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Recursions for compound phase distributions

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Abstract

In this paper, we study a simple recursion procedure for a compound phase-type distribution. This procedure is only based on the rationality of the characteristic function of the phase-type distribution of the claim number.

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1. Introduction

Since Panjer's (1981) well-known recursion scheme (Panjer, 1981) for the distribution of a compound variable

$$S = \sum_{t=1}^{T} Y_t, \tag{1.1}$$

where $(Y_t)_{t\geq 0}$ are positive iid variables, independent of T, the number variable which is supposed to satisfy Panjer's relation

$$\mathbb{P}_T(t) = \left(a + \frac{b}{t}\right)\mathbb{P}_T(t-1), \quad t = 1, 2, \ldots, \tag{1.2}$$

a huge number of extensions and modifications appeared, see the survey article (Sundt, 2002), (Klugman et al., 1998, Chapter 4) or (Rolski et al., 2000, Chapter 4). (1.2) is known to be equivalent to T being either binomial, negative binomial or Poisson. In its simplest equidistant form, the recursion reads as

$$\mathbb{P}_S(t) = \sum_{u=1}^{t} \mathbb{P}_S(t-u)\mathbb{P}_Y(u)\left(a + b\frac{u}{t}\right), \quad t \geq 1. \tag{1.3}$$

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Important generalizations of (1.2) are given by Sundt (1992, 2003), for example the case where (1.2) is replaced by

$$\mathbb{P}_T(t) = \sum_{s=1}^{r\wedge t} \left(a_s + \frac{b_s}{t}\right) \mathbb{P}_T(t-s), \quad t \geq 1. \tag{1.4}$$

In this case, the recursion procedure is

$$\mathbb{P}_S(t) = \sum_{u=1}^{t} \mathbb{P}_S(t-u) \sum_{s=u}^{r\wedge t} \mathbb{P}_Y^{*s}(u) \left(a_s + \frac{b_s}{s}\frac{u}{t}\right), \quad t \geq 1. \tag{1.5}$$

In Hess et al. (2002), (1.2) and (1.3) are extended to Panjer distributions of order k, i.e. (1.2) holds only for $t \geq k$. New interesting distributions appear. Multivariate extensions can be found by Hesselager (1996) and Sundt (1999).

In the context with phase distributions, Hipp (2004) studied the case where the claim size variables Y_t are of phase-type (in the sequel simply called phase variables), assuming T to satisfy Panjer's relation (1.2). He showed that in this case the recursion scheme (1.3) is of a local character. This means in the discrete case, that at level t only the values $\mathbb{P}_S(t-u)$ for $u = 1, \ldots, d \wedge t$ are involved, where d is the dimension of the phase variables Y_t. In the continuous case, the usual convolution integral can be replaced by two linear differential equations of order d.

Here, we look at the different case where T is a phase variable while the Y_t continue to be arbitrary positive random variables, discrete or continuous. We find recursion formulas which involve convolutions of Y up to order d, the dimension of the phase variable T. They are based on the rationality of the generating function of T; no additional relation like (1.2) is needed.

In Section 2 we introduce the necessary notations and properties of phase variables. The recursion schemes are given in Section 3.

2. Phase variables

A phase distribution is defined as the distribution of the survival time of a time-homogeneous Markov chain with a finite state space having an attracting and absorbing state. The phase distribution may be discrete, having values in $\mathbb{N}_0 = \{0, 1, \ldots\}$ or continuous with values in $[0, \infty)$.

In the discrete case, we have a timely homogeneous Markov chain $(X(t))_{t\in\mathbb{N}_0}$ with state space $\tilde{D} = \{0, 1, 2, \ldots, d\}$, a transition matrix $\tilde{P} = (\tilde{P}_{jk})_{0\leq j,k\leq d}$ and a starting distribution $\tilde{\pi} = (\pi_0, \pi_1, \ldots, \pi_d) = (\pi_0, \pi)$, where we assume that the state $0 \in \tilde{D}$ is as well absorbing as attracting: i.e.

