Central limit theorem for a class of relativistic diffusions

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Two similar Minkowskian diffusions have been considered, on one hand by Debbasch and co-workers [J. Math. Phys. **40**, 2891 (2001); Eur. Phys. J. **19**, 37 (2001); **23**, 487 (2001); J. Stat. Phys. **88**, 945 (1997); **90**, 1179 (1998)], and on the other hand by Dunkel and Hänggi [Phys. Rev. E **71**, 016124 (2005); **72**, 036106 (2005)]. We address here two questions, asked by Debbasch and Rivet [J. Stat. Phys. **90**, 1179 (1998)] and by Dunkel and Hänggi [Phys. Rev. E **71**, 016124 (2005); **72**, 036106 (2005)], respectively, about the asymptotic behavior of such diffusions. More generally, we establish a central limit theorem for a class of Minkowskian diffusions, to which the two above ones belong. As a consequence, we correct a partially wrong guess by Dunkel and Hänggi [Phys. Rev. E **71**, 016124 (2005)].
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I. INTRODUCTION

Debbasch *et al.* introduced in Ref. 7 a relativistic diffusion in Minkowski space, which they called relativistic Ornstein-Uhlenbeck process (ROUP), to describe the motion of a point particle surrounded by a heat bath, or relativistic fluid, with respect to the rest frame of the fluid, in which the particle diffuses. This ROUP was then studied in Refs. 1–3 and 8 and extended to the curved case in Ref. 4. Then Dunkel and Hänggi introduced and discussed in Refs. 5 and 6 a similar process, in Minkowski space, which they called relativistic Brownian motion.

Note that independently, a relativistic diffusion on any Lorentz manifold was defined in Ref. 10 as the only diffusion whose law possesses the relativistic invariance under the whole isometry group of the manifold. Accordingly, the particular case of the Schwarzschild-Kruskal-Szekeres manifold was studied. The case of Gödel's universe was recently studied in Ref. 9.

In Ref. 8, Debbasch and Rivet argue qualitatively that the so-called hydrodynamical limit of their ROUP should behave in a Brownian way. They stress that a mathematical rigorous proof remains needed to confirm such not much intuitive statement.

In Refs. 5 and 6, Dunkel and Hänggi ask the question of the asymptotic behavior of the variance, or "mean square displacement," of their diffusion. Indeed, compared to the nonrelativistic case and after numerical computations, they guess that this variance, normalized by time, should converge to some constant, for which they conjecture an empirical formula.

We answer here these two questions, asked by Debbasch and Rivet in Ref. 8 and by Dunkel and Hänggi in Refs. 5 and 6, and indeed a more general one. We establish in fact rigorously in this article a central limit theorem for a class of Minkowskian diffusions, to which the two above mentioned ones, ROUP and Dunkel-Hänggi (DH) diffusion, belong. As a consequence of our main result, we establish for this whole class the convergence of the normalized variance, guessed (for their particular case) in Refs. 5 and 6. Getting the exact expression for this limiting variance and particularizing to the DH diffusion, we can then invalidate and correct the wrong conjecture made in Ref. 5 on its expression and asymptotic behavior (as the noise parameter goes to infinity).

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To summarize the content, we begin by describing in Sec. II below the class of Minkowskian diffusions we consider, which contains both ROUP and DH diffusions as particular cases.

Then in Sec. III, we present our study, leading to the following main result:

Theorem 1: *Let* $(\mathbf{x}_t,\mathbf{p}_t)_{t\geq 0}=(x_t^i,p_t^i)_{1\leq i\leq d,t\geq 0}$ *be an* $\mathbb{R}^d\times\mathbb{R}^d$*-valued diffusion solving the stochastic differential system: For* $1\leq i\leq d$,

$$(\star)\begin{cases} \mathrm{d}x_t^i = f(|\mathbf{p}_t|)p_t^i\mathrm{d}t \\ \mathrm{d}p_t^i = -b(|\mathbf{p}_t|)p_t^i\mathrm{d}t + \sigma(|\mathbf{p}_t|)(\beta[1+\eta(|\mathbf{p}_t|)^2])^{-1/2}[\mathrm{d}W_t^i + \eta(|\mathbf{p}_t|)(p_t^i/|\mathbf{p}_t|)\mathrm{d}w_t]. \end{cases}$$

Then, under some hypotheses $(\mathcal{H})$ *on the continuous functions* f, b, σ, *and* η, *the law of the process* $(t^{-1/2}\mathbf{x}_{at})_{a\geq 0}$ *converges, as* $t\to\infty$, *to the law of* $(\Sigma_\beta\mathcal{B}_a)_{a\geq 0}$, *in* $C(\mathbb{R}^+,\mathbb{R}^d)$ *endowed with the topology of uniform convergence on compact sets of* $\mathbb{R}_+$. *Here* W *and* $\mathcal{B}$ *are standard* d*-dimensional Brownian motions*, w *is a standard real Brownian motion, independent of* W, $\beta>0$ *is an inverse noise or heat parameter, and* Σ_β *is a constant, displayed in Proposition 2 below*.

The following is then deduced. Recall that the symbol $\mathbb{E}$ stands for expected or mean value, with respect to the underlying probability measure (or distribution) $\mathbb{P}$ (governing the given Brownian motions W, w).

Corollary 2: Under the same hypotheses as in the above theorem, from any starting point, the normalized variance (mean square displacement) $t^{-1}\mathbb{E}[|\mathbf{x}_t|^2]$ *goes, as* $t\to\infty$, *towards* $d\times\Sigma_\beta^2$.

The method we use in Sec. III to get our Theorem 1 and Corollary 2 was already known. It appears in Ref. 14, in the general framework of a Harris positively recurrent Markov process, where are established in particular general versions of our Lemma 1 and Propositions 2 and 3 below, in a somewhat less direct way however. To apply such general result, the difficulty is mainly to get our Proposition 1 below, which in Ref. 14 is replaced by a hypothesis made on the range of the infinitesimal generator.

In Sec. IV, we study the behaviors of the limiting variance Σ_β^2 as the inverse noise parameter β goes to 0 or to ∞, and we also support our rigorous result by numerical simulations. Focusing on the particular case of the DH diffusion, for $d=1$, we get the following, which, though confirming a nonclassical variance behavior, shows a behavior near 0 which differs from the one implied by the wrong guess made in Ref. 5 about the expression of Σ_β^2.

Proposition 3: Consider the DH case, for $d=1$, *as in Ref. 5, Then, we have* $\Sigma_\beta^2\sim 2/\beta$ *as* $\beta\nearrow\infty$, *and, as* $\beta\searrow 0$, $\Sigma_\beta^2\sim 2\log(1/\beta)$ *(in the backward approach) or* $\Sigma_\beta^2\sim A/\log(1/\beta)$, *for some explicit constant* $A>0$ *(in the Itô forward approach)*.

Finally, we detail in Secs. V and VI two somewhat involved proofs. We thank Reinhard Schäfke for his kind and decisive help for the proof of Secs. V A and V B.

II. A CLASS OF MINKOWSKIAN DIFFUSIONS

Let $\mathbb{R}^{1,d}$, where $d\geq 1$ is an integer, denote the usual Minkowski space of special relativity. In its canonical basis, denote by $x=(x^\mu)=(x^0,x^i)=(x^0,\mathbf{x})$ the coordinates of the generic point, with Greek indices running $0,\ldots,d$ and latin indices running $1,\ldots,d$. The Minkowskian pseudometric is given by $\mathrm{d}s^2=|\mathrm{d}x^0|^2-\Sigma_{i=1}^d|\mathrm{d}x^i|^2$.

The world line of a particle having mass m is a timelike path in $\mathbb{R}^{1,d}$, which we can always parametrize by its arc length or proper time s. So the moves of such particle are described by a path $s\mapsto(x_s^\mu)$, having momentum $p=(p_s)$ given by

$$p=(p^\mu)=(p^0,\ p^i)=(p^0,\mathbf{p}),\quad \text{where } p_s^\mu := m\frac{\mathrm{d}x_s^\mu}{\mathrm{d}s},$$

and satisfying $|p^0|^2-\Sigma_{i=1}^d|p^i|^2=m^2$.

We shall consider here world lines of type $(t,\mathbf{x}(t))_{t\geq 0}$ and take $m=1$. Introducing the velocity $\mathbf{v}=(v^1,\ldots,v^d)$ and polar coordinates (r,Θ) by setting

$$v^i := \frac{\mathrm{d}x^i}{\mathrm{d}t}, \quad r := |\mathbf{p}| = \left(\sum_{i=1}^{d} |p^i|^2\right)^{1/2}, \quad \text{and } \Theta := \frac{\mathbf{p}}{r} =: (\theta^1, \ldots, \theta^d) \in \mathbb{S}^{d-1},$$

we get at once

$$p^0 = \frac{\mathrm{d}t}{\mathrm{d}s} = \sqrt{1+r^2} = (1-|\mathbf{v}|^2)^{-1/2} \quad \text{and } p = \sqrt{1+r^2}(1,\mathbf{v}).$$

Thus, a full space-time trajectory

$$(x(t),p(t))_{t\geq 0} = (t,\mathbf{x}(t),p^0(t),\mathbf{p}(t))_{t\geq 0}$$

is determined by the mere knowledge of its spacial component $(\mathbf{x}(t),\mathbf{p}(t))$.

We can, therefore, from now on focus on spacial trajectories $t \mapsto (\mathbf{x}_t,\mathbf{p}_t) \in \mathbb{R}^d \times \mathbb{R}^d$.

