi_2(gT_z) = g(z)$, and $u = g\theta_\infty = gT_z\theta_\infty = T_{z'}g'\theta_\infty$.

Let us identify g to $(\phi, \sigma, r) \in [0, \pi] \times \mathbb{S}^{d-1} \times SO_d$, the point $(\phi, \sigma) \in \mathbb{S}^d$ being the image under g of the north pole of \mathbb{S}^d . An elementary computation shows that $u = \sigma \cot g(\phi/2)$, and similarly $u' = \sigma' \cot g(\phi'/2)$ if $g' \equiv (\phi', \sigma', r')$ and $u' = g'\theta_\infty$. It also shows $T_{z'}g'\theta_\infty = x' + y'\sigma' \cot g(\phi'/2)$. Whence $u = x' + y'u'$. \square

Remark 6. – If $g \in \text{SO}_d$, which is equivalent to $u(g)$ being ∞ , we have $g_z \in \text{SO}_d$ by Lemma 1, and then if $F = F \circ \pi_1$ we get $Q_t^\Phi F(\xi g) = \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \times \mathbb{E}[F \times \Phi \circ \pi_2(\xi T_g(Z_t^0))] = Q_t^\Phi F(\xi)$, which shows that Q_t^Φ acts on $T^1\mathcal{M}$.

By definition of $Q_t^\Phi F$ and of $\overline{Q_t^\Phi F}$, we have:

$$\begin{aligned} Q_t^\Phi F(\xi) &= \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \mathbb{E}[F \times \Phi \circ \pi_2(\xi_t^0)] \quad \text{and} \\ \overline{Q_t^\Phi F}(\pi_2(\xi)) &= \int \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi g)} \mathbb{E}[F \times \Phi \circ \pi_2(\xi g T_{Z_t^0})] dv_\xi(g) \\ &= \int \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \mathbb{E}[F \times \Phi \circ \pi_2(\xi T_g(Z_t^0) g_{Z_t^0})] dv_\xi(g) \\ &= \int \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \mathbb{E}[F \times \Phi \circ \pi_2(\xi T_{Z_t^0} g_{g^{-1}(Z_t^0)})] dv_\xi(g) \\ &= \int \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \mathbb{E}[\Phi \circ \pi_2(\xi_t^0) \times F(\xi_t^0 g_t')] dv_\xi(g), \end{aligned}$$

where $g_t' := g_{g^{-1}(Z_t^0)}$. Therefore we have:

$$Q_t^\Phi F(\xi) - \overline{Q_t^\Phi F}(\pi_2(\xi)) = \int \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \mathbb{E}[\Phi \circ \pi_2(\xi_t^0) \times (F(\xi_t^0) - F(\xi_t^0 g_t'))] dv_\xi(g).$$

Let us fix $0 < \varepsilon < 2\delta - d$, and set for $t \geq 0$ and $g \in \text{SO}_{d+1}$:

$$a_t^1 := 1_{\{y_t^0 > \exp[(\delta-d-\varepsilon)t/2]\}}, \quad a_t^2 := (1 - a_t^1) \times 1_{\{|x_t^0 - u(g)| \leq y_t^0 e^{\varepsilon t/6}\}}, \quad a_t^3 := 1 - a_t^1 - a_t^2,$$

and for $1 \leq j \leq 3$:

$$A_t^j(\xi) := \int \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \mathbb{E}[\Phi \circ \pi_2(\xi_t^0) \times (F(\xi_t^0) - F(\xi_t^0 g_t')) \times a_t^j] dv_\xi(g).$$

Of course we have $\|Q_t^\Phi F - \overline{Q_t^\Phi F} \circ \pi_2\|_{L^2(v)} \leq \sum_{j=1}^3 \|A_t^j\|_{L^2(v)}$.

Moreover, using that $Q_t^\Phi 1 = 1$, we have:

$$\|A_t^j\|_{L^2(v)}^2 \leq 4\|F\|_\infty^2 \times \int \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \mathbb{E}[\Phi \circ \pi_2(\xi_t^0) \times a_t^j] dv_\xi(g) dv(\xi).$$

Using Proposition 1, we get first:

$$\begin{aligned} \|A_t^1\|_{L^2(v)}^2 &\leq 4\|F\|_\infty^2 \times \mathbb{E}\left[a_t^1 \times e^{-\lambda_0 t} \int \frac{\Phi \circ \pi_2(\xi_t^0)}{\Phi \circ \pi_2(\xi)} dv(\xi)\right] \\ &= 4\|F\|_\infty^2 \times e^{-\lambda_0 t} \times \mathbb{E}[a_t^1 \times (y_t^0)^{d-\delta}] = 4\|F\|_\infty^2 \times e^{-(d-\delta)^2 t/2} \\ &\quad \times \int_{(\delta-\varepsilon)t/2}^\infty e^{(d-\delta)y - y^2/2t} dy / \sqrt{2\pi t} \end{aligned}$$

$$= 4\|F\|_\infty^2 \times \int_{(2\delta-d-\varepsilon)\sqrt{t}/2}^\infty e^{-y^2/2} dy/\sqrt{2\pi} = \mathcal{O}(\exp[-(2\delta-d-\varepsilon)^2t/8]).$$

Then for dealing with A_t^2 we need the following:

LEMMA 2. – We have for any $\xi \in \mathcal{OM}$ and any $z \in \mathbb{R}^d \times \mathbb{R}_+^*$:

$$\frac{\Phi \circ \pi_2(\xi T_z)}{\Phi \circ \pi_2(\xi)} = \int_{\text{SO}_{d+1}} p^\delta(z, u(g)) p^{-\delta}(e_0, u(g)) dv_\xi(g),$$

where $u(g) \in \mathbb{R}^d \cup \{\infty\}$ denotes the extremity $g\theta_\infty$ of the half-geodesic $g\theta_{\mathbb{R}_+}$.

(Let us recall that we have in $\mathbb{R}^d \times \mathbb{R}_+^*$: $\text{SO}_{d+1} \equiv T_{e_0}^1(\mathbb{R}^d \times \mathbb{R}_+^*)$, with $e_0 = (0, 1)$.)

Proof. – Let us choose the half-space model for \mathbb{H} , in such a way that ξ be the unit element of G . So that in particular $u(g) = \xi g\theta_\infty$, and then

$$\begin{aligned} \Phi \circ \pi_2(\xi) \int p^\delta(z, u(g)) p^{-\delta}(e_0, u(g)) dv_\xi(g) &= \int p^\delta(z, u) p^{-\delta}(e_0, u) d\mu_{e_0}(u) \\ &= \int d\mu_z = \Phi(z). \quad \square \end{aligned}$$

This lemma implies the following corollary, analogous to the celebrated Sullivan’s “shadow lemma”.