(i) $\tilde{P}_{0j} = 0$ for all $j \in D := \{1, 2, \ldots, d\}$ (absorption at 0),
(ii) there exists a power τ of the transition matrix $\tilde{P}$ such that $\tilde{P}^{\tau}_{j0} > 0$ for all $j \in D$ (attraction of 0).

According to the Perron–Frobenius theorem, these conditions imply that the maximal real positive eigenvalue $\lambda_0 = 1$ of $\tilde{P}$ is simple and the submatrice $P = (P_{jk})_{1\leq j,k\leq d}$ is submarkovian having a real maximal eigenvalue $\lambda_1 < 1 : |\lambda| \leq \lambda_1$ for all eigenvalues of P, which are also the eigenvalues of $\tilde{P}$.

The phase variable T is defined as the 'life time' of the Markov process $X(t)$, interpreting the attracting and absorbing state 0 as 'cemetery' of the process:

$$T = \min\{t \in \mathbb{N}_0, X(t) = 0\}. \tag{2.1}$$

The distribution of T is the discrete phase distribution

$$\mathbb{P}_T(t) = \mathbb{P}(T = t). \tag{2.2}$$

$\mathbb{P}_T$ is described by the characteristics (D, P, π) of the underlying Markov process, since we know

$$\tilde{P}_{0j} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{else} \end{cases}, \qquad \tilde{P}_{j0} = 1 - \sum_{k=1}^{d} P_{jk} \quad \text{and} \quad \pi_0 = 1 - \sum_{k=1}^{d} \pi_k.$$

We have

$$\mathbb{P}_T(0) = \pi_0 \tag{2.3}$$

and since $P^t_{jk} = \mathbb{P}(X(t) = k|X(0) = l)$, we find

$$\mathbb{P}_T(\{s \in \mathbb{N}_0, s \geq t+1\}) = \pi P^t \eta, \tag{2.4}$$

where

$$\eta = (1, 1, \ldots, 1) \in \mathbb{R}^d. \tag{2.5}$$

Therefore, for $t \geq 1$

$$\mathbb{P}_T(t) = \pi P^{t-1}(Id - P)\eta. \tag{2.6}$$

The generating function $z \mapsto \Phi_T(z) = \mathbb{E}(z^T)$ of a discrete phase distribution is given by

$$\Phi_T(z) = \pi_0 + \sum_{t=1}^{\infty} z^t(\pi P^{t-1}(Id - P)\eta) = \pi_0 + \pi(Idz^{-1} - P)^{-1}(Id - P)\eta \tag{2.7}$$

and well defined for $z \in \mathbb{C}$, $|z| < 1/\lambda_1$ (see Hipp, 2004). By Cramer's rule, the elements A^{-1}_{kl} of the inverse of an invertible matrix A can be calculated by

$$A^{-1}_{kl} = (-1)^{k+l} \frac{\det(A(l,k))}{\det(A)},$$

where $A(l,k)$ denotes the adjoint matrix which we get by erasing the lth line and the kth row of A. Applying this result to the matrix $(Idz^{-1} - P)$, we see that

$$(Idz^{-1} - P)^{-1}_{kl} = (-1)^{k+l} \frac{(z^{-(d-1)} + a_2(k,l)z^{-(d-2)} + \cdots + a_d(k,l))}{\det(Idz^{-1} - P)} = (-1)^{k+l} \frac{(z + a_2(k,l)z^2 + \cdots + a_d(k,l)z^d)}{1 + b_1 z^1 + \cdots + b_d z^d},$$

where $\det(Idy - P) = y^d + b_1 y^{d-1} + \cdots + b_d$ is the characteristic polynomial of P. It has the coefficients

$$b_i = (-1)^i \sum_{1 \leq k_1 < k_2 < \cdots < k_{d-i} \leq d} \det(P(k_1, k_2, \ldots, k_{d-i})), \tag{2.8}$$

where $P(k_1, k_2, .., k_{d-i})$ denotes the matrix which we get by suppressing the k_1th, k_2th, up to the k_{d-i}th rows and columns. In particular

$$b_1 = -\mathrm{trace}(P) \quad \text{and} \quad b_d = (-1)^d \det(P).$$

Therefore, the generating function $\Phi_T(z)$ is rational and may be written as

$$\Phi_T(z) = \pi_0 + \frac{a_1 z + \cdots + a_d z^d}{1 + b_1 z + \cdots + b_d z^d} \tag{2.9}$$

(see also Asmussen, 2000, Proposition 6.1 or Hipp, 2005, Theorem 3). Since the coefficients b_i are given by (2.8), we can get the coefficients a_i by the following recursion scheme.