The Minkowskian diffusions we consider here are associated as above to Euclidian diffusions $t \mapsto (\mathbf{x}_t,\mathbf{p}_t) = (x_t^i,p_t^i)_{1\leq i \leq d}$, which are the solution to a stochastic differential system of the following type:

$$(\star)\begin{cases} \mathrm{d}x_t^i = f(r_t)p_t^i \mathrm{d}t \\ \mathrm{d}p_t^i = -b(r_t)p_t^i \mathrm{d}t + \sigma(r_t)(\beta[1+\eta(r_t)^2])^{-1/2}[\mathrm{d}W_t^i + \eta(r_t)\theta_t^i \mathrm{d}w_t] \end{cases} \quad \text{for } 1 \leq i \leq d,$$

where $\mathbf{W} := (W^1, \ldots, W^d)$ denotes a standard d-dimensional Euclidian Brownian motion, w denotes a standard real Brownian motion, independent of $\mathbf{W}$, $\beta > 0$ is an inverse noise or heat parameter, and the real functions f, b, σ, and η are continuous on $\mathbb{R}_+$ and satisfy the following hypotheses, for some fixed $\varepsilon > 0$:

$$(\mathcal{H}) \quad \sigma \geq \varepsilon \text{ on } \mathbb{R}_+, \quad g(r) := \frac{2rb(r)}{\sigma^2(r)} \geq \varepsilon \text{ for large } r, \quad \lim_{r\to\infty} e^{-\varepsilon' r} f(r) = 0 \text{ for some } \varepsilon' < \frac{\beta\varepsilon}{2}.$$

Of course, in the particular case of the constant functions f, b, σ, and $\eta=0$, the process $(\mathbf{x}_t)$ is an integrated Ornstein-Uhlenbeck process. The process considered by Debbasch and co-workers,[1–3,7,8] which they call the ROUP, corresponds to

$$\text{(ROUP) } f(r) = b(r) = (1+r^2)^{-1/2}, \quad \sigma(r) = \sqrt{2}, \quad \eta = 0, \quad g(r) = r(1+r^2)^{-1/2},$$

and the relativistic process considered by Dunkel and Hänggi[5,6] corresponds to

$$\text{(DH) } f(r) = (1+r^2)^{-1/2}, \quad b(r) = 1 - \mathrm{d}\beta^{-1}(1+r^2)^{-1/2}, \quad \sigma(r) = \sqrt{2\sqrt{1+r^2}}, \quad \eta(r) = r.$$

Note that Dunkel and Hänggi consider in fact three "approaches" of their stochastic differential equation, namely, according to viewing the stochastic integration, either in the Itô, the Stratonovitch, or the backward (they call Hänggi-Klimontovitch) sense. These three meanings are of course included in our present general setting; merely by modifying our function $b(r)$, the Itô acceptation corresponds to $b(r) \equiv 1$, and the Itô correction, from Itô to backward, is precisely $-\mathrm{d}\beta^{-1}(1+r^2)^{-1/2}$, which we wrote just above, preferring it since it leads to the physically more satisfying Jüttner equilibrium distribution (see Sec. III B below); the Itô correction, from Itô to Stratonovitch, would be exactly half.

These processes are intended to describe the motion of a point particle surrounded by a heat bath, or relativistic fluid, with respect to the rest frame of the fluid, in which the particle diffuses. The Minkowskian diffusion $(\mathbf{x}_t,\mathbf{p}_t)$ solving the stochastic differential system $(\star)$ is isotropic precisely when $\eta \equiv 0$ (for $d \geq 2$; when $d=1$, η does not matter). If $\eta \neq 0$, the momentum $(\mathbf{p}_t)$ undergoes a radial drift.

In Ref. 8 (Sec. IV), Debbasch and Rivet argue heuristically that the so-called hydrodynamical limit of their ROUP should behave in a Brownian way and ask the question of a mathematical proof confirming such not much intuitive statement.

In Refs. 5 and 6, Dunkel and Hänggi ask the question of the convergence, as t goes to infinity, of the normalized variance (or mean square displacement):

$$\Sigma^2(t) := t^{-1}\mathbb{E}\left(\sum_{i=1}^{d}|x_t^i|^2\right) = \mathbb{E}[|\mathbf{x}_t|^2/t].$$

We shall answer these two questions by means of the more general one we address, which is the asymptotic behavior, as $t\to\infty$, of the process

$$(\mathbf{x}_a^t)_{a\geq 0} := (t^{-1/2}\mathbf{x}_{at})_{a\geq 0} = t^{-1/2}(x_{at}^1, \ldots, x_{at}^d)_{a\geq 0},$$

where the diffusion $(\mathbf{x}_t,\mathbf{p}_t)_{t\geq 0}$ solves $(\star)$, under the hypotheses $(\mathcal{H})$.

III. ASYMPTOTIC BEHAVIOR OF THE PROCESS $(\mathbf{x}_a^t)_{a\geq 0}$

A. An auxiliary function F

Let us look for a function $F=(F^1,\ldots,F^d)\in C^2(\mathbb{R}^d,\mathbb{R}^d)$ such that for $1\leq i\leq d$,

$$\mathrm{d}F^i(\mathbf{p}_t) + \mathrm{d}x_t^i - \mathrm{d}M_t^i = 0 \tag{1}$$

for some martingale $\mathbf{M}_t=(M_t^1,\ldots,M_t^d)$, so that $t^{-1/2}x_{at}^i=t^{-1/2}M_{at}^i-t^{-1/2}F^i(\mathbf{p}_{at})$.

Now, Itô's formula gives

$$\mathrm{d}F^i(\mathbf{p}_t) = \left[-\sum_{j=1}^{d}\partial_j F^i(\mathbf{p}_t)p_t^j b(r_t) + \frac{\sigma^2(r_t)}{2\beta[1+\eta(r_t)^2]}\sum_{1\leq j,\ k\leq d}(\delta_{jk}+\eta(r_t)^2\theta_t^j\theta_t^k)\partial_{jk}^2F^i(\mathbf{p}_t)\right]\mathrm{d}t + \mathrm{d}M_t^i,$$

with

$$\mathrm{d}M_t^i = (\beta[1+\eta(r_t)^2])^{-1/2}\sigma(r_t)\sum_{j=1}^{d}\partial_jF^i(\mathbf{p}_t)[\mathrm{d}W_t^j + \eta(r_t)\theta_t^j\mathrm{d}w_t].$$

Note that, in other words, this means that the so-called infinitesimal generator of the momentum diffusion $(\mathbf{p}_t)$ is

$$\frac{\sigma^2(r)}{2\beta[1+\eta(r)^2]}\left(\Delta + \eta(r)^2\sum_{1\leq j,k\leq d}\theta^j\theta^k\frac{\partial^2}{\partial p^j\partial p^k}\right) - b(r)\sum_{j=1}^{d}p^j\frac{\partial}{\partial p^j},$$

Δ denoting the usual Euclidian Laplacian of $\mathbb{R}^d$. Hence a function F satisfying Eq. (1) must solve

$$\frac{\sigma^2(r)}{2\beta[1+\eta(r)^2]}\sum_{1\leq j,\ k\leq d}(\delta_{jk}+\eta(r)^2\theta^j\theta^k)\partial_{jk}^2F^i(\mathbf{p}) - b(r)\sum_{j=1}^{d}p^j\partial_jF^i(\mathbf{p}) = -p^i\times f(r). \tag{2}$$

Let us take F^i of the form

$$F^i(\mathbf{p}) = \theta^i\times\psi_\beta(r) = p^i\times\psi_\beta(r)/r$$

and set the following for $r\in\mathbb{R}_+$:

$$g(r) := \frac{2rb(r)}{\sigma^2(r)}, \quad h(r) := \frac{2rf(r)}{\sigma^2(r)}, \quad \text{and } G(r) := \int_0^r g(\rho)\mathrm{d}\rho. \tag{3}$$

Then a direct computation shows that

$$\mathrm{d}M_t^i = \frac{\sigma(r_t)}{\sqrt{\beta[1+\eta(r_t)^2]}}\left[\psi_\beta'(r_t)\mathrm{d}W_t^i + \left[\frac{\psi_\beta(r_t)}{r_t} - \psi_\beta'(r_t)\right]\sum_{j=1}^{d}\left[\delta_{ij} - \theta_t^i\theta_t^j\right]\mathrm{d}W_t^j + \eta(r_t)\psi_\beta'(r_t)\theta_t^i\mathrm{d}w_t\right], \tag{4}$$

and that Eq. (2) is equivalent to

$$\psi_\beta''(r) - \left(\beta g(r) - \frac{d-1}{r[1+\eta(r)^2]}\right)\psi_\beta'(r) - \frac{d-1}{r^2[1+\eta(r)^2]}\psi_\beta(r) + \beta h(r) = 0. \tag{5}$$

Note that if $b\equiv f$, or equivalently, if $g\equiv h$, then Eq. (5) admits the trivial solution $\psi_\beta(r)=r$. If $d=1$, Eq. (5) is easily solved, too (see Sec. VI). But it is not easily solved in the general case we are considering, and not even in the case of the diffusion (DH) considered in Ref. 5 (and isotropically extended to higher dimensions) or in Ref. 6.

However, we have the following, whose delicate proof is postponed to Sec. V.

Proposition 1: Under hypotheses $(\mathcal{H})$, *Eq. (5) admits a solution* $\psi_\beta \in C^2(\mathbb{R}_+,\mathbb{R})$ *such that* $\psi_\beta(0)=0$, *and* $|\psi_\beta(r)|=\mathcal{O}(e^{\varepsilon' r})$, $|\psi_\beta'(r)|=\mathcal{O}(e^{\varepsilon' r})$ *near infinity, for some* $\varepsilon' < \varepsilon\beta/2$. *Moreover, if* $0 \leq f \leq b$, *then we have* $0 \leq \psi_\beta \leq Id$.

B. Polar decomposition of the process $(\mathbf{p}_t)$ and equilibrium distribution ν

Since the diffusion $(\mathbf{x}_t,\mathbf{p}_t)$ solves $(\star)$, the radial process $r_t=|\mathbf{p}_t|$ solves

$$\mathrm{d}r_t = \left(\frac{(d-1)\sigma^2(r_t)}{2\beta[1+\eta(r_t)^2]r_t} - r_t b(r_t)\right)\mathrm{d}t + \sigma(r_t)\beta^{-1/2}\mathrm{d}B_t,$$

with

$$\mathrm{d}B_t := (1+\eta(r_t)^2)^{-1/2}\left[\sum_{i=1}^{d}\theta_t^i\mathrm{d}W_t^i + \eta(r_t)\mathrm{d}w_t\right]. \tag{6}$$

As $\langle B,B\rangle_t=t$, B is a standard real Brownian motion. Consider then the angular process $\tilde{\Theta}_s = (\tilde{\theta}_s^1,\ldots,\tilde{\theta}_s^d) \in \mathbb{S}^{d-1}$ defined by the time change $\tilde{\Theta}_s := \Theta_{C^{-1}(s)}$, i.e., by $\mathbf{p}_t = r_t \times \tilde{\Theta}_{C_t}$, by means of the clock

$$C_t = C(t) := \int_0^t \frac{\sigma^2(r_s)}{\beta r_s^2}\mathrm{d}s.$$

The process $(\tilde{\Theta}_s) \in \mathbb{S}^{d-1}$ is a spherical Brownian motion, since it solves

$$\mathrm{d}\tilde{\theta}_s^i = \left(\frac{1-d}{2}\right)\tilde{\theta}_s^i\mathrm{d}s + \sum_{j=1}^{d}(\delta_{ij} - \tilde{\theta}_s^i\tilde{\theta}_s^j)\mathrm{d}\tilde{W}_s^j,$$

for some standard Brownian motion $\tilde{W}=(\tilde{W}^1,\ldots,\tilde{W}^d)\in\mathbb{R}^d$. Hence the infinitesimal generator of the diffusion $(r_t,\Theta_t)=(r_t,\tilde{\Theta}_{C_t})$ is

$$\mathcal{A} := \mathcal{L}_r + \frac{\sigma^2(r)}{2\beta r^2}\Delta_{\mathbb{S}^{d-1}}, \quad \text{with } \mathcal{L}_r := \frac{\sigma^2(r)}{2\beta}\left(\partial_r^2 + \left[\frac{d-1}{[1+\eta(r)^2]r} - \beta g(r)\right]\partial_r\right). \tag{7}$$

Under this form, it appears that the anisotropy function η results in a radial drift.