COROLLARY 4. – We have for any $\xi \in \mathcal{OM}$, $a \geq 1$, and $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}_+^*$:

$$v_\xi(\{g \in \text{SO}_{d+1} \mid |u(g) - x| \leq ay\}) \leq \frac{\Phi \circ \pi_2(\xi T_z)}{\Phi \circ \pi_2(\xi)} \times (2y)^\delta \times a^{2\delta}.$$

Proof. – We merely use Lemma 2, observing that:

$$\begin{aligned} p^\delta(z, u) p^{-\delta}(e_0, u) &= y^{-\delta} (1 + |x - u|^2 y^{-2})^{-\delta} (1 + |u|^2)^\delta \geq 2^{-\delta} y^{-\delta} a^{-2\delta} \quad \text{on} \\ &\{ |u - x| \leq ay \}. \quad \square \end{aligned}$$

Applying this to $\|A_t^2\|_2^2$, we obtain:

$$\begin{aligned} \|A_t^2\|_2^2 &\leq 4\|F\|_\infty^2 \int \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \mathbb{E}[\Phi \circ \pi_2(\xi_t^0) \\ &\quad \times (1 - a_t^1) \times v_\xi(\{g \mid |u(g) - x_t^0| \leq y_t^0 e^{\varepsilon t/6}\})] dv(\xi) \\ &\leq 2^{2+\delta} \|F\|_\infty^2 \times e^{-\lambda_0 t} \int \mathbb{E} \left[\frac{\Phi^2 \circ \pi_2(\xi_t^0)}{\Phi^2 \circ \pi_2(\xi)} \times (y_t^0)^\delta e^{\varepsilon \delta t/3} \times 1_{\{y_t^0 \leq e^{(\delta-d-\varepsilon)t/2}\}} \right] dv(\xi) \\ &\leq 2^{2+\delta} \|F\|_\infty^2 \times e^{-\varepsilon \delta t/6} \int \mathbb{E} \left[\frac{\Phi^2 \circ \xi(Z_t^0)}{\Phi^2 \circ \xi(Z_0^0)} \right] dv(\xi) \\ &= 2^{2+\delta} \|F\|_\infty^2 \times e^{-\varepsilon \delta t/6} \int \mathbb{E}_{\pi_2(\xi)} \left[\frac{\Phi^2(Z_t^0)}{\Phi^2(Z_0^0)} \right] dv(\xi) \\ &= 2^{2+\delta} \|F\|_\infty^2 \times e^{-\varepsilon \delta t/6} \int P_t \Phi^2 dV = 2^{2+\delta} \|F\|_\infty^2 \times e^{-\varepsilon \delta t/6} \times \|\Phi\|_2^2 = \mathcal{O}(e^{-\varepsilon \delta t/6}). \end{aligned}$$

Finally, using that $F = F \circ \pi_1$, and that $u(g_t^0) = (u(g) - x_t^0)/y_t^0$, we get:

$$\begin{aligned} |A_t^3|^2(\xi) &\leq \int \frac{e^{-\lambda_0 t}}{\Phi \circ \pi_2(\xi)} \mathbb{E}[\Phi \circ \pi_2(\xi_t^0) \times |F(\xi_t^0) - F(\xi_t^0 g_t')|^2 1_{\{|u(g_t')| > e^{\epsilon t/6}\}}] d\nu_\xi(g) \\ &\leq \mathbb{E} \left[\sup_{g \in \text{SO}_{d+1}} |F(\xi_t^\Phi) - F(\xi_t^\Phi g)|^2 \times 1_{\{d(g, \text{SO}_d) = \mathcal{O}(e^{-\epsilon t/6})\}} \right] = \mathcal{O}(e^{-\epsilon t/3}), \end{aligned}$$

by the Hölderian hypothesis made on the function F (in Theorem 2).

So far, we just proved the following:

PROPOSITION 4. – *For any bounded measurable function F on $T^1\mathcal{M}$ which is rotationally Hölderian, there exists some $\varrho > 0$ such that we have:*

$$\|Q_t^\Phi F - \overline{Q_t^\Phi F} \circ \pi_2\|_{L^2(\nu)} \leq \varrho^{-1} e^{-\varrho t} \quad \text{for all } t \geq 0.$$

We now complete the proof of Theorem 2.

On one hand we have

$$\begin{aligned} \|Q_t^\Phi F\|_2 &\leq \|Q_{t/2}^\Phi(Q_{t/2}^\Phi F - \overline{Q_{t/2}^\Phi F} \circ \pi_2)\|_2 + \|Q_{t/2}^\Phi(\overline{Q_{t/2}^\Phi F} \circ \pi_2)\|_2 \\ &\leq \|Q_{t/2}^\Phi F - \overline{Q_{t/2}^\Phi F} \circ \pi_2\|_2 + \|P_{t/2}^\Phi(\overline{Q_{t/2}^\Phi F} \circ \pi_2)\|_{L^2(\nu^\Phi)} \\ &= \mathcal{O}(e^{-\varrho t}) + \|P_{t/2}^\Phi(\overline{Q_{t/2}^\Phi F} \circ \pi_2)\|_{L^2(\nu^\Phi)}, \end{aligned}$$

by Propositions 3 and 4, and on the other hand we have:

$$\int \overline{Q_{t/2}^\Phi F} \circ \pi_2 dV^\Phi = \int \overline{Q_{t/2}^\Phi F} d\nu = \int Q_{t/2}^\Phi F d\nu' = \int F d\nu' = \int F d\nu = 0,$$

whence by the spectral gap property of \mathcal{M} (see [12] and [7]), there exists $\eta > 0$ such that for any $t \geq 0$

$$\begin{aligned} \|P_{t/2}^\Phi(\overline{Q_{t/2}^\Phi F} \circ \pi_2)\|_{L^2(\nu^\Phi)} &\leq e^{-\eta t} \|\overline{Q_{t/2}^\Phi F} \circ \pi_2\|_{L^2(\nu^\Phi)} = e^{-\eta t} \|Q_{t/2}^\Phi F\|_{L^2(\nu)} \\ &\leq e^{-\eta t} \|Q_{t/2}^\Phi F^2\|_{L^1(\nu)}^{1/2} = e^{-\eta t} \|F\|_{L^2(\nu)}. \end{aligned}$$

6. From geodesic flow to stochastic flow

Let us consider a bounded Borelian function f on $T^1\mathcal{M}$, that is to say a SO_d -invariant function on \mathcal{OM} , such that $\int f dm = 0$ and such that $\mathcal{L}_j f$ and $\mathcal{L}_j^2 f$ are bounded for $1 \leq j \leq d$.

6.1. The conjugate functions

We construct here functions f_j on \mathcal{OM} which are conjugate to the function f , in order to get on \mathcal{OM} a 1-form ω which has f as first coordinate and has a closed restriction to each stable leaf. This will be crucial to replace geodesics by diffusion paths.

DEFINITION 4. –

- (i) Denote by $\{\mathcal{L}'_j \mid 0 \leq j \leq d\}$ the dual basis (in $\Lambda^1(\mathcal{OM})$) of $\{\mathcal{L}_j \mid 0 \leq j \leq d\}$.
- (ii) Set $\mathcal{U}^q \phi(\xi) := \int_0^\infty e^{-qs} \phi(\xi \theta_s) ds$, for any $q \in \mathbb{N}^*$, $\xi \in \mathcal{OM}$, and ϕ bounded measurable on \mathcal{OM} .
- (iii) For $1 \leq j \leq d$, set $f_j := -\mathcal{U}^1 \mathcal{L}_j f$, and set also (for convenience) $f_0 := f$.
- (iv) Set $\omega := \sum_{j=0}^d f_j \mathcal{L}'_j$. (This is a bounded 1-form on \mathcal{OM} .)