Proposition 2.1. *We have in* (2.9)

$$a_1 = \mathbb{P}_T(1) \tag{2.10}$$

and recursively for $i = 2, \ldots, d$

$$a_i = \mathbb{P}_T(i) + \sum_{l=1}^{i-1} b_l \mathbb{P}_T(i-l). \tag{2.11}$$

Proof. Rewriting (2.9), we get

$$\Phi_T(z) = \pi_0 + \sum_{i=1}^{d} a_i z^i + \sum_{i=1}^{d} b_i z^i (\pi_0 - \Phi_T(z))$$

and since $\mathbb{P}_T(0) = \pi_0$

$$\sum_{t\geq 1} \mathbb{P}_T(t) z^t = \sum_{i=1}^{d} a_i z^i - \sum_{i=1}^{d} b_i z^i \left(\sum_{t\geq 1} \mathbb{P}_T(t) z^t \right) = \sum_{i=1}^{d} a_i z^i - \sum_{t\geq 2} z^t \left(\sum_{i=1}^{d\wedge(t-1)} b_i \mathbb{P}_T(t-i) \right).$$

Comparing the first d coefficients gives (2.10) and (2.11). □

Example 2.2.

(i) Let $\tilde{D} = \{0,\ 1\}$, $\tilde{\pi} = (\pi_0, 1 - \pi_0)$ and

$$\tilde{P} = \begin{pmatrix} 1 & 0 \\ 1-p & p \end{pmatrix}$$

with $p \in (0, 1)$. Evidently $\zeta_P(y) = y - p$ and $b_1 = -p$. (2.6) is now

$$\mathbb{P}_T(t) = \begin{cases} \pi_0 & \text{for } t = 0, \\ (1-\pi_0)(1-p)p^{t-1} & \text{for } t \geq 1. \end{cases}$$

Moreover,

$$a_1 = (1 - \pi_0)(1 - p),$$

such that

$$\Phi_T(z) = \pi_0 + \frac{(1-\pi_0)(1-p)z}{1-pz}. \tag{2.12}$$

In the case $\pi_0 = 1 - p$, we have for $t \geq 0$

$$\mathbb{P}_T(t) = (1-p)p^t \quad \text{and} \quad \Phi_T(z) = \frac{1-p}{1-pz}.$$

This is the negative binomial distribution with shape parameter $\alpha = 1$ (i.e. the geometric distribution starting at 0).

(ii) Set $\tilde{D} = \{0, 1, 2\}$, $\tilde{\pi} = (\pi_0, \pi_1, \pi_2)$ and

$$\tilde{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p_1 & 1-p_1 \\ 1-p_2 & 0 & p_2 \end{pmatrix}$$

with $p_1, p_2 \in (0, 1)$. The characteristic polynomial is

$$\zeta_P(y) = (y - p_1)(y - p_2) = y^2 - (p_1 + p_2) + p_1 p_2,$$

such that $b_1 = -(p_1 + p_2)$ and $b_2 = p_1 p_2$. Since for $t \geq 1$

$$P^t = \begin{pmatrix} p_1^t & (1-p_1)\sum_{l=0}^{t-1} p_1^{t-1-l} p_2^l \\ 0 & p_2^t \end{pmatrix},$$

we find

$$\mathbb{P}_T(0) = \pi_0, \qquad \mathbb{P}_T(1) = \pi_2(1 - p_2)$$

and for $t \geq 2$

$$\mathbb{P}_T(t) = \pi_1(1-p_1)[p_1^{t-1} + (1-p_1)\sum_{l=0}^{t-2} p_1^{t-2-l} p_2^l - p_2^{t-1}] + \pi_2(1-p_2)p_2^{t-1}$$
$$= \pi_1(1-p_1)\left[\sum_{l=0}^{t-2} p_1^{t-2-l} p_2^l - p_2^{t-1} - \sum_{l=1}^{t-1} p_1^{t-2-l} p_2^l\right] + \pi_2(1-p_2)p_2^{t-1}$$
$$= \pi_1(1-p_1)(1-p_2)\sum_{l=0}^{t-2} p_1^{t-2-l} p_2^l + \pi_2(1-p_2)p_2^{t-1}.$$