Set for $r\in\mathbb{R}_+$

$$\mu(r) \coloneqq \exp\left[\int_1^r \frac{\mathrm{d}s}{s[1+\eta(s)^2]}\right] \in \mathbb{R}_+. \tag{8}$$

Note that $\min\{r,1\} \leqslant \mu(r) \leqslant \max\{r,1\}$ and that $0 \leqslant r \leqslant s \Rightarrow 1 \leqslant \mu(s)/\mu(r) \leqslant s/r$.

The radial process (r_t) admits the invariant measure $\nu(r)\mathrm{d}r$, having density on $\mathbb{R}_+$:

$$\nu(r) \coloneqq \sigma(r)^{-2}\mu(r)^{d-1}e^{-\beta G(r)}. \tag{9}$$

Note that this equilibrium distribution equals the so-called Jüttner one in the ROUP and DH cases. We have indeed $G(r)=\sqrt{1+r^2}-1$ and $\mu(r)=r$ in the ROUP case and $G(r)=\sqrt{1+r^2}-1-(d/2\beta)\log(1+r^2)$ and $\mu(r)=(r\sqrt{2})/\sqrt{1+r^2}$ in the DH case.

The hypotheses $(\mathcal{H})$ ensure that ν is finite and then that the radial process (r_t) is ergodic. Denoting by $\mathrm{d}\Theta$ the uniform probability measure on the sphere $\mathbb{S}^{d-1}$ and setting

$$\pi(\mathrm{d}r,\mathrm{d}\Theta) \coloneqq \left(\int_0^\infty \nu\right)^{-1} \times \nu(r)\mathrm{d}r\mathrm{d}\Theta, \tag{10}$$

it is easily seen that π is an invariant probability measure (or equivalently, equilibrium distribution, meaning that the operator $\mathcal{A}$ is symmetrical with respect to π: $\int\Phi_1\mathcal{A}\Phi_2\mathrm{d}\pi=\int\Phi_2\mathcal{A}\Phi_1\mathrm{d}\pi$ for any test functions Φ_1, Φ_2 on $\mathbb{R}_+\times\mathbb{S}^{d-1}$) for the process $(r_t,\tilde{\Theta}_{C_t})=(r_t,\Theta_t)$, which is then a symmetrical ergodic diffusion on $\mathbb{R}_+\times\mathbb{S}^{d-1}$.

Lemma 1: For any starting point $\mathbf{p}_0=r_0\Theta_0$, *uniformly with respect to* $a\geqslant 0$, *we have*

$$t^{-1}\mathbb{E}_{\mathbf{p}_0}[|F^i(\mathbf{p}_{at})|^2] \to 0, \quad \textit{as } t\to\infty, \quad \textit{for } 1\leqslant i\leqslant d.$$

Proof: Since $|F^i(\mathbf{p})|^2=|\theta^i\times\psi_\beta(r)|^2\leqslant\psi_\beta^2(r)=\mathcal{O}(e^{2\varepsilon' r})$ by Proposition 1, we have $\mathbb{E}_{\mathbf{p}_0}[|F^i(\mathbf{p}_{at})|^2]\leqslant C\mathbb{E}_{r_0}[e^{2\varepsilon' r_{at}}]$, for some constant C.

Let (Q_t) denote the semigroup of the radial diffusion (r_t), solution to $\partial_t Q_t=\mathcal{L}_r Q_t$. It is known (see, for example, Ref. 11, Sec. 4.11) that $Q_t(r_0,r)$ is a continuous function of (t,r) and that $Q_1(r_0,r)=q_1(r)\nu(r)$ for some bounded function q_1. Hence on one hand we have

$$\mathbb{E}_{r_0}[e^{2\varepsilon' r_s}] \leqslant \sup_{0\leqslant s\leqslant 1} Q_s(e^{2\varepsilon'\cdot})(r_0) < \infty \quad \text{for } 0\leqslant s\leqslant 1,$$

and on the other hand, by the Markov property, for $s\geqslant 1$ we have

$$\mathbb{E}_{r_0}[e^{2\varepsilon' r_s}] = Q_1Q_{s-1}(e^{2\varepsilon'\cdot})(r_0) = \int_0^\infty \mathbb{E}_\rho[e^{2\varepsilon' r_{s-1}}]q_1(\rho)\nu(\rho)\mathrm{d}\rho \leqslant \|q_1\|_\infty\int_0^\infty e^{2\varepsilon'\rho}\nu(\rho)\mathrm{d}\rho < +\infty$$

by $(\mathcal{H})$ and since $2\varepsilon' < \beta\varepsilon$. This shows that $s\mapsto\mathbb{E}_{r_0}[e^{2\varepsilon' r_s}]$ is bounded, whence the result. □

C. Asymptotic study of the martingale M

By formula (1) and Lemma 1, we are now left with the study of the martingale part $(\mathbf{M}_t)$. Recall that the coordinates M^i of the martingale $\mathbf{M}$ are given by Eq. (4).

1. Asymptotic independence of the martingales M^i

Lemma 2: For $1\leqslant i$, $l\leqslant d$, *as* $t\to\infty$, *we have almost surely*

$$\lim_{t\to\infty}\frac{\langle M^i,M^l\rangle_t}{t} = \delta_{il}\Sigma_\beta^2, \quad \textit{with } \Sigma_\beta^2 \coloneqq \frac{1}{\beta d}\left[\int |\psi'_\beta|^2\sigma^2\mathrm{d}\pi + (d-1)\int \psi_\beta^2(1+\eta^2)^{-1}Id^{-2}\sigma^2\mathrm{d}\pi\right].$$

Proof: The computation of brackets gives easily

$$\beta\langle M^i, M^l\rangle_t = \delta_{il} S_t^i - (1-\delta_{il}) T_t^{i,\,l},$$

with

$$S_t^i := \int_0^t \sigma^2(r_s)\psi_\beta'(r_s)^2|\theta_s^i|^2 \mathrm{d}s + \int_0^t [1+\eta(r_s)^2]^{-1} r_s^{-2}\sigma^2(r_s)\psi_\beta^2(r_s)(1-|\theta_s^i|^2)\mathrm{d}s,$$

$$T_t^{i,\,l} := \int_0^t \sigma^2(r_s)[\psi_\beta^2(r_s)[1+\eta(r_s)^2]^{-1} r_s^{-2} - \psi_\beta'(r_s)^2]\theta_s^i\theta_s^l \mathrm{d}s.$$

Setting

$$k^i(r,\Theta) := \sigma^2(r)\psi_\beta'(r)^2|\theta^i|^2 + [1+\eta(r)^2]^{-1} r^{-2}\sigma^2(r)\psi_\beta^2(r)(1-|\theta^i|^2)$$

and

$$\ell^{i,\,l}(r,\,\Theta) := \sigma^2(r)[\psi_\beta^2(r)[1+\eta(r)^2]^{-1} r^{-2} - \psi_\beta'(r)^2]\theta^i\theta^l$$

and noticing that these functions are π integrable by Proposition 1, using Sec. III B we can apply the ergodic theorem to get the following almost sure convergences:

$$\lim_{t\to\infty} S_t^i/t = \int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} k^i \mathrm{d}\pi, \quad \lim_{t\to\infty} T_t^{i,\,l}/t = \int_{\mathbb{R}_+\times\mathbb{S}^{d-1}} \ell^{i,\,l}\mathrm{d}\pi.$$

Now the spherical symmetry with respect to Θ implies that for $1\leqslant i\neq l\leqslant d$,

$$\int k^i \mathrm{d}\pi = \mathrm{d}^{-1}\int |\psi_\beta'|^2\sigma^2\mathrm{d}\pi + (1-\mathrm{d}^{-1})\int \psi_\beta^2(1+\eta^2)^{-1} I\mathrm{d}^{-2}\sigma^2 \mathrm{d}\pi =: \beta\Sigma_\beta^2 \quad \text{and} \quad \int \ell^{i,\,l}\mathrm{d}\pi = 0.$$

Therefore we have got

$$t^{-1}\langle M^i, M^l\rangle_t \xrightarrow{p.s.} \delta_{il}\times\Sigma_\beta^2 = \delta_{il}(\beta d)^{-1}[\pi(\sigma^2|\psi_\beta'|^2) + (d-1)\pi(\sigma^2\psi_\beta^2/[(1+\eta^2)Id^2])].$$

□

Consider now the martingale $\mathbf{M}^t$ defined by

$$\mathbf{M}_a^t := (M_a^{1,\,t},\ldots,M_a^{d,\,t}) := t^{-1/2}\mathbf{M}_{at}$$

and the Dambis-Dubins-Schwarz Brownian motions $B^{i,t}$, such that

$$M_a^{i,\,t} = B^{i,\,t}(\langle M^{i,\,t}, M^{i,\,t}\rangle_a) = B^{i,\,t}(t^{-1}\langle M^i,\, M^i\rangle_{at}).$$

Applying the asymptotic Knight theorem (see, for example, Ref. 13, Theorem 2.3 and Corollary 2.4, pp. 524–525), we deduce now from Lemma 2 the asymptotic independence of the martingales M^i and M^l, for $1\leqslant i\neq l\leqslant d$, in the following sense.

Corollary 1: The process $(B^{1,t},\ldots,B^{d,t})$ *converges in law, as t goes to infinity, towards a standard d-dimensional Brownian motion* $\mathcal{B}$.