(v) For each $\xi \in \mathcal{OM}$, let $\tilde{\xi}$ denote the map from \mathbb{H} into \mathcal{OM} defined by $\tilde{\xi}(z) := \xi T_z$, and let $\omega_{\tilde{\xi}} := \tilde{\xi}^* \omega$ denote the pull-back of ω by $\tilde{\xi}$.

LEMMA 3. – The 1-form ω_{ξ} is closed and bounded on \mathbb{H} , for each $\xi \in \mathcal{OM}$. Moreover the $\mathcal{L}_i f_j$ exist and are bounded on \mathcal{OM} for $0 \leq i \leq d$ and $1 \leq j \leq d$, and we have:

$$\omega_{\tilde{\xi}}(z) = y^{-1} f(\xi T_z) dy + y^{-1} \sum_{j=1}^d f_j(\xi T_z) dx^j.$$

Proof. – By (5) we have $y \frac{\partial}{\partial y} (f \circ \tilde{\xi}) = (\mathcal{L}_0 f) \circ \tilde{\xi}$, and then $y \frac{\partial}{\partial y} \circ \tilde{\xi}^* = \tilde{\xi}^* \circ \mathcal{L}_0$, whence by duality $y^{-1} dy = \tilde{\xi}^* \mathcal{L}'_0$, and then $(f \circ \tilde{\xi}) y^{-1} dy = \tilde{\xi}^* (f \mathcal{L}'_0)$. Since the same works with $(f_j, dx^j, \mathcal{L}_j)$, we get indeed the right expression for $\omega_{\tilde{\xi}}$.

Then the commutation relation (1) between the flows implies that:

$$\mathcal{L}_j \mathcal{U}^q \phi(\xi) = \frac{d_0}{dt} \int_0^\infty e^{-qs} \phi(\xi \theta_t^j \theta_s) ds = \int_0^\infty e^{-qs} \frac{d_0}{dt} \phi(\xi \theta_s \theta_{te^{-s}}^j) ds = \mathcal{U}^{q+1} \mathcal{L}_j \phi(\xi)$$

for $1 \leq j \leq d$, and that:

$$\mathcal{L}_0 \mathcal{U}^q \phi(\xi) = \int_0^\infty e^{-qs} \frac{d}{ds} \phi(\xi \theta_s) ds = q \mathcal{U}^q \phi(\xi) - \phi(\xi).$$

This implies existence and boundedness of the $\mathcal{L}_i f_k$, by taking $\phi = -\mathcal{L}_k f$, and also that $\mathcal{L}_j f_k - \mathcal{L}_k f_j = \mathcal{U}^2 [\mathcal{L}_k, \mathcal{L}_j] f = 0$, and that $\mathcal{L}_0 f_j - \mathcal{L}_j f - f_j = 0$.

Therefore, using (5) again, we have finally:

$$\begin{aligned} y^2 d\omega_{\xi} &= \sum_{j=1}^d (\mathcal{L}_0 f_j - \mathcal{L}_j f - f_j)(\xi T_z) dy \wedge dx^j \\ &+ \sum_{1 \leq j < k \leq d} (\mathcal{L}_j f_k - \mathcal{L}_k f_j)(\xi T_z) dx^j \wedge dx^k = 0. \quad \square \end{aligned}$$

6.2. A contour deformation

We take here advantage of the closedness of the form ω_{ξ} to change the integration path in $\int_0^t f(\xi \theta_s) ds$: we substitute the diffusion path $\{\xi_s^\Phi \mid 0 \leq s \leq (\delta - d/2)^{-1} t\}$ for the geodesic $\xi[0, t] := \{\xi \theta_s \mid 0 \leq s \leq t\}$. In this contour deformation three residual terms appear, that we prove to be negligible.

PROPOSITION 5. – For any real a , the following difference goes to 0 as $t \rightarrow \infty$:

$$c(\delta) \int_{T^1 \mathcal{M}} \exp\left(\frac{a\sqrt{-1}}{\sqrt{t}} \int_0^t f(\xi \theta_s) ds\right) dm(\xi) - \mathbb{E} \left[\int \exp\left(\frac{a\sqrt{-1}}{\sqrt{(\delta - d/2)t}} \int_{Z^\delta[-t, 0]} \omega_{\xi}\right) d\nu(\xi) \right].$$

Proof. – Let us set $t' := (\delta - d/2)t$. Using Corollary 2, we have almost surely:

$$\begin{aligned}
 c(\delta) \int \exp\left(\frac{\sqrt{-1}}{\sqrt{t'}} \int_0^{t'} f(\xi\theta_s) ds\right) dm(\xi) &= \lim_{S \rightarrow \infty} \int \exp\left(\frac{\sqrt{-1}}{\sqrt{t'}} \int_{\log y_S^\delta}^{t'+\log y_S^\delta} f(\xi\theta_s) ds\right) d\nu(\xi) \\
 &= \lim_{S \rightarrow \infty} \int \exp\left(\frac{\sqrt{-1}}{\sqrt{t'}} \int_{(0, y_S^\delta)}^{(0, y_S^\delta e^{t'})} \omega_\xi\right) d\nu(\xi).
 \end{aligned}$$

Then, using Lemma 3, we get:

$$\int_{(0, y_S^\delta)}^{(0, y_S^\delta e^{t'})} \omega_\xi = \int_{Z_S^\delta}^{Z_{S+t}^\delta} \omega_\xi + \int_{(0, y_S^\delta)}^{Z_S^\delta} \omega_\xi - \int_{(0, y_{S+t}^\delta)}^{Z_{S+t}^\delta} \omega_\xi - \int_{(0, y_S^\delta e^{t'})}^{(0, y_{S+t}^\delta)} \omega_\xi,$$

and since f_j is bounded:

$$\begin{aligned}
 \int_{(0, y_u^\delta)}^{Z_u^\delta} \omega_\xi &= (y_u^\delta)^{-1} \sum_{j=1}^d \int_0^1 (x_u^\delta)^j \times f_j(\xi\theta_{s(x_u^\delta)^j} \theta_{\log y_u^\delta}) ds = \mathcal{O}\left(\left|\frac{x_u^\delta}{y_u^\delta}\right|\right) \\
 &\stackrel{\text{law}}{=} |W_1| \times \mathcal{O}\left(\int_0^u (y_s^\delta)^2 (y_u^\delta)^{-2} ds\right)^{1/2} \stackrel{\text{law}}{=} |W_1| \times \mathcal{O}\left(\int_0^u e^{2w_s - (2\delta-d)s} ds\right)^{1/2},
 \end{aligned}$$

which is almost surely bounded. Hence, uniformly with respect to $S \geq 0$ and to $\xi \in T^1\mathcal{M}$, $t^{-1/2} \int_{(0, y_S^\delta)}^{Z_S^\delta} \omega_\xi$ and $t^{-1/2} \int_{(0, y_{S+t}^\delta)}^{Z_{S+t}^\delta} \omega_\xi$ go to zero in probability as $t \rightarrow \infty$.