The coefficients in (2.9) are now

$$a_1 = \pi_2(1-p_2) \quad \text{and}$$
$$a_2 = \pi_1(1-p_1)(1-p_2) + \pi_2(1-p_2)p_2 - \pi_2(p_1+p_2)(1-p_2) = \pi_1(1-p_1)(1-p_2) - \pi_2 p_1(1-p_2),$$

such that

$$\Phi_T(z) = \pi_0 + \frac{\pi_2(1-p_2)z + [\pi_1(1-p_1)(1-p_2) - \pi_2 p_1(1-p_2)]z^2}{(1-p_1 z)(1-p_2 z)}.$$

In the special case $p_1 = p_2 = p$, $\pi_0 = (1-p)^2$, $\pi_1 = 2(1-p)^2 p$ and $\pi_2 = 2p$, we find – setting $\alpha = 2$

$$\Phi_T(z) = \left(\frac{1-p}{1-pz}\right)^{\alpha}$$

and

$$\mathbb{P}_T(t) = (t+1)(1-p)^{\alpha} p^t = \binom{\alpha - 1 + t}{t}(1-p)^{\alpha} p^t,$$

which is the negative binomial distribution with shape parameter $\alpha = 2$, for which Panjer's recursion holds for general α (see also Remark 3.3(ii)).

3. Compound phase variables

As in the situation of Panjer's recursion, we are now studying compound variables S with a frequency phase distribution:

$$S = \sum_{t=1}^{T} Y_t, \tag{3.1}$$

where T is a discrete phase variable with characteristics (D, P, π) and $(Y_t)_{t\geq 1}$ are iid variables with values in $(0, \infty)$ and distribution $\mathbb{P}_Y$.

3.1. Discrete case

We first assume that the Y_i are discrete, i.e. with values at $\mathbb{N} = \{1, 2, \ldots\}$. Similar to Panjer's recursion for compound equidistant claims, we get the following recursion for the distribution of S.

Proposition 3.1. *We have*

$$\mathbb{P}_S(0) = \pi_0 \tag{3.2}$$

and for $t \geq 1$

$$\mathbb{P}_S(t) = \sum_{j=1}^{d\wedge t} a_j \mathbb{P}_Y^{*j}(t) - \sum_{j=1}^{d\wedge(t-1)} b_j \left(\sum_{u=1}^{t-1} \mathbb{P}_S(u)\mathbb{P}_Y^{*j}(t-u) \right). \tag{3.3}$$

Here, $\mathbb{P}_Y^{*j}$ *is the distribution of the jth convolution of Y.*

Proof. (3.2) is evident. The generating function of the compound phase variable is known to be

$$\Phi_S(z) = \Phi_T(\Phi_Y(z)) = \pi_0 + \frac{\sum_{j=1}^d a_j \Phi_Y^j(z)}{1 + \sum_{j=1}^d b_j \Phi_Y^j(z)},$$

therefore

$$\Phi_S(z) = \pi_0 + \sum_{j=1}^d a_j \Phi_Y^j(z) - \sum_{j=1}^d b_j(\Phi_S(z) - \pi_0)\Phi_Y^j(z).$$

Using (3.2) and the fact that $\Phi_Y^j(z) = \Phi_{Y*j}(z)$, we get

$$\sum_{t\geq 1} z^t \mathbb{P}_S(t) = \sum_{t\geq 1} z^t \left(\sum_{j=1}^d a_j \mathbb{P}_Y^{*j}(t) \right) - \sum_{t\geq 1} z^t \left(\sum_{u=1}^{t-1} \mathbb{P}_S(u) \left(\sum_{j=1}^d b_j \mathbb{P}_Y^{*j}(t-u) \right) \right)$$
$$= \sum_{t\geq 1} z^t \left[\sum_{j=1}^{d\wedge t} a_j \mathbb{P}_Y^{*j}(t) - \sum_{j=1}^{d\wedge(t-1)} b_j \left(\sum_{u=1}^{t-1} \mathbb{P}_S(u)\mathbb{P}_Y^{*j}(t-u) \right) \right].$$