2. Convergence of the finite-dimensional marginal laws

Proposition 2: The finite-dimensional marginal laws of the martingale $\mathbf{M}^t$ *converge, as t goes to infinity, to those of the Brownian motion* $\Sigma_\beta\times\mathcal{B}$, *where* Σ_β *is the (positive) constant given by*

$$\Sigma_\beta^2 = \left[d\int_0^\infty e^{-\beta G(r)}\mu(r)^{d-1}\sigma(r)^{-2}\mathrm{d}r\right]^{-1} \times \int_0^\infty \psi_\beta(r) e^{-\beta G(r)}\mu(r)^{d-1}h(r)\mathrm{d}r. \tag{11}$$

Recall that ψ_β *comes from Proposition 1, and that we set in formulas (3) and (8):*

$$G(r) = \int_0^r g(\rho)\mathrm{d}\rho, \quad h(r) = \frac{2rf(r)}{\sigma^2(r)}, \quad \mu(r) = \exp\left[\int_1^r \frac{\mathrm{d}s}{s[1+\eta(s)^2]}\right]. \tag{12}$$

Proof: Fix any integer $N \geq 1$, positive numbers $0 < a_1 < \cdots < a_N$, and consider the vector random processes

$$X^t := (\langle M^{i,\ t}, M^{i,\ t}\rangle_{a_k}, B_s^{j,t})_{1\leq i,\ j\leq d, 1\leq k\leq N, s\geq 0}, \quad X^\infty := (\Sigma_\beta^2 a_k, \mathcal{B}_s^j)_{1\leq i,\ j\leq d, 1\leq k\leq N, s\geq 0}.$$

By Sec. III C 1, X^t converges in law, as t goes to infinity, to X^∞. By the Skorokhod coupling theorem (see, for example, Ref. 12, Theorem 4.30, p. 78), there exist vector random processes $\tilde{X}^t$ and $\tilde{X}^\infty$ satisfying the following identities in law:

$$(\tilde{X}^t) \overset{d}{=} (X^t), \quad \tilde{X}^\infty \overset{d}{=} X^\infty,$$

and such that $\tilde{X}^t$ converges almost surely to $\tilde{X}^\infty$. As a consequence, we get the following convergence in distribution:

$$(B^{i,t}(\langle M^{i,\ t},\ M^{i,\ t}\rangle_{a_k}))_{1\leq i\leq d, 1\leq k\leq N} \overset{d}{\longrightarrow} (\mathcal{B}^i(\Sigma_\beta^2 a_k))_{1\leq i\leq d, 1\leq k\leq N}$$

or, equivalently,

$$(M_{a_k}^{i,\ t})_{1\leq i\leq d, 1\leq k\leq N} \overset{d}{\longrightarrow} \Sigma_\beta \times (\mathcal{B}_{a_k}^i)_{1\leq i\leq d, 1\leq k\leq N}.$$

Note that from formulas (9) and (10) and Lemma 2, we get directly the following expression for Σ_β:

$$\Sigma_\beta^2 = \frac{\int_0^\infty \psi_\beta'(r)^2 e^{-\beta G(r)}\mu(r)^{d-1}\mathrm{d}r + (d-1)\int_0^\infty \psi_\beta(r)^2 e^{-\beta G(r)}\mu(r)^{d-1}[1+\eta(r)^2]^{-1}r^{-2}\mathrm{d}r}{\beta d\int_0^\infty e^{-\beta G(r)}\mu(r)^{d-1}\sigma(r)^{-2}\mathrm{d}r}.$$

It remains to derive from this expression Eq. (11) of the statement for Σ_β. This is achieved as follows, integrating by parts and using Proposition 1, which implies that $\lim_{r\to\infty}[\psi_\beta(r)\psi_\beta'(r)e^{-\beta G(r)}\mu(r)^{d-1}]=0$, together with Eq. (5):

$$\begin{aligned}
\int_0^\infty \psi_\beta'(r)^2 e^{-\beta G(r)}\mu(r)^{d-1}\mathrm{d}r &= [\psi_\beta(r)\psi_\beta'(r)e^{-\beta G(r)}\mu(r)^{d-1}]_0^\infty - \int_0^\infty \psi_\beta(r)\frac{\mathrm{d}}{\mathrm{d}r}[\psi_\beta'(r)e^{-\beta G(r)}\mu(r)^{d-1}]\mathrm{d}r \\
&= \int_0^\infty \psi_\beta(r)\left[\left(\frac{(d-1)/r}{1+\eta(r)^2} - \beta g(r)\right)\psi_\beta'(r) - \frac{(d-1)\psi_\beta(r)}{r^2[1+\eta(r)^2]} + \beta h(r) - \psi_\beta'(r)\right. \\
&\quad \left.\times\left(\frac{(d-1)/r}{1+\eta(r)^2} - \beta g(r)\right)\right]e^{-\beta G(r)}\mu(r)^{d-1}\mathrm{d}r \\
&= \beta\int_0^\infty \psi_\beta(r)e^{-\beta G(r)}\mu(r)^{d-1}h(r)\mathrm{d}r - (d-1)\int_0^\infty \psi_\beta(r)^2 e^{-\beta G(r)}\mu(r)^{d-1} \\
&\quad \times[1+\eta(r)^2]^{-1}r^{-2}\mathrm{d}r.
\end{aligned}$$

□

3. Tightness

Proposition 3: The family of martingales $\mathbf{M}^t$ *is tight, in* $C(\mathbb{R}_+,\mathbb{R}^d)$*, endowed with the topology of uniform convergence on compact sets of* $\mathbb{R}_+$.

Proof: Fix any $T>0$ and use the Arzelà-Ascoli theorem (see, for example, Ref. 12, Theorem 16.5, p. 311): the family $(t^{-1/2}\mathbf{M}_{at}, a\in[0,T])$ is tight, in $C([0,T],\mathbb{R}^d)$, if and only if

$$\lim_{h\to 0}\lim_{t\to\infty}\sup \mathbb{E}[t^{-1/2}\sup_{\substack{0\leq a\leq b\leq T\\ b-a\leq h}}|\mathbf{M}_{at}-\mathbf{M}_{bt}|]=0.$$

Fix $i\in\{1,\ldots,d\}$, $h>0$, and denote by n the integral part of T/h. There exists a standard Brownian motion $\widetilde{W}^i$ such that (S^i being as in Sec. III C)

$$\frac{1}{\sqrt{ht}}\sup_{\substack{0\leq a\leq b\leq T\\ b-a\leq h}}|M^i_{at}-M^i_{bt}|=\frac{1}{\sqrt{\beta ht}}\sup_{\substack{0\leq a\leq b\leq T\\ b-a\leq h}}|\widetilde{W}^i(S^i_{bt}-S^i_{at})|.$$

Setting $\widetilde{W}^{i*}(u)=\sup_{0\leq s\leq u}|\widetilde{W}^i_s|$, we also have

$$\frac{1}{\sqrt{ht}}\sup_{\substack{0\leq a\leq b\leq T\\ b-a\leq h}}|\widetilde{W}^i(S^i_{bt}-S^i_{at})|=\widetilde{W}^{i*}\left(\sup_{0\leq a\leq T-h}\frac{1}{ht}(S^i_{a+ht}-S^i_a)\right)\leq\widetilde{W}^{i*}(A^i_{ht}),$$

where

$$A^i_{ht}\coloneqq\sup_{0\leq j\leq n}A^{i,j}_{ht}\quad\text{and }A^{i,j}_{ht}\coloneqq\frac{1}{ht}(S^i_{(j+2)ht}-S^i_{jht}).$$

By the ergodic theorem we have (as in the proof of Lemma 2) the following convergence, as $t\to\infty$, valid almost surely and in L^1 norm as well: $A^{i,j}_{ht}\to 2\Sigma^2_\beta$, which implies the uniform integrability of $\{A^j_{ht}, ht\geq 1\}$.

Otherwise, by Doob's inequality (applied to the martingale $\int_0^s 1_{\{u\leq A_{ht}\}}d\widetilde{W}^i_u$), we have

$$\mathbb{E}[\widetilde{W}^{i*}(A^i_{ht})]\leq\|\widetilde{W}^{i*}(A^i_{ht})\|_2\leq 2\sqrt{\mathbb{E}[A^i_{ht}]},$$

whence

$$\mathbb{E}[t^{-1/2}\sup_{\substack{0\leq a\leq b\leq T\\ b-a\leq h}}|M^i_{at}-M^i_{bt}|]\leq 2\beta^{-1/2}\sqrt{h\times\mathbb{E}[A^i_{ht}]}.$$

Now, as for fixed h and for any $\lambda>2\Sigma^2_\beta$ we have:

$$\mathbb{E}[A^i_{ht}]=\int_0^\infty\mathbb{P}(A^i_{ht}\geq s)ds\leq\lambda+\int_\lambda^\infty\mathbb{P}(A^i_{ht}\geq s)ds\leq\lambda+\sum_{j=0}^n\int_\lambda^\infty\mathbb{P}(A^{i,j}_{ht}\geq s)ds,$$

we deduce that

$$\mathbb{E}[A^i_{ht}]\leq\lambda+\sum_{j=0}^n\mathbb{E}[A^{i,j}_{ht}\times 1_{\{A^{i,j}_{ht}\geq\lambda\}}]\to\lambda,\quad\text{as } t \text{ goes to infinity}.$$

Hence, $\limsup_{t\to\infty}\mathbb{E}[A^i_{ht}]\leq 2\Sigma^2_\beta$, and then

$$\lim_{h\to 0}\limsup_{t\to\infty} t^{-1/2}\mathbb{E}[\sup_{\substack{0\le a\le b\le T\\ b-a\le h}} |M^i_{at}-M^i_{bt}|]=0.$$

□

4. Main result

Gathering formula (1), Lemma 1, and Propositions 2 and 3, we get at once the following main result of this article.

Theorem 1: *Let $(\mathbf{x}_t,\mathbf{p}_t)=(x^i_t,p^i_t)_{1\le i\le d}$ be an $\mathbb{R}^d\times\mathbb{R}^d$-valued diffusion solving the stochastic differential system*

$$(\star)\begin{cases} \mathrm{d}x^i_t = f(r_t)p^i_t\mathrm{d}t \\ \mathrm{d}p^i_t = -b(r_t)p^i_t\mathrm{d}t + \sigma(r_t)(\beta[1+\eta(r_t)^2])^{-1/2}[\mathrm{d}W^i_t + \eta(r_t)\theta^i_t\mathrm{d}w_t] \end{cases}, \quad \textit{for } 1\le i\le d,$$

where $\mathbf{W}:=(W^1,\ldots,W^d)$ denotes a standard d-dimensional Euclidian Brownian motion, w denotes a standard real Brownian motion, independent of $\mathbf{W}$, $\beta>0$ is an inverse noise or heat parameter, and the real functions f, b, σ, are η are continuous on $\mathbb{R}_+$ and satisfy the following hypotheses for some fixed $\varepsilon>0$:

$$(\mathcal{H})\quad \sigma\ge\varepsilon \textit{ on } \mathbb{R}_+,\quad g(r):=\frac{2rb(r)}{\sigma^2(r)}\ge\varepsilon \textit{ for large } r,\quad \lim_{r\to\infty} e^{-\varepsilon' r}f(r)=0 \textit{ for some } \varepsilon'<\frac{\beta\varepsilon}{2}.$$

Then the law of the process $(t^{-1/2}\mathbf{x}_{at})_{a\ge 0}$ converges, as $t\to\infty$, to the law of $(\Sigma_\beta\mathcal{B}_a)_{a\ge 0}$, in $C(\mathbb{R}_+,\mathbb{R}^d)$, endowed with the topology of uniform convergence on compact sets of $\mathbb{R}_+$. Here $\mathcal{B}$ is a standard d-dimensional Brownian motion, and the constant Σ_β is given by formula (11). This result holds from any starting point $(\mathbf{x}_0,\mathbf{p}_0)$ ($\mathbf{p}_0$ can also obey the equilibrium law π).