Then we have:

$$\int_{(0, y_S^\delta e^{t'})}^{(0, y_{S+t}^\delta)} \omega_\xi = \int_{w_S + (\delta-d/2)(S+t)}^{w_{S+t} + (\delta-d/2)(S+t)} f(\xi\theta_s) ds = \int_0^{w_{S+t} - w_S} f(\xi\theta_{s+w_S + (\delta-d/2)(S+t)}) ds,$$

and thus for some standard Brownian motion w' independent of w we have:

$$\mathbb{E} \left[\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t}} \int_{(0, y_S^\delta e^{t'})}^{(0, y_{S+t}^\delta)} \omega_\xi\right) d\nu(\xi) \right] = \mathbb{E} \left[\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t}} \int_0^{w'_t} f(\xi\theta_{s+w_S + (\delta-d/2)(S+t)}) ds\right) d\nu(\xi) \right]$$

which, thanks to Corollary 2, goes as $S \rightarrow \infty$ to

$$c(\delta) \mathbb{E} \left[\int \exp\left(\frac{\sqrt{-1}}{\sqrt{t}} \int_0^{w'_t \sqrt{t}} f(\xi\theta_s) ds\right) dm(\xi) \right] \xrightarrow{t \rightarrow \infty} c(\delta) \mathbb{E} \left[\int \exp\left(\sqrt{-1} w'_1 \int f dm\right) dm \right]$$

by ergodicity, the last quantity being equal to $\|\Phi\|_2^2$, since $\int f dm = 0$. Finally

$$\begin{aligned} & \lim_{t \rightarrow \infty} c(\delta) \mathbb{E} \left[\int \exp \left(\frac{\sqrt{-1}}{\sqrt{t}} \int_0^t f(\xi \theta_s) ds \right) dm(\xi) \right] \\ &= \lim_{t \rightarrow \infty} \lim_{S \rightarrow \infty} \mathbb{E} \left[\int \exp \left(\frac{\sqrt{-1}}{\sqrt{t'}} \int_{Z_S^\delta}^{Z_{S+t}^\delta} \omega_\xi \right) d\nu(\xi) \right] \\ &= \lim_{t \rightarrow \infty} \mathbb{E} \left[\int \exp \left(\frac{\sqrt{-1}}{\sqrt{t'}} \int_{Z_{-t}^\delta}^{Z_0^\delta} \omega_\xi \right) d\nu(\xi) \right] \end{aligned}$$

by Corollary 3 . \square

Now observe that by Definition 4 we have:

$$\int_{Z^\delta[-t,0]} \omega_\xi = \int_{\tilde{\xi}(Z^\delta[-t,0])} \omega = \int_{\xi^\delta[-t,0]} \omega.$$

Hence by reversing the time we deduce from Corollary 3 and Proposition 5 the following:

COROLLARY 5. – We have for any real a ,

$$\lim_{t \rightarrow \infty} \left\{ c(\delta) \int_{T^1\mathcal{M}} \exp \left(\frac{a\sqrt{-1}}{\sqrt{t}} \int_0^t f(\xi \theta_s) ds \right) dm(\xi) - \nu \otimes \mathbb{E} \left[\exp \left(\frac{-a\sqrt{-1}}{\sqrt{(\delta - d/2)t}} \int_{\xi^\Phi[0,t]} \omega \right) \right] \right\} = 0.$$

7. The central limit theorem on $T^1\mathcal{M}$

Let us fix a Borelian function f of class C^2 on $T^1\mathcal{M}$, with bounded and Hölderian derivatives, and such that $\int f dm = 0$.

Remark 7. – A careful reading of our arguments below shows that indeed the following slightly weaker regularity hypothesis is sufficient to guarantee our result:

- (H) f and $\mathcal{L}_0 f$ are bounded, rotationally Hölderian on $T^1\mathcal{M}$, and continuous along the stable leaves, and, for $1 \leq j \leq d$, $\mathcal{L}_j f$ and $\mathcal{L}_j^2 f$ are bounded and Hölderian on OM .

Recall that “ F rotationally Hölderian on $T^1\mathcal{M}$ ” means: there exists some $r > 0$ such that $d(g, SO_d)^{-r} |F(\xi g) - F(\xi)|$ is bounded independently from $g \in SO_{d+1} - SO_d$ and $\xi \in T^1\mathcal{M}$, d denoting here any distance on SO_{d+1} , and that “ F Hölderian on OM ” precisely means: there exists some $r > 0$ such that $\text{dist}(\xi, \xi')^{-r} |F(\xi) - F(\xi')|$ is bounded on $\{(\xi, \xi') \in OM^2 \mid 0 < \text{dist}(\xi, \xi') < 1\}$.

7.1. The divergence of ω with respect to ξ_t^Φ

LEMMA 4. – If ϕ is a bounded Hölderian function on OM , then $\mathcal{U}^1\phi$ and $\mathcal{U}^2\phi$ are bounded and Hölderian on OM .

Proof. – Let us consider $\varphi_s := \exp[s \sum_{j=1}^d a_j \mathcal{L}_{0,j}]$, for $s > 0$ and for fixed real a_j 's. Fix an Hölder exponent for ϕ , say $r \in]0, 1[$. We have for $q = 1$ or 2 :

$$s^{-r} |\mathcal{U}^q \phi(\xi \varphi_s) - \mathcal{U}^q \phi(\xi)| \leq \int_0^\infty e^{-qt} s^{-r} |\phi(\xi \varphi_s \theta_t) - \phi(\xi \theta_t)| dt,$$

which is bounded with respect to (s, ξ) if $\int_0^\infty e^{(r-q)t} s^{-r} |\phi(\xi \varphi_{se^{-t}} \theta_t) - \phi(\xi \theta_t)| dt$ is, and thus if $s^{-r} |\phi(\xi \varphi_{se^{-t}} \theta_t) - \phi(\xi \theta_t)|$ is bounded with respect to (s, t, ξ) .

Now this will be so if we prove that $D(\theta_{-t} \varphi_{se^{-t}} \theta_t, \text{Id}) = \mathcal{O}(s)$, uniformly with respect to t , D denoting some metric on G , which we can choose left invariant.

Then observe that it is sufficient to consider the case of $\varphi_s = \exp[s \mathcal{L}_{0,j}]$.