Comparison of coefficients gives (3.3). □

In the recursion procedure, of course, the computation of $\mathbb{P}_S(t)$, $\mathbb{P}_Y^{*j}(t)$ and the last sum can be done in one scheme. Using for simplicity the convention that $\sum_{l=m}^n(\cdots) = 0$ whenever $n < m$, we get the following proposition.

Proposition 3.2.

$$\mathbb{P}_S(0) = \pi_0$$

and recursively for $t \geq 1$

$$\mathbb{P}^{*1}(t) = \mathbb{P}_Y(t), \quad \text{for } k = 2, \ldots, d,$$

$$\mathbb{P}^{*k}(t) = \sum_{l=k-1}^{t-1} \mathbb{P}^{*k-1}(l)\mathbb{P}_Y(t-l) \quad \text{and} \quad \mathbb{P}^{\circ k-1}(t) = \sum_{l=k-1}^{t-1} \mathbb{P}^{*k-1}(l)\mathbb{P}_S(t-l),$$

$$\mathbb{P}^{\circ d}(t) = \sum_{l=d}^{t-1} \mathbb{P}^{*d}(l)\mathbb{P}_S(t-l), \qquad \mathbb{P}_S(t) = \sum_{i=1}^d a_i \mathbb{P}^{*i}(t) - \sum_{i=1}^d b_i \mathbb{P}^{\circ i}(t).$$

Remark 3.3.

(i) Since a phase variable T has a rational generating function (2.9), it follows that $\frac{d}{dz}\ln(\Phi_T(z))$ can be expressed as the quotient of two polynomials, a polynomial of degree $\leq 2d-1$ in the nominator and one of degree $2d$ and with constant term 1 in the denominator. By Sundt (2002, Theorem 3.2), T satisfies the relation (1.4) with $r = 2d$, and a recursion procedure like (1.5) follows. In general, this procedure is more complicated than (3.3).

(ii) The condition of Theorem 3.2. (Sundt, 2002), i.e.

$$\frac{d}{dz}\ln(\Phi_{\tilde{T}}(z)) \text{ is rational} \tag{3.4}$$

is satisfied for the composition

$$\tilde{T} = \sum_{t=1}^{N} T_t \tag{3.5}$$

of a Panjer variable N, satisfying (1.2), with iid phase variables $(T_t)_{t\geq 1}$. Thus, $\tilde{T}$ admits a recursion principle. By this way, we find a unified procedure for Panjer variables and phase variables which nevertheless rests different from (3.3). For simple Panjer variables, (1.3) is the easier procedure, while for simple phase variables, (3.3) seems to be the natural one.

3.2. Continuous case

Now, we assume $\mathbb{P}_Y$ to be an absolutely continuous loss distribution on $(0, \infty)$:

$$\mathbb{P}_Y(\mathrm{d}y) = g_Y(y)\,\mathrm{d}y. \tag{3.6}$$

Evidently, the distribution $\mathbb{P}_S$ of the compound phase variable S from (3.1) has a discrete part at 0 and a continuous part on $(0, \infty)$:

$$\mathbb{P}_S(\mathrm{d}y) = \pi_0\delta_0(\mathrm{d}y) + g_S(y)\,\mathrm{d}y. \tag{3.7}$$

Proposition 3.4. *The* (*non-normalized*) *density g_S of S on* $(0, \infty)$ *satisfies*

$$g_S(y) = \sum_{j=1}^{d} a_j g_Y^{*j}(y) - \sum_{j=1}^{d} b_j \int_0^y g_S(u) g_Y^{*j}(y-u)\,\mathrm{d}u, \tag{3.8}$$

where again g_Y^{*j} denotes the jth convolution of g_Y.

Proof. The Laplace transform of S (with argument $-r$)

$$\Psi_S(r) = \mathbb{E}(\exp(rS)) = \pi_0 + \int