We deduce now the result conjectured in Refs. 5 and 6 and an expression of the limit.

Corollary 2: Under the same hypotheses as in the above theorem, for any starting point, the normalized variance (mean square displacement) $t^{-1}\mathbb{E}[|\mathbf{x}_t|^2]$ goes, as $t\to\infty$, towards $d\times\Sigma^2_\beta$.

Proof: By Theorem 1, we have convergence in law of the random variable $t^{-1}|\mathbf{x}_t|^2$, towards $\Sigma^2_\beta|\mathcal{B}_1|^2$. By formula (1) and Lemma 1, we have only to make sure that for $1\le i\le d$, the following holds:

$$t^{-1}\mathbb{E}[|M^i_t|^2]=(\beta t)^{-1}\mathbb{E}[S^i_t]\to\Sigma^2_\beta.$$

Now, on one hand, we already noticed (recall the proof of Lemma 2) that, by ergodicity, we have $t^{-1}S^i_t\overset{p.s.}{\longrightarrow}\int k^i\mathrm{d}\pi=\beta\Sigma^2_\beta$. And, on the other hand, exactly the same reasoning as in the proof of Lemma 1 (to show that $s\mapsto\mathbb{E}_{r_0}[e^{2\varepsilon' r_{at}}]$ is bounded), merely using the semigroup (P_t) of the diffusion (r_t,Θ_{C_t}), solution to $\partial_t P_t=\mathcal{A}P_t$, instead of the radial semigroup (Q_t), shows that $s\mapsto\mathbb{E}_{\mathbf{p}_0}[k^i(\mathbf{p}_s)]$ is bounded. Moreover, in the same spirit, by the Markov property and by the proof of Lemma 2, we have

$$\mathbb{E}_{\mathbf{p}_0}\left[\frac{S^i_t}{t}\right]=\frac{1}{t}\int_0^1 P_s(k^i)(\mathbf{p}_0)\mathrm{d}s+\int\left(\frac{1}{t}\int_0^{t-1}P_s(k^i)(\mathbf{p})\mathrm{d}s\right)\tilde{q}_1(\mathbf{p})\pi(\mathrm{d}\mathbf{p}),$$

$\tilde{q}_1$ being the bounded density of $P_1(\mathbf{p}_0,\mathrm{d}\mathbf{p})$ with respect to $\pi(\mathrm{d}\mathbf{p})$. It is clear that the first term of the right hand side goes to 0. Finally, by the Chacon-Ornstein theorem and by dominated convergence, the second term goes indeed to $\beta\Sigma^2_\beta$. □

IV. BEHAVIORS OF Σ^2_β AS $\beta\searrow 0$ AND AS $\beta\nearrow\infty$

Theorem 1 and Corollary 2 show up the interest of the limiting constant $d\times\Sigma^2_\beta$. Recall then from Secs. II and III A that the processes considered by Debbasch and co-workers and by Dunkel and Hänggi correspond, respectively, to

$$\text{(ROUP)}\ h(r) = g(r) = r(1+r^2)^{-1/2}, \quad G(r) = \sqrt{1+r^2} - 1, \quad \eta = 0, \quad \sigma(r)^2 = 2, \quad \psi_\beta(r) = r,$$

$$\text{(DH)}\ h(r) = \frac{r}{1+r^2}, \quad \mu(r) = \frac{r\sqrt{2}}{\sqrt{1+r^2}}, \quad G(r) = \sqrt{1+r^2} - 1 - \frac{d}{2\beta}\log(1+r^2), \quad \eta(r) = r$$

for some positive (noise or heat) inverse parameter β. It is natural to wonder, as in Ref. 5, how the limiting variance Σ_β^2 behaves as $\beta \searrow 0$ and as $\beta \nearrow \infty$.

In the ROUP case, we have simply $d \times \Sigma_\beta^2 = 2d/\beta$. The variance behavior is Euclidian.

In the DH case of Refs. 5 and 6, we have by formula (11)

$$d \times \Sigma_\beta^2 = 2 \times \frac{\int_0^\infty \psi_\beta(r) e^{-\beta\sqrt{1+r^2}}(1+r^2)^{-1/2} r^d \mathrm{d}r}{\int_0^\infty e^{-\beta\sqrt{1+r^2}} r^{d-1} \mathrm{d}r}. \tag{13}$$

Note that the precise value of ψ_β is given in Sec. V B:

$$\psi_\beta(r) = \zeta_1(r)\int_0^r \zeta_2(\rho) w_\beta(\rho)^{-1} h(\rho)\mathrm{d}\rho + \zeta_2(r)\int_r^\infty \zeta_1(\rho) w_\beta(\rho)^{-1} h(\rho)\mathrm{d}\rho,$$

with functions ζ_1, ζ_2, w_β given in Sec. V A.

In Ref. 5, for $d=1$, after numerical simulations, Dunkel and Hänggi conjecture that Σ_β^2 could be equal to $2/(2+\beta)$. The expression we got above for Σ_β^2 invalidates this conjecture and even the asymptotic behavior near 0 it implies. However, it is true that a nonclassical variance behavior occurs. We have indeed the following, whose technical proof is postponed to Sec. VI.

Proposition 4: Consider the DH case, for $d=1$, *as in Ref. 5 (backward approach). Then, we have*

$$\Sigma_\beta^2 \sim 2/\beta \text{ as } \beta \nearrow \infty \text{ and } \Sigma_\beta^2 \sim 2\log(1/\beta) \text{ as } \beta \searrow 0.$$

Remark 1: For their simulation, Dunkel and Hänggi consider in fact the Itô approach of their stochastic differential equation, that is to say, $b(r) \equiv 1$ instead of $b(r)$ as above. This changes formula (13) and then the asymptotic behavior got in the above Proposition 4. However, in this case, too, the conjecture of Dunkel and Hänggi remains quantitatively wrong. Indeed, proceeding as for Proposition 4, we obtain the following.

Consider the DH case, for $d=1$, *as in Ref. 5, but with* $b(r) \equiv 1$ *(according to their Itô approach and simulation). Then, setting*

$$A := 2\int_0^\infty \left[\int_x^\infty \frac{e^{-u}\mathrm{d}u}{u}\right]^2 e^x \mathrm{d}x \approx 3.2048\cdots,$$

we have

$$\Sigma_\beta^2 \sim \frac{2}{\beta} \text{ as } \beta \nearrow \infty \text{ and } \Sigma_\beta^2 \sim \frac{A}{\log(1/\beta)} \text{ as } \beta \searrow 0.$$

Numerical simulations

To confirm the validity of our rigorous estimates in Proposition 4 and Remark 1, invalidating the conjecture of Ref. 5, we performed numerical simulations relating to the DH diffusion, in the case $d=1$. We used the Monte Carlo method, with $N=1000$ simulations. For different values of β (from 10^{-5} to 10^5), we computed $\mathbf{x}_j(t)_{j=1\cdots N}$ for $0 \leq t \leq T=100$, with mesh $\mathrm{d}t=10^{-4}$, and then the quantity $\overline{\mathbf{x}^2}(T) = N^{-1}\Sigma_{j=1\cdots N}\mathbf{x}_j^2(T)$.

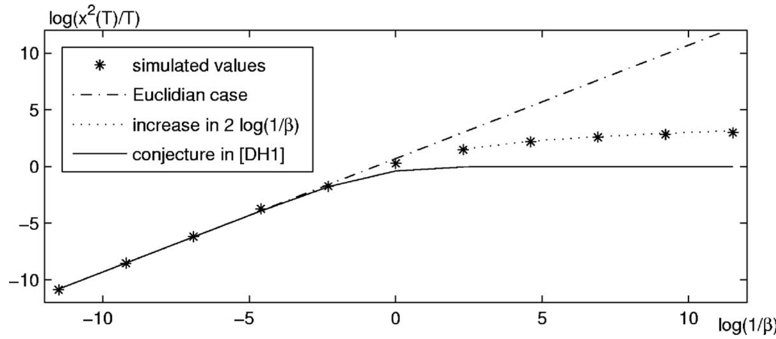

FIG. 1. Backward approach.

1. The program used for the simulations

function res=DH1 (N,dt,T,r,M) (written in "matlab")

N is the iteration number in the Monte Carlo method. The process $x(t)$ is simulated on $[0,T]$, with mesh d*t, for $\beta=10^k$ and $k\in\{-r,-r+1,\ldots,r-1,r\}$. If $M=0$, the postpoint discretization rule is performed (in the backward approach); if $M=1$, the prepoint discretization rule is performed (in the Itô forward approach).*

Vector var *contain the values of $\overline{\mathbf{x}}^2(T)/T$ corresponding to the values of β* valbeta

```
valbeta=zeros(1,2*s+1); var=zeros(1,2*s+1); n=T*floor(1./dt);
```

Loop for the different values of beta:

```
  for k=-r: 1: r beta=10^k;
```

Arrays p *and* x *contain the values of $p(t)$ and $x(t)$ for $0\leqslant t\leqslant T$,*

```
p=zeros(N,n); x=zeros(N,n);
```

Simulation of Brownian motionW:

```
W=sqrt(dt).*randn(N,n);
```

Simulation of processes $p(t)$ and $x(t)$:

```
if M=0 for j=1 : 1 : n-1
     gam=sqrt(1+p(:  ,j).*p(:  ,j));
p(:  ,j+1)=p(:  ,j)-p(:  ,j).*(1-1./(beta.*gam))dt
     +sqrt((2./beta).*gam).*W(:  ,j);
         x(:  ,j+1)=x(:  ,j)+(p(:  ,j)./gam).*dt; end
else for j=1 : 1 : n-1
       gam=sqrt(1+p(:  ,j).*p(:  ,j));
  p(:  ,j+1)=p(:  ,j)-p(:  ,j).*dt
     +sqrt((2./beta).*gam).*W(:  ,j);
  x(:  ,j+1)=x(:  ,j)+(p(:  ,j)./gam)*dt; end
end
```

Computation of the mean of $\overline{\mathbf{x}}^2(T)$, normalized by T:

```
  variance=mean(x(:  ,n).*x(:  ,n))./T;
    var(1,1+s+k)=variance; valbeta(1,s+k+1)=beta;
end
```

End of program.