Now we have the Campbell–Hausdorff formula: $\frac{d}{ds}(\theta_{-t} \exp(s \mathcal{L}_{0,j}) \theta_t) = \exp[\text{ad}(-t \mathcal{L}_0)] \mathcal{L}_{0,j}$, and (4) gives the matrix of $\text{ad}(\mathcal{L}_0)$ in the base $(\mathcal{L}_{0,j}, \mathcal{L}_j)$, which has its square equal to the unit matrix. So we see that $\exp[\text{ad}(-t \mathcal{L}_0)] \mathcal{L}_{0,j} = e^t \mathcal{L}_{0,j} - (\text{sh } t) \mathcal{L}_j$, and thus that

$$\frac{d}{ds}(\theta_{-t} \varphi_{se^{-t}} \theta_t) = \theta_{-t} \varphi_{se^{-t}} \theta_t \times \frac{d}{ds}(\theta_{-t} \varphi_{se^{-t}} \theta_t) = \theta_{-t} \varphi_{se^{-t}} \theta_t \times \left(\mathcal{L}_{0,j} - \left(\frac{1 - e^{-2t}}{2} \right) \mathcal{L}_j \right),$$

which by left invariance of D shows the boundedness of $\frac{d}{ds} D(\theta_{-t} \varphi_{se^{-t}} \theta_t, \text{Id})$. This shows that $\mathcal{U}^q \phi$ is rotationally Hölderian on \mathcal{OM} . Finally, observe that we get from (4):

$$\frac{d}{ds}(\theta_{-t} \exp(s \mathcal{L}_j) \theta_t) = e^{-t} \mathcal{L}_j$$

if $1 \leq j \leq d$ (and \mathcal{L}_0 if $j = 0$). Thus we get also the boundedness of $\frac{d}{ds} D(\theta_{-t} \exp(se^{-t} \mathcal{L}_j) \theta_t, \text{Id})$, and of $\frac{d}{ds} D(\theta_{-t} \exp(se^{-t} \mathcal{L}_{i,k}) \theta_t, \text{Id})$, for $1 \leq i < k \leq d$, since $[\mathcal{L}_0, \mathcal{L}_{i,k}] = 0$. The result follows. \square

Recall that the diffusion ξ_t^Φ admits the generator $\frac{1}{2} D^\Phi = \frac{1}{2} \sum_{j=0}^d \mathcal{L}_j^2 + \sum_{j=0}^d h_j \mathcal{L}_j$, where $h_j := \mathcal{L}_j(\log \Phi \circ \pi_2) - \frac{d}{2} 1_{\{j=0\}}$, for $0 \leq j \leq d$.

Now we have the following general lemma, more or less known:

LEMMA 5. – Consider a 1-form Ω of class C^1 on \mathcal{OM} . We have for any $t \geq 0$:

$$\int_{\xi^\Phi_{[0,t]}} \Omega = M_t^\Omega + \int_0^t \text{div } \Omega(\xi_s^\Phi) ds, \quad \text{with} \quad \text{div } \Omega = \frac{1}{2} \sum_{j=0}^d \mathcal{L}_j(\Omega(\mathcal{L}_j)) + \sum_{j=0}^d h_j \Omega(\mathcal{L}_j);$$

M_t^Ω is a continuous martingale having increasing process:

$$\langle M_t^\Omega \rangle = \int_0^t \sum_{j=0}^d \Omega(\mathcal{L}_j)^2(\xi_s^\Phi) ds.$$

Proof. – By linearity, it is sufficient to consider $\Omega = G dF$, for F of class C^1 and G of class C^2 on OM . Now we have by Itô formula:

$$dF(\xi_s^\Phi) = dM_s^F + \frac{1}{2} D^\Phi F(\xi_s^\Phi) ds, \quad \text{with} \quad d\langle M_s^F \rangle = \sum_{j=0}^d (\mathcal{L}_j F)^2(\xi_s^\Phi) ds$$

and then

$$\int_{\xi^\Phi[0,t]} \Omega = \int_0^t G(\xi_s^\Phi) \circ dF(\xi_s^\Phi) = \int_0^t G(\xi_s^\Phi) dM_s^F + \int_0^t \frac{G \times D^\Phi F}{2}(\xi_s^\Phi) ds + \frac{1}{2} \langle M_t^G, M_t^F \rangle.$$

Therefore we get the formula of the statement, with:

$$2 \operatorname{div} \Omega = G \times D^\Phi F + \sum_{j=0}^d \mathcal{L}_j G \times \mathcal{L}_j F \quad \text{and} \quad d\langle M_s^\Omega \rangle = \sum_{j=0}^d (G \times \mathcal{L}_j F)^2(\xi_s^\Phi) ds.$$

This gives the wanted formula, since $\Omega = G dF$ and $D^\Phi = \sum_{j=0}^d \mathcal{L}_j^2 + 2 \sum_{j=0}^d h_j \mathcal{L}_j$ imply:

$$\begin{aligned} \Omega(\mathcal{L}_j) &= G \mathcal{L}_j F \quad \text{and} \\ \sum_{j=0}^d \mathcal{L}_j(\Omega(\mathcal{L}_j)) &= G \sum_{j=0}^d \mathcal{L}_j^2 F + \sum_{j=0}^d \mathcal{L}_j G \mathcal{L}_j F = 2 \operatorname{div} \Omega - 2 \sum_{j=0}^d h_j \Omega(\mathcal{L}_j). \quad \square \end{aligned}$$

Let us apply this Lemma 5 to ω (see Definition 4(iv)). Observe that this is licit by Lemma 4. We get

$$(F) \quad \int_{\xi^\Phi[0,t]} \omega = M_t^\omega + \int_0^t Kf(\xi_s^\Phi) ds,$$

with

$$(F') \quad Kf := \frac{1}{2} \sum_{j=0}^d \mathcal{L}_j f_j + \sum_{j=0}^d (\mathcal{L}_j \log \Phi \circ \pi_2) f_j - \frac{d}{2} f = \left(\frac{d}{2} - \delta\right) f - \frac{1}{2} \sum_{j=0}^d \mathcal{L}_j f_j - \sum_{j=0}^d \mathcal{L}_j^* f_j,$$

where $\mathcal{L}_j^* := -\mathcal{L}_j - \mathcal{L}_j(\log \Phi \circ \pi_2) + 1_{\{j=0\}}(d - \delta)$ is the adjoint of \mathcal{L}_j with respect to ν (this is the infinitesimal version of Proposition 1), and where M_t^ω is a continuous martingale having its increasing process given by

$$(F'') \quad \langle M_t^\omega \rangle = \int_0^t \sum_{j=0}^d f_j^2(\xi_s^\Phi) ds.$$

LEMMA 6. – *The function Kf is SO_d -invariant, bounded, and Hölderian.*

Proof. – We already observed in Lemma 3 that the f_j and the $\mathcal{L}_j f_j$ are bounded. Moreover we have also that the $\mathcal{L}_k \log \Phi \circ \pi_2$ and $\mathcal{L}_{0,j} \mathcal{L}_k \log \Phi \circ \pi_2$ are bounded, showing the boundedness of Kf . Indeed, by (4) and since $\mathcal{L}_{0,j} \log \Phi \circ \pi_2 = 0$, it is sufficient to verify the first assertion.

Now it is straightforward to see that $|y \frac{\partial}{\partial y} \log p(z, v)| \leq 1$ and $|y \frac{\partial}{\partial x^j} \log p(z, v)| \leq 1$, which in turn implies that $|\mathcal{L}_j \log \Phi \circ \pi_2| \leq \delta$, using that by (5)

$$\mathcal{L}_j \log \Phi \circ \pi_2(\xi) = \frac{\partial_0}{\partial x^j} \int \frac{p^\delta(\xi((x, 1)), u)}{p^\delta(e_0, u)} d\mu_{e_0}(u) = \frac{\partial_0}{\partial x^j} \int \frac{p^\delta((x, 1), \xi^{-1}(u))}{p^\delta(\xi^{-1}(e_0), \xi^{-1}(u))} d\mu_{e_0}(u).$$

Then the SO_d -invariance follows from the observation that ξ_t^Φ is SO_d -invariant, and that the form ω is equivariant with respect to the SO_d -action on \mathcal{OM} .