2. Results of the simulations

The diagrams Fig. 1 and 2 represent our results in logarithmic coordinates. Thus, the horizontal axis represents $\log(1/\beta)$; the points blue $*$ represent the simulated values $\log[\overline{\mathbf{x}^2}(T)/T]$ as a

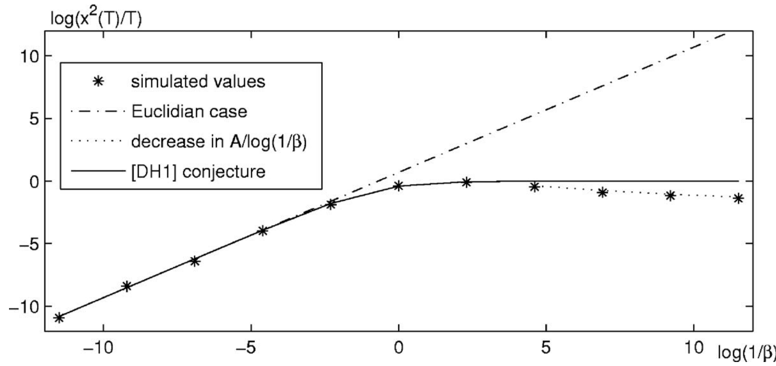

FIG. 2. Itô foward approach.

function of $\log(1/\beta)$. The straight line corresponds to the Euclidian behavior, and the continuous curve to the function $\beta \mapsto 2/(\beta+2)$.

Backward approach. When the postpoint discretization rule is performed, the behavior of $\log[\overline{\mathbf{x}^2}(T)/T]$ as a function of $\log(1/\beta)$ is drawn in Fig. 1. Here, the dashed curve corresponds to an increase in $2\log(1/\beta)$ (for small β) (see Fig. 1). The simulations confirm the Euclidian behavior of the DH diffusion as $\beta \gg 1$. For $\beta \geqslant 1$, a divergence appears clearly. On the contrary, the $2\log(1/\beta)$-like asymptotic behavior of the limit, which we established rigorously, appears as confirmed.

Itô (forward) approach. When the prepoint discretization rule is performed, the simulations still confirm the Euclidian behavior of the DH diffusion as $\beta \gg 1$. For small β, the expression conjectured in Ref. 5 is a good approximation as long as $\beta > 1/10$; however, for smaller β, a divergence appears again. Our rigorous behavior in $A \log(1/\beta)^{-1}$ appears also as confirmed (see Fig. 2).

V. PROOF OF PROPOSITION 1

We are indebted to Reinhard Schäfke for this proof, who kindly indicated to us how to proceed for Secs. V A and V B below. We thank him warmly. Consider first the homogeneous equation associated with Eq. (5):

$$\zeta''(r) + \left(\frac{d-1}{r[1+\eta(r)^2]} - \beta g(r)\right)\zeta'(r) - \frac{d-1}{r^2[1+\eta(r)^2]}\zeta(r) = 0. \tag{14}$$

It has a pole of order 2 at 0 (except for $d=1$) and a pole at infinity. Using the fixed point method, we construct two solutions ζ_1 and ζ_2 of Eq. (14), bounded, respectively, near infinity and near 0. Using these two solutions of the homogeneous equation, a solution ψ_β to Eq. (5) is then deduced, which vanishes at 0. Finally, we establish the wanted control on ψ_β, ψ'_β.

Recall from formula (8) that we set

$$\mu(r) = \exp\left[\int_1^r \frac{\mathrm{d}s}{s[1+\eta(s)^2]}\right],$$

so that μ increases and $\min\{r,1\} \leqslant \mu(r) \leqslant \max\{r,1\}$ and $0 \leqslant r \leqslant s \Rightarrow 1 \leqslant \mu(s)/\mu(r) \leqslant s/r$.

A. Constructing solutions to the homogeneous [Eq. (14)]

1. Constructing a solution ζ_1 to Eq. (14), bounded near ∞

Using hypotheses $(\mathcal{H})$, fix $\varepsilon>0$ and $r_0\geqslant 1$ such that $g\geqslant\varepsilon$ on $[r_0,\infty[$. For $r\geqslant r_0$, set

$$\lambda(r) \coloneqq \int_r^\infty \mu(\rho)^{1-d}e^{\beta G(\rho)}\left[\int_\rho^\infty e^{-\beta G(s)}\mu(s)^{d-1}[1+\eta(s)^2]^{-1}s^{-2}\mathrm{d}s\right]\mathrm{d}\rho.$$

We have

$$\lambda(r) \leqslant \int_r^\infty \left[\int_\rho^\infty e^{-\beta\varepsilon(s-\rho)}\left[\frac{s}{\rho}\right]^{d-1}s^{-2}\mathrm{d}s\right]\mathrm{d}\rho = \int_r^\infty \left[\int_0^\infty e^{-\beta\varepsilon s}(1+s/\rho)^{d-3}\mathrm{d}s\right]\rho^{-2}\mathrm{d}\rho$$
$$\leqslant \frac{1}{r}\int_0^\infty e^{-\beta\varepsilon s}\max\{1,(1+s/r_0)^{d-3}\}\mathrm{d}s = \mathcal{O}(1/r).$$

As $r\to\infty$, $\lambda(r)$ decreases to 0, so that (up to increase r_0) we can suppose that $\lambda(r_0)\leqslant 1/(2d)$. On $[r_0,\infty[$, let us define by induction on $n\in\mathbb{N}$ the functions $\varphi_0\equiv 1$ and

$$\varphi_{n+1}(r) \coloneqq 1+(d-1)\int_r^\infty \mu(\rho)^{1-d}e^{\beta G(\rho)}\left[\int_\rho^\infty e^{-\beta G(s)}\mu(s)^{d-1}\varphi_n(s)[1+\eta(s)^2]^{-1}s^{-2}\mathrm{d}s\right]\mathrm{d}\rho.$$

We have the following for $r\geqslant r_0$:

$$1\leqslant \varphi_{n+1}(r)\leqslant 1+(d-1)\|\varphi_n\|_{L^\infty[r_0,\infty[}\times\lambda(r)\leqslant 1+\tfrac{1}{2}\|\varphi_n\|_{L^\infty[r_0,\infty[},$$

whence $1\leqslant\varphi_n\leqslant\|\varphi_n\|_{L^\infty[r_0,\infty[}<2$, for any $n\in\mathbb{N}$. Then similarly,

$$\|\varphi_{n+1}-\varphi_n\|_{L^\infty[r_0,\infty[}\leqslant\|\varphi_n-\varphi_{n-1}\|_{L^\infty[r_0,\infty[}\times(d-1)\lambda(r_0),$$

which allows to apply the fixed point method to get $\zeta_1\coloneqq\lim_{n\to\infty}^{L^\infty[r_0,\infty[}\varphi_n$, which satisfies

$$1\leqslant\zeta_1(r)=1+(d-1)\int_r^\infty \mu(\rho)^{1-d}e^{\beta G(\rho)}\left[\int_\rho^\infty e^{-\beta G(s)}\mu(s)^{d-1}\zeta_1(s)[1+\eta(s)^2]^{-1}s^{-2}\mathrm{d}s\right]\mathrm{d}\rho. \tag{15}$$

In particular, as $r\to\infty$ we have

$$\zeta_1(r)\leqslant 1+(d-1)\|\zeta_1\|_{L^\infty[r_0,\infty[}\lambda(r)\to 1.$$

Hence $\lim_{r\to\infty}\zeta_1(r)=1$ and

$$\zeta_1'(r)=(1-d)\mu(r)^{1-d}e^{\beta G(r)}\int_r^\infty e^{-\beta G(s)}\mu(s)^{d-1}\zeta_1(s)[1+\eta(s)^2]^{-1}s^{-2}\mathrm{d}s<0.$$

Then

$$\zeta_1''(r)+\left(\frac{d-1}{r[1+\eta(r)^2]}-\beta g(r)\right)\zeta_1'(r)-\frac{d-1}{r^2[1+\eta(r)^2]}\zeta_1(r)=0.$$

This solution can be continued over the whole $\mathbb{R}_+^*$, yielding ζ_1 still satisfying Eqs. (14) and (15) on $\mathbb{R}_+^*$. We also have $\lim_0\zeta_1=+\infty$.

2. Constructing a solution ζ_2 to Eq. (14), bounded near 0

$$\text{For } r \in [0,\ 1], \quad \text{set } \Lambda(r) \coloneqq \beta\int_0^r \mu(\rho)^{-d-1}\rho^{-2}e^{\beta G(\rho)}\Big[\int_0^\rho e^{-\beta G(s)}\mu(s)^{d+1}|g(s)|s\mathrm{d}s\Big]\mathrm{d}\rho.$$

We have $0 \leqslant \Lambda'(r) \leqslant \beta e^{2\beta\int_0^r |g|}\int_0^1 |g(rs)|s\mathrm{d}s \to 0$ as $r \to 0$, by hypotheses $(\mathcal{H})$. We can then fix $r_1 \in]0,1]$ such that $\Lambda(r_1) \leqslant 1/2$. On $]0,r_1]$, let us define by induction on $n \in \mathbb{N}$ the functions $\phi_0 \equiv 1$ and

$$\phi_{n+1}(r) \coloneqq 1 + \beta\int_0^r \mu(\rho)^{-d-1}\rho^{-2}e^{\beta G(\rho)}\left[\int_0^\rho e^{-\beta G(s)}\mu(s)^{d+1}g(s)\phi_n(s)s\mathrm{d}s\right]\mathrm{d}\rho.$$