It remains to show that Kf is Hölderian. Now, by the beginning of this proof, we only have to ensure that $\mathcal{U}^2 \mathcal{L}_j^2 f$ and $\mathcal{U}^1 \mathcal{L}_j f$ are Hölderian. But this follows immediately from Lemma 4. \square

PROPOSITION 6. – We have $\int Kf dv = 0$.

Proof. – By the very definition (4(iii)) of f_j , we have:

$$\begin{aligned} f_j(\xi) &= - \lim_{S \rightarrow \infty} \int_0^S e^{-s} \mathcal{L}_j f(\xi \theta_s) ds = - \lim_{S \rightarrow \infty} \int_0^S e^{-s} \frac{d_0}{dt} f(\xi \theta_s \theta_t^j) ds \\ &= - \lim_{S \rightarrow \infty} \int_0^S e^{-s} \frac{d_0}{dt} f(\xi \theta_{te^j} \theta_s) ds = - \lim_{S \rightarrow \infty} \int_0^S \frac{d_0}{dt} f(\xi \theta_t^j \theta_s) ds \\ &= - \lim_{S \rightarrow \infty} \mathcal{L}_j \left(\int_0^S f(\cdot \theta_s) ds \right) (\xi). \end{aligned}$$

Whence by formula (6) (defining D^δ , in Section 4.2)

$$\begin{aligned} \sum_{j=1}^d \mathcal{L}_j f_j &= - \lim_{S \rightarrow \infty} \sum_{j=1}^d \mathcal{L}_j^2 \left(\int_0^S f(\cdot \theta_s) ds \right) = \lim_{S \rightarrow \infty} [-D^\delta + \mathcal{L}_0^2 + (2\delta - d)\mathcal{L}_0] \left(\int_0^S f(\cdot \theta_s) ds \right) \\ &= \lim_{S \rightarrow \infty} \left(-D^\delta \left(\int_0^S f(\cdot \theta_s) ds \right) + \int_0^S [\mathcal{L}_0^2 + (2\delta - d)\mathcal{L}_0] f(\cdot \theta_s) ds \right) \\ &= \lim_{S \rightarrow \infty} \left(-D^\delta \left(\int_0^S f(\cdot \theta_s) ds \right) + \mathcal{L}_0 f(\cdot \theta_s) - \mathcal{L}_0 f + (2\delta - d)f(\cdot \theta_s) - (2\delta - d)f \right). \end{aligned}$$

Therefore:

$$\left(\frac{d}{2} - \delta \right) f - \frac{1}{2} \sum_{j=0}^d \mathcal{L}_j f_j = \lim_{S \rightarrow \infty} \left(\frac{1}{2} D^\delta \left(\int_0^S f(\cdot \theta_s) ds \right) - \frac{1}{2} \mathcal{L}_0 f(\cdot \theta_s) + \left(\frac{d}{2} - \delta \right) f(\cdot \theta_s) \right).$$

Finally, using the duality with respect to ν , that is to say Proposition 2, the fact that $D^\Phi 1 = \mathcal{L}_j 1 = 0$, Corollary 2, and the hypothesis (H) on f , we deduce that:

$$\int Kf dv = \lim_{S \rightarrow \infty} \left(\left(\frac{d}{2} - \delta \right) \int f d\theta_S^* \nu - \frac{1}{2} \int \mathcal{L}_0 f d\theta_S^* \nu \right)$$

$$= \left(\frac{d}{2} - \delta\right) \int f \, dm - \frac{1}{2} \int \mathcal{L}_0 f \, dm = 0. \quad \square$$

7.2. End of proof of the main result (the theorem in Section 1)

Let us fix a sequence $\{F_n \mid n \in \mathbb{N}\}$ of compactly supported and smooth functions on $T^1\mathcal{M}$, such that:

- (i) F_n converges in $L^2(\nu)$ towards Kf as $n \rightarrow \infty$;
- (ii) each F_n is Hölderian, with the same constants as Kf ;
- (iii) the F_n are uniformly bounded and have zero mean with respect to ν .

Note that this is possible by using Lemma 6 and Proposition 6, and by using a partition of unity and some convolution. Observe then that by Theorem 2 and its proof (see Section 5), there exists some $\varrho > 0$ such that

$$(7) \quad \|Q_t^\Phi F\|_2 \leq \varrho^{-1} e^{-\varrho t} \quad \text{for } t \geq 0 \text{ and } F = (Kf \text{ or any } F_n).$$

DEFINITION 5. – Set $V_b F := \int_0^b Q_t^\Phi F \, dt$, for $0 < b < \infty$ and $F = (Kf \text{ or any } F_n)$, and write $V F$ for $V_\infty F$.

Formula (7) shows that the convergence of $V F$ holds in $L^2(\nu)$. Moreover, it shows that for $b > 0$ and $F = (Kf \text{ or } F_n)$, we have:

$$\|(V - V_b)F\|_2 \leq \int_b^\infty \|Q_t^\Phi F\|_2 \, dt \leq \varrho^{-2} e^{-\varrho b}$$

and in the same vein:

$$\begin{aligned} \|V Kf - V F_n\|_2 &\leq \int_0^\infty \|Q_t^\Phi (Kf - F_n)\|_2 \, dt \leq \int_0^\infty \min\{\|Kf - F_n\|_2, 2\varrho^{-1} e^{-\varrho t}\} \, dt \\ &= \varrho^{-1} \|Kf - F_n\|_2 \log(2e\varrho^{-1} \|Kf - F_n\|_2^{-1}). \end{aligned}$$

This shows that $V_b F_n$ converges towards $V Kf$ in $L^2(\nu)$ as $b, n \uparrow \infty$.

LEMMA 7. – For any smooth bounded function F with bounded derivatives on $O\mathcal{M}$ and for any $t \geq 0$, $Q_t^\Phi F$ is also smooth on $O\mathcal{M}$, with bounded \mathcal{L}_j -derivatives.

Proof. – Let us first observe that for any $n, k \in \mathbb{N}$, $\xi \in O\mathcal{M}$ and $t \geq 0$, we have $\mathbb{E}(|x_t^0|^n (y_t^0)^{-k} \Phi \circ \pi_2(\xi_t^0)) < \infty$. (Recall from Section 4.1 that $Z_t^0 = (x_t^0, y_t^0)$ and from Remark 4 that $\xi_t^0 = \xi T_{Z_t^0}$.) Indeed we use Schwarz inequality, and the two following facts:

- on one hand $\mathbb{E}(\Phi^2 \circ \pi_2(\xi_t^0)) = P_t \Phi^2 \circ \pi_2(\xi)$ is continuous and λ' -integrable, by invariance of the Brownian semi-group P_t , and thus finite;
- on the other hand it is clear from the expression of y_t^0 that $\mathbb{E}((y_t^0)^k)$ is finite for any $k \in \mathbb{Z}$, and for any $n \in \mathbb{N}$, we have using Doob inequality:

$$\begin{aligned} \mathbb{E}(|x_t^0|^{2n}) &\leq c_n \mathbb{E} \left[\left(\int_0^t \exp[2w_s - ds] \, ds \right)^n \right] \leq c_n t^n \mathbb{E} \left[\sup_{0 \leq s \leq t} \exp[2nw_s - nds] \right] \\ &\leq c_n(t) \mathbb{E} \left[\left(\sup_{0 \leq s \leq t} \exp[nw_s - n^2s/2] \right)^2 \right] \leq c'_n(t) \mathbb{E} \left[\exp[2nw_t - n^2t] \right] < \infty. \end{aligned}$$

Then, recall from the proof of Lemma 6 that the \mathcal{L}_j -derivatives of $\log \Phi \circ \pi_2$ are bounded by δ . Thus, we see that $\mathcal{L}_j(\Phi \circ \pi_2 \times F)/\Phi \circ \pi_2$ is smooth bounded for $0 \leq j \leq d$, and that $\Phi \circ \pi_2(\xi \theta_s^j) \leq e^\delta \Phi \circ \pi_2(\xi)$ for $0 \leq j \leq d$ (here $\theta_s^0 := \theta_s$, $|s| \leq 1$ and $\xi \in \mathcal{OM}$).