We have $\phi_n \in C^2(]0,r_1])$, $\|\phi_{n+1}\|_{L^\infty]0,r_1]} \leqslant 1 + \Lambda(r_1)\|\phi_n\|_{L^\infty]0,r_1]}$ so that $\|\phi_n\|_{L^\infty}]0,r_1] < 2$, and

$$\|\phi_{n+1} - \phi_n\|_{L^\infty 0,\ r_1} \leqslant \|\phi_n - \phi_{n-1}\|_{L^\infty 0,\ r_1} \times \Lambda(r_1),$$

which allows to apply the fixed point method to get $\tilde{\phi} \coloneqq \lim_{n\to\infty}^{L^\infty]0,r_1]}\phi_n$, which satisfies

$$\tilde{\phi}(r) = 1 + \beta\int_0^r \mu(\rho)^{-d-1}\rho^{-2}e^{\beta G(\rho)}\left[\int_0^\rho e^{-\beta G(s)}\mu(s)^{d+1}g(s)\tilde{\phi}(s)s\mathrm{d}s\right]\mathrm{d}\rho = 1 + \mathcal{O}[\Lambda(r)] \qquad (16)$$

for any $r \in]0,r_1]$. Hence,

$$\tilde{\phi}'(r) = \beta\mu(r)^{-d-1}r^{-2}e^{\beta G(r)}\int_0^r e^{-\beta G(s)}\mu(s)^{d+1}g(s)\tilde{\phi}(s)s\mathrm{d}s = \mathcal{O}[\Lambda'(r)] \to 0, \quad \text{as } r \to 0.$$

Therefore, $\tilde{\phi}(0)=1$, $\tilde{\phi}(0)=0$, and for any $r \in]0,r_1]$

$$\tilde{\phi}''(r) + \left[\frac{d+1}{r[1+\eta(r)^2]} + \frac{2}{r} - \beta g(r)\right]\tilde{\phi}'(r) - \beta\frac{g(r)}{r}\tilde{\phi}(r) = 0. \qquad (17)$$

This function $\tilde{\phi}$ can be continued on the whole $\mathbb{R}_+$ into a function $\tilde{\phi}$ satisfying still Eqs. (16) and (17). Set now $\zeta_2(r) \coloneqq r\tilde{\phi}(r)$. It is immediate that ζ_2 solves Eq. (14) on $\mathbb{R}_+$ and satisfies

$$\zeta_2(0) = 0, \quad \zeta_2'(0) = 1.$$

3. The Wronskian w_β of ζ_1, ζ_2

Consider the Wronskian

$$w_\beta \coloneqq \zeta_1\zeta_2' - \zeta_1'\zeta_2 \quad \text{on } \mathbb{R}_+^*.$$

We have

$$w_\beta' = \zeta_1\zeta_2'' - \zeta_1''\zeta_2 = \left(\beta g - \frac{d-1}{[1+\eta^2]Id}\right) \times w_\beta,$$

so that

$$w_\beta(r) = a_\beta\mu(r)^{1-d}e^{\beta G(r)} \quad \text{for any } r > 0$$

and for some constant a_β. As $\zeta_1 \geqslant 1$, $\zeta_2' > 0$, $\zeta_1' < 0$, and $\zeta_2 > 0$ near 0, we must have $a_\beta > 0$.

B. Constructing a solution ψ_β to Eq. (5) on $\mathbb{R}_+$

For any continuous function k on $\mathbb{R}_+$, such that $\lim_{r\to\infty}e^{-\varepsilon' r}k(r)=0$ for some $\varepsilon' < \beta\varepsilon/2$ and for any $0 < r < \infty$, set

$$\Psi(k)(r) := \zeta_1(r)\int_0^r \zeta_2(\rho)w_\beta(\rho)^{-1}k(\rho)\mathrm{d}\rho + \zeta_2(r)\int_r^\infty \zeta_1(\rho)w_\beta(\rho)^{-1}k(\rho)\mathrm{d}\rho.$$

Note that $\Psi(k)$ is well defined; we have indeed, using that $g \geqslant \varepsilon$ on $[r_0,\infty[$,

$$\int_{r_0}^\infty \zeta_1(\rho)w_\beta(\rho)^{-1}|k(\rho)|\mathrm{d}\rho = \mathcal{O}(1)\int_{r_0}^\infty \mu(\rho)^{1-d}e^{-\beta G(\rho)}e^{\beta\varepsilon\rho/2}\mathrm{d}\rho < \infty.$$

Note also that by $(\mathcal{H})$, we can take, in particular, $k=\beta h$ [recall formula (3) defining h]. Moreover, again for $0<r<\infty$, we have

$$\Psi(k)'(r) = \zeta_1'(r)\int_0^r \zeta_2(\rho)w_\beta(\rho)^{-1}k(\rho)\mathrm{d}\rho + \zeta_2'(r)\int_r^\infty \zeta_1(\rho)w_\beta(\rho)^{-1}k(\rho)\mathrm{d}\rho$$

and

$$\Psi(k)''(r) = \zeta_1''(r)\int_0^r \psi_2(\rho)w_\beta(\rho)^{-1}k(\rho)\mathrm{d}\rho + \zeta_2''(r)\int_r^\infty \zeta_1(\rho)w_\beta(\rho)^{-1}k(\rho)\mathrm{d}\rho - k(r),$$

so that $\Psi(k)$ solves Eq. (5) on $\mathbb{R}_+^*$, with k instead of βh. Near 0, we have $\zeta_2(r)\sim r$. Otherwise, noticing that $(\zeta_1/\zeta_2)'=-w_\beta/\zeta_2^2$, we have for $r>0$

$$\int_r^1 \frac{w_\beta}{\zeta_2^2} = \frac{\zeta_1}{\zeta_2}(r) - \frac{\zeta_1}{\zeta_2}(1), \quad \text{i.e.,} \quad \zeta_1(r) = \frac{\zeta_1}{\zeta_2}(1)\zeta_2(r) + \zeta_2(r)\int_r^1 \frac{w_\beta}{\zeta_2^2}.$$

Hence, near 0 we have $\zeta_1(r)\sim ra_\beta\int_r^1\mu(s)^{1-d}s^{-2}\mathrm{d}s \leqslant a_\beta\mu(r)^{1-d}$, and then $\Psi(k)(r)=\mathcal{O}(r)$. In particular, we have $\Psi(k)(0)=0$. Using $\zeta_1(s)=\mathcal{O}(\mu(s)^{1-d})$ in the expression of ζ_1' (recall Sec. V A 1), we get at once $|\zeta_1'(r)|=\mathcal{O}(\mu(r)^{1-d}/r)$ near 0. We have therefore near 0

$$|\zeta_1'(r)|\int_0^r \zeta_2(\rho)w_\beta(\rho)^{-1}k(\rho)\mathrm{d}\rho = \mathcal{O}(\mu(r)^{1-d}/r)\int_0^r \mu(\rho)^{d-1}\rho\mathrm{d}\rho = \mathcal{O}(r)$$

and

$$\zeta_2'(r)\int_r^\infty \zeta_1(\rho)w_\beta(\rho)^{-1}k(\rho)\mathrm{d}\rho = \mathcal{O}(1),$$

whence $\Psi(k)'(0)=\int_0^\infty\zeta_1(\rho)w_\beta(\rho)^{-1}k(\rho)\mathrm{d}\rho \in \mathbb{R}$. Using Eq. (5), we also get $\psi_\beta''(0)\in\mathbb{R}$.

Setting $\psi_\beta := \beta\Psi(h)$, we have thus $\psi_\beta \in C^2(\mathbb{R}_+)$, $\psi_\beta(0)=0$, and ψ_β solves Eq. (5) on $\mathbb{R}_+$.

C. Estimates for ψ_β and ψ_β' near ∞

Recall from Sec. V B that

$$\psi_\beta(r) = \beta\zeta_1(r)\int_0^r \zeta_2(\rho)w_\beta(\rho)^{-1}h(\rho)\mathrm{d}\rho + \beta\zeta_2(r)\int_r^\infty \zeta_1(\rho)w_\beta(\rho)^{-1}h(\rho)\mathrm{d}\rho$$

and

$$\psi_\beta'(r) = \beta\zeta_1'(r)\int_0^r \zeta_2(\rho)w_\beta(\rho)^{-1}h(\rho)\mathrm{d}\rho + \beta\zeta_2'(r)\int_r^\infty \zeta_1(\rho)w_\beta(\rho)^{-1}h(\rho)\mathrm{d}\rho.$$

Near infinity, we have on one hand $\zeta_1\sim 1$ and (by Sec. V A 1)

$$|\zeta_1'(r)| = \mathcal{O}(\mu(r)^{1-d})\int_r^\infty e^{-\beta\varepsilon(s-r)}\mu(s)^{d-1}s^{-2}\mathrm{d}s = \mathcal{O}(r^{-2})\int_0^\infty e^{-\beta\varepsilon s}(1+s/r)^{d-1}\mathrm{d}s = \mathcal{O}(r^{-2}).$$

On the other hand, by $(\mathcal{H})$ we also have near infinity

$$w_\beta(r) - \frac{2}{\beta\varepsilon}w_\beta'(r) = \left(1 - \frac{2}{\varepsilon}g(r) + \frac{2(d-1)}{\beta\varepsilon[1+\eta(r)^2]r}\right)w_\beta(r) < 0,$$

So that

$$\int_{r_0}^r w_\beta \leqslant \frac{2}{\beta\varepsilon}(w_\beta(r) - w_\beta(r_0)) = \mathcal{O}(w_\beta(r))$$

Noticing that $(\zeta_2/\zeta_1)' = w_\beta/\zeta_1^2$, we have for any $r>0$

$$\zeta_2(r) = \frac{\zeta_2}{\zeta_1}(1)\zeta_1(r) + \zeta_1(r)\int_1^r \frac{w_\beta}{\zeta_1^2}.$$

This implies that $\zeta_2(r) = \mathcal{O}(w_\beta(r))$ near infinity. Therefore, by $(\mathcal{H})$ there exists an $\varepsilon' < \varepsilon\beta/2$ such that

$$\int_0^r \zeta_2(\rho)w_\beta(\rho)^{-1}h(\rho)\mathrm{d}\rho = \int_0^r \mathcal{O}(e^{\varepsilon'\rho})\mathrm{d}\rho = \mathcal{O}(e^{\varepsilon' r}).$$

To control the second integral in the expressions for ψ_β and ψ_β', observe similarly that

$$\begin{aligned} w_\beta(r)\int_r^\infty \zeta_1(\rho)w_\beta(\rho)^{-1}h(\rho)\mathrm{d}\rho &= \mu(r)^{1-d}\int_r^\infty e^{-\beta[G(\rho)-G(r)]}\mu(\rho)^{d-1}\mathcal{O}(e^{\varepsilon'\rho})\mathrm{d}\rho \\ &= \mathcal{O}(e^{\varepsilon' r})\mu(r)^{1-d}\int_r^\infty e^{-\varepsilon\beta(\rho-r)/2}\mu(\rho)^{d-1}\mathrm{d}\rho \\ &= \mathcal{O}(e^{\varepsilon' r})\int_0^\infty e^{-\varepsilon\beta\rho/2}(1+\rho/r)^{d-1}\mathrm{d}\rho = \mathcal{O}(e^{\varepsilon' r}). \end{aligned}$$

Then by definition of w_β, we have $\zeta_2' = w_\beta/\zeta_1 + (\zeta_2/\zeta_1)\zeta_1'$, whence $\zeta_2' \sim w_\beta$ near infinity.