Observe yet that by the commutation formula (2) we have: $\theta_s^j T_{Z_t^0} = T_{Z_t^0} \theta_{s/y_t^0}^j$ and $\theta_s T_{Z_t^0} = T_{Z_t^0} \theta_{(e^s - 1)x_t^0/y_t^0}^+$. Therefore we can differentiate under the expectation and get by using Remark 4 and the above:

$$\begin{aligned} \mathcal{L}_0 Q_t^\Phi F(\xi) &= \frac{e^{-\lambda_0 t}}{\Phi(\xi)} \times \left(\mathbb{E} \left[\left(\mathcal{L}_0 + (y_t^0)^{-1} \sum_{j=1}^d (x_t^0)^j \mathcal{L}_j \right) (\Phi F)(\xi_t^0) \right] \right. \\ &\quad \left. - \mathbb{E}[(\Phi F)(\xi_t^0)] \mathcal{L}_0 \log \Phi(\xi) \right) \end{aligned}$$

(we write here Φ for $\Phi \circ \pi_2$) and for $1 \leq j \leq d$:

$$\mathcal{L}_j Q_t^\Phi F(\xi) = e^{-\lambda_0 t} \Phi(\xi)^{-1} \times (\mathbb{E}[(y_t^0)^{-1} \mathcal{L}_j(\Phi F)(\xi_t^0)] - \mathbb{E}[(\Phi F)(\xi_t^0)] \mathcal{L}_j \log \Phi(\xi)).$$

This proves the existence of the first-order derivatives, and also of the higher-order derivatives, since they will have the same form as the first-order ones above, but with powers of $(x_t^0)^j$ and $(y_t^0)^{-1}$.

To prove that $\mathcal{L}_j Q_t^\Phi F$ is bounded, we use again that $\mathcal{L}_0 \log \Phi \circ \pi_2$ is bounded by δ , to get the uniform estimate: $\Phi \circ \pi_2(\xi_t^0)/\Phi \circ \pi_2(\xi) \leq \exp[\delta \times \text{dist}_{\mathcal{M}}(\pi_2(\xi_t^0), \pi_2(\xi))]$. Now recall from Section 4.1 that $\pi_2(\xi_t^0) = \xi(Z_t^0)$ is a Brownian motion on \mathcal{M} starting from $\pi_2(\xi) = \xi(e_0)$, so using that ξ is an isometry, we get:

$$\Phi \circ \pi_2(\xi_t^0)/\Phi \circ \pi_2(\xi) \leq \exp[\delta \times \text{dist}_{\mathbb{H}}(e_0, Z_t^0)].$$

Hence we see, using the above expressions for $\mathcal{L}_j Q_t^\Phi F$ and the Schwarz inequality, that the proof will be complete if we show that $\mathbb{E}(\exp[2\delta \times \text{dist}_{\mathbb{H}}(e_0, Z_t^0)])$ is finite for any t . Now, we have (using the classical formula for the distance, see [13]):

$$\exp[2\delta \times \text{dist}_{\mathbb{H}}(e_0, Z_t^0)] \leq 4^\delta ch^{2\delta} [\text{dist}_{\mathbb{H}}(e_0, Z_t^0)] = (|x_t^0|^2 + (y_t^0)^2 + 1)^{2\delta} \times (y_t^0)^{-2\delta}$$

and thus we only have to use again that $\mathbb{E}(|x_t^0|^n (y_t^0)^k)$ is finite, for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. \square

This Lemma 7 allows to write for any $b > 0$ and $n \in \mathbb{N}$:

$$Q_b^\Phi F_n - F_n = \int_0^b \frac{d}{dt} Q_t^\Phi F_n dt = \frac{1}{2} \int_0^b D^\Phi Q_t^\Phi F_n dt = \frac{1}{2} D^\Phi V_b F_n,$$

whence:

$$(8) \quad D^\Phi V_b F_n = 2(Q_b^\Phi F_n - F_n).$$

Thus replacing ω by $\omega + d(V_b F_n)$ in the formula (F) of Section 7.1, we get:

$$(9) \quad \int_{\xi^\Phi[0,t]} \omega + V_b F_n(\xi_t^\Phi) - V_b F_n(\xi) = M_t^{b,n} + \int_0^t (Kf - F_n + Q_b^\Phi F_n)(\xi_s^\Phi) ds,$$

where $M_t^{b,n}$ is a continuous martingale having as increasing process:

$$(10) \quad \langle M_t^{b,n} \rangle = \int_0^t \sum_{j=0}^d (f_j + \mathcal{L}_j V_b F_n)^2 (\xi_s^\Phi) ds.$$

We want to go to the limit in the formula (9) above, as $b, n \uparrow \infty$. But for that, we need to control also the \mathcal{L}_j -derivatives, in the convergence of $V_b F_n$ to $V K f$.

LEMMA 8. – *The potential $V K f$ admits \mathcal{L}_j -derivatives in $L^2(v')$, and $\mathcal{L}_j V K f$ is the limit in $L^2(v')$ of $\mathcal{L}_j V_b F_n$ as $b, n \uparrow \infty$, for $0 \leq j \leq d$.*

Proof. – Let us first observe that by using the adjoint \mathcal{L}_j^* mentioned in the formula (F') of Section 7.1, we have: $\sum_{j=0}^d \mathcal{L}_j^* \mathcal{L}_j = -(D^\Phi + D^\delta)/2$, and that by using Proposition 2 this implies: $\int \sum_{j=0}^d |\mathcal{L}_j F|^2 dv' = -\int F \times D^\Phi F dv'$.

Using this and formulas (8), (7), we get on one hand, for any $b, c > 0$ and $n \in \mathbb{N}$:

$$\begin{aligned} \int \sum_{j=0}^d |\mathcal{L}_j (V_b - V_c) F_n|^2 dv' &= - \int (V_b - V_c) F_n \times D^\Phi (V_b - V_c) F_n dv' \\ &= \int (V_c - V_b) F_n \times 2(Q_b^\Phi - Q_c^\Phi) F_n dv' \\ &\leq \| (V_b - V_c) F_n \|_2 \times 2(\| Q_b^\Phi F_n \|_2 + \| Q_c^\Phi F_n \|_2) \\ &\leq \| V F_n \|_2 \times 2Q^{-1}(e^{-qb} + e^{-qc}) \leq 4Q^{-3}(e^{-qb} + e^{-qc}); \end{aligned}$$

and on the other hand, for any $b > 0$ and $n, p \in \mathbb{N}$:

$$\begin{aligned} \int \sum_{j=0}^d |\mathcal{L}_j V_b (F_n - F_p)|^2 dv' &= - \int V_b (F_n - F_p) \times D^\Phi V_b (F_n - F_p) dv' \\ &= - \int V_b (F_n - F_p) \times 2(Q_b^\Phi (F_n - F_p) - (F_n - F_p)) dv' \\ &\leq 2(\| V_b F_n \|_2 + \| V_b F_p \|_2) \times (\| Q_b^\Phi (F_n - F_p) \|_2 + \| (F_n - F_p) \|_2) \\ &\leq 8Q^{-2} \times \| (F_n - F_p) \|_2. \end{aligned}$$

This shows that $\mathcal{L}_j V_b F_n$ is Cauchy in $L^2(v')$, and then proves the result. \square

We can now go to the limit in formulas (9) and (10), thereby showing the following:

PROPOSITION 7. – *We have $\int_{\xi^\Phi[0,t]} \omega = V K f(\xi) - V K f(\xi_t^\Phi) + M_t$, where M_t is a continuous martingale, with as increasing process*

$$\langle M_t \rangle = \int_0^t \sum_{j=0}^d (f_j + \mathcal{L}_j V K f)^2 (\xi_s^\Phi) ds.$$

COROLLARY 6. – *As $t \rightarrow \infty$, the law of $t^{-1/2} \int_{\xi^\Phi[0,t]} \omega$ converges towards the centered Gaussian law with variance $\mathcal{V}(f) := \int \sum_{j=0}^d (f_j + \mathcal{L}_j V K f)^2 dv'$, which vanishes if and only if f equals $\mathcal{L}_0 h$, for some $h \in L^2(T^1 \mathcal{M}, \nu)$.*

Proof. – Firstly, $t^{-1/2}(VKf(\xi) - VKf(\xi_t^\Phi))$ goes to zero in $L^2(v \otimes \mathbb{P})$ -probability as $t \rightarrow \infty$, since $VKf(\xi_t^\Phi)$ is stationary. Hence, applying Proposition 7, we see that we have to deal with $\lim_{t \rightarrow \infty} v' \otimes \mathbb{E}[\exp(a\sqrt{-1}M_t/\sqrt{t})]$. Now, observe that by Lemma 8 we have $\lim_{\{b,n \uparrow \infty\}} \|(M_t - M_t^{b,n})/\sqrt{t}\|_{L^2(v' \otimes \mathbb{P})}^2 = \lim_{\{b,n \uparrow \infty\}} \sum_{j=0}^d \|\mathcal{L}_j VKf - \mathcal{L}_j V_b F_n\|_{L^2(v')}^2 = 0$. Thus

$$\begin{aligned} & \lim_{t \rightarrow \infty} v' \otimes \mathbb{E}[\exp(a\sqrt{-1}M_t/\sqrt{t})] \\ (11) \quad &= \lim_{b,n \uparrow \infty} \lim_{t \rightarrow \infty} v' \otimes \mathbb{E}[\exp(a\sqrt{-1}M_t^{b,n}/\sqrt{t})] \\ &= \lim_{b,n \uparrow \infty} \lim_{t \rightarrow \infty} v' \otimes \mathbb{E}[\exp(a\sqrt{-1}M_t^{b,n}/\sqrt{t} + a^2\langle M_t^{b,n} \rangle/2t) \exp(-a^2\langle M_t^{b,n} \rangle/2t)]. \end{aligned}$$

We now need some ergodic property for the degenerate diffusion ξ_t^Φ , which is not clear in the present context. So let us show that $\langle M_t^{b,n} \rangle/t$ converges in $L^2(v' \otimes \mathbb{P})$ as $t \rightarrow \infty$, towards $\mathcal{V}_{b,n}(f) := \int \sum_{j=0}^d (f_j + \mathcal{L}_j V_b F_n)^2 dv'$, for every $b > 0$ and $n \in \mathbb{N}$.

Indeed, we see from formula (10) that $\langle M_t^{b,n} \rangle/t - \mathcal{V}_{b,n}(f) = t^{-1} \int_0^t H_{b,n}(\xi_s^\Phi) ds$, with $H_{b,n}$ smooth bounded on \mathcal{OM} by Lemma 7, and such that $\int H_{b,n} dv' = 0$.

Then applying Itô's Formula to $V_c H_{b,n}(\xi_t^\Phi)$, and going to the limit as $c \rightarrow \infty$, we get as for the proof of Proposition 7:

$$\int_0^t H_{b,n}(\xi_s^\Phi) ds = V H_{b,n}(\xi) - V H_{b,n}(\xi_t^\Phi) + M_t^{b,n,\infty},$$

where $M_t^{b,n,\infty}$ is a continuous martingale having as increasing process

$$\langle M_t^{b,n,\infty} \rangle = \int_0^t \sum_{j=0}^d (\mathcal{L}_j V H_{b,n})^2(\xi_s^\Phi) ds.$$

Therefore

$$\left\| t^{-1} \int_0^t H_{b,n}(\xi_s^\Phi) ds \right\|_{L^2(v' \otimes \mathbb{P})}^2 \leq 4t^{-2} \|V H_{b,n}\|_2^2 + 2t^{-1} \sum_{j=0}^d \|\mathcal{L}_j V H_{b,n}\|_2^2 = \mathcal{O}(t^{-1}).$$

As a consequence, we get the convergence in $L^1(v' \otimes \mathbb{P})$ of $\exp(-a^2\langle M_t^{b,n} \rangle/2t)$ towards $\exp(-a^2\mathcal{V}_{b,n}(f)/2)$. Finally, we know from Lemma 7 that for each $b > 0, n \in \mathbb{N}$, $\langle M_t^{b,n} \rangle/t$ is uniformly bounded, and thus that the continuous martingale $\exp(a\sqrt{-1}M_t^{b,n}/\sqrt{t} + a^2\langle M_t^{b,n} \rangle/2t)$ is bounded and has expected value one.

Hence we conclude from formula (11) and Lemma 8, by:

$$\lim_{t \rightarrow \infty} v' \otimes \mathbb{E}[\exp(a\sqrt{-1}M_t/\sqrt{t})] = \lim_{b,n \uparrow \infty} \exp(-a^2\mathcal{V}_{b,n}(f)/2) = \exp(-a^2\mathcal{V}(f)/2). \quad \square$$

Finally, the central limit theorem stated in the introduction follows from Corollaries 1, 5 and 6, even under the slightly weaker assumption (H) of Remark 7: f and $\mathcal{L}_0 f$ are bounded, rotationally Hölderian on $T^1\mathcal{M}$, and continuous along the stable leaves, and, for $1 \leq j \leq d$, $\mathcal{L}_j f$ and $\mathcal{L}_j^2 f$ are bounded and Hölderian on \mathcal{OM} .

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