As a conclusion, gathering the above, we have indeed the following for some $\varepsilon' < \varepsilon\beta/2$: $|\psi_\beta(r)| = \mathcal{O}(e^{\varepsilon' r})$ and $|\psi_\beta'(r)| = \mathcal{O}(e^{\varepsilon' r})$ for large r.

D. We have $\beta\Psi(g) = Id$ on $\mathbb{R}_+$ and $0 \leqslant \psi_\beta \leqslant Id$ if $0 \leqslant f \leqslant b$

Set $\tilde{\psi}_\beta := \beta\Psi(g)$ and note that the identical function Id solves Eq. (5) if $g=h$. Hence, the function $r \mapsto \tilde{\psi}_\beta(r) - r$ solves the homogeneous equation [Eq. (14)], so that

$$\tilde{\psi}_\beta(r) - r = c\zeta_2(r) + c'\zeta_1(r) \quad \text{for some real constants } c,\ c' \text{ and for any } r > 0.$$

As $\tilde{\psi}_\beta(0) = 0$, we must have $c' = 0$. Then, as $\zeta_2(r) \sim \int_1^r w_\beta \geqslant e^{\beta G(r)/2} \geqslant r$ for large r, we must have $c \geqslant 0$. Otherwise, integrating by parts, near infinity we have

$$\beta\int_r^\infty \mu(\rho)^{d-1}e^{-\beta G(\rho)}g(\rho)\mathrm{d}\rho = \mu(r)^{d-1}e^{-\beta G(r)} + \mathcal{O}(r^{-1})\int_r^\infty e^{-\beta G(\rho)}\mu(\rho)^{d-1}\mathrm{d}\rho \sim \mu(r)^{d-1}e^{-\beta G(r)},$$

so that

$$\beta\int_r^\infty \zeta_1(\rho)\mu(\rho)^{d-1}e^{-\beta G(\rho)}g(\rho)\mathrm{d}\rho \sim \beta\int_r^\infty \mu(\rho)^{d-1}e^{-\beta G(\rho)}g(\rho)\mathrm{d}\rho \sim \mu(r)^{d-1}e^{-\beta G(r)}.$$

Hence, by definition of $\tilde{\psi}_\beta$ and Ψ and by the above, we have near infinity

$$\tilde{\psi}_\beta(r) = \mathcal{O}(1)\int_0^r g(\rho)\mathrm{d}\rho + \mathcal{O}(w_\beta(r))\mu(r)^{d-1}e^{-\beta G(r)} = \mathcal{O}(G(r)+1) = o(e^{\beta G(r)/2}),$$

whence

$$ce^{\beta G(r)/2} = o(c\zeta_2(r)) = o(\tilde{\psi}_\beta(r) - r) = o(e^{\beta G(r)/2}), \quad \text{which forces } c=0.$$

Therefore $\tilde{\psi}_\beta(r)=r$ on $\mathbb{R}_+$, as wanted. Finally, if $0\leqslant f\leqslant b$, then $0\leqslant h\leqslant g$, so that $\Psi(h)\geqslant 0$ and $\Psi(g-h)\geqslant 0$ by Sec. V B above, and then by linearity of Ψ, $Id=\beta\Psi(g)\geqslant\beta\Psi(h)=\psi_\beta$. □

VI. PROOF OF PROPOSITION 4

For $d=1$ (and any η), it is immediate from Eq. (5) that

$$\psi_\beta(r) = \beta\int_0^r\left[\int_\rho^\infty e^{-\beta G}h\right]e^{\beta G(\rho)}\mathrm{d}\rho,$$

whence in the DH case,

$$\psi_\beta(r) = \beta\int_0^r\left[\int_\rho^\infty e^{-\beta\sqrt{1+u^2}}\frac{u}{\sqrt{1+u^2}}\mathrm{d}u\right]e^{\beta\sqrt{1+\rho^2}}\frac{\mathrm{d}\rho}{\sqrt{1+\rho^2}} = \int_0^r\frac{\mathrm{d}\rho}{\sqrt{1+\rho^2}}.$$

Hence, in this case, using formula (13) and integrating by parts, we get

$$\Sigma_\beta^2 = 2\times\frac{\int_0^\infty \psi_\beta(r)e^{-\beta\sqrt{1+r^2}}(1+r^2)^{-1/2}r\mathrm{d}r}{\int_0^\infty e^{-\beta\sqrt{1+r^2}}\mathrm{d}r} = \frac{2}{\beta}\times\frac{\int_0^\infty e^{-\beta\sqrt{1+r^2}}(1+r^2)^{-1/2}\mathrm{d}r}{\int_0^\infty e^{-\beta\sqrt{1+r^2}}\mathrm{d}r} =: \frac{2}{\beta}\times\frac{J_\beta^1}{J_\beta^0},$$

where

$$J_\beta^j := \int_0^\infty e^{-\beta(\sqrt{1+r^2}-1)}(1+r^2)^{-j/2}\mathrm{d}r.$$

A. Behavior as $\beta\to\infty$

Setting $x=\beta(\sqrt{1+r^2}-1)$, we get

$$J_\beta^j = \beta^{-1/2}\int_0^\infty e^{-x}\left(\frac{x}{\beta}+1\right)^{1-j}\frac{\mathrm{d}x}{\sqrt{(2+x/\beta)x}} \sim \beta^{-1/2}\int_0^\infty e^{-x}\frac{\mathrm{d}x}{\sqrt{2x}} = \sqrt{\frac{\pi}{2\beta}}$$

by dominated convergence. Hence $\Sigma_\beta^2\sim 2/\beta$.

B. Behavior as $\beta\to 0$

We have on one hand, by dominated convergence,

$$J_\beta^0 = \beta^{-1}\int_0^\infty e^{-x}\sqrt{\frac{x}{x+2\beta}}\mathrm{d}x + \int_0^\infty \frac{e^{-x}\mathrm{d}x}{\sqrt{x(x+2\beta)}} = \beta^{-1}[1+\mathcal{O}(1)] + J_\beta^1.$$

On the other hand, we have

$$J_\beta^1 = \int_0^\infty \frac{e^{-x}\mathrm{d}x}{\sqrt{x(x+2\beta)}} = 2\int_0^\infty \frac{e^{-\beta u^2}\mathrm{d}u}{\sqrt{u^2+2}} = 2\left[\int_0^1 \frac{e^{-\beta u^2}\mathrm{d}u}{\sqrt{u^2+2}} + \int_0^{1/\sqrt{\beta}} \frac{e^{-1/t^2}\mathrm{d}t}{t\sqrt{1+2\beta t^2}}\right],$$

where we performed the change of variable $\beta u^2 = 1/t^2$. Hence, as $\beta \to 0$, we have

$$\begin{aligned} J_\beta^1 &= 2\left[\int_0^1 \frac{e^{-\beta u^2}\mathrm{d}u}{\sqrt{u^2+2}} + \int_0^1 \frac{e^{-1/t^2}\mathrm{d}t}{t\sqrt{1+2\beta t^2}} + \int_1^{1/\sqrt{\beta}} \frac{\mathrm{d}t}{t\sqrt{1+2\beta t^2}} + \int_1^{1/\sqrt{\beta}} \frac{(e^{-1/t^2}-1)\mathrm{d}t}{t\sqrt{1+2\beta t^2}}\right] \\ &= 2\int_1^{1/\sqrt{\beta}} \frac{\mathrm{d}t}{t\sqrt{1+2\beta t^2}} + C_1 + o(1) \end{aligned}$$

for a constant C_1. Integrating by parts then yields

$$\int_1^{1/\sqrt{\beta}} \frac{\mathrm{d}t}{t\sqrt{1+2\beta t^2}} = \left[\frac{\log t}{\sqrt{1+2\beta t^2}}\right]_1^{1/\sqrt{\beta}} + 2\beta\int_1^{1/\sqrt{\beta}} \frac{t\log t\mathrm{d}t}{(1+2\beta t^2)^{3/2}}.$$

Setting $u = \sqrt{\beta}t$, we get

$$\begin{aligned} \int_1^{1/\sqrt{\beta}} \frac{\mathrm{d}t}{t\sqrt{1+2\beta t^2}} &= \frac{1}{\sqrt{3}}\log\left(\frac{1}{\sqrt{\beta}}\right) + 2\int_{\sqrt{\beta}}^1 \frac{u\log(u/\sqrt{\beta})\mathrm{d}u}{(1+2u^2)^{3/2}} \\ &= \frac{1}{\sqrt{3}}\log\left(\frac{1}{\sqrt{\beta}}\right) + 2\log\left(\frac{1}{\sqrt{\beta}}\right)\int_{\sqrt{\beta}}^1 \frac{u\mathrm{d}u}{(1+2u^2)^{3/2}} + 2\int_{\sqrt{\beta}}^1 \frac{u\log u\mathrm{d}u}{(1+2u^2)^{3/2}}. \end{aligned}$$

Now, as $\beta \to 0$ we have

$$\int_{\sqrt{\beta}}^1 \frac{u\mathrm{d}u}{(1+2u^2)^{3/2}} = \left[\frac{-1}{2\sqrt{1+2u^2}}\right]_{\sqrt{\beta}}^1 = \frac{1}{2}\left(\frac{1}{\sqrt{1+2\beta}} - \frac{1}{\sqrt{3}}\right) \to \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right),$$

and

$$2\int_{\sqrt{\beta}}^1 \frac{u\log u\mathrm{d}u}{(1+2u^2)^{3/2}} \to C_2 := 2\int_0^1 \frac{u\log u\mathrm{d}u}{(1+2u^2)^{3/2}} \in \mathbb{R}_-.$$

Hence,

$$\int_1^{1/\sqrt{\beta}} \frac{\mathrm{d}t}{t\sqrt{1+2\beta t^2}} = \log(1/\sqrt{\beta}) \times [1+o(1)] + C_2 \quad \text{and } J_\beta^1 \sim \log(1/\beta).$$

Finally,

$$\Sigma_\beta^2 = \frac{2}{\beta} \times \frac{J_\beta^1}{J_\beta^0} = \frac{2J_\beta^1}{1+o(1)+\beta J_\beta^1} \sim 2\log(1/\beta).$$

□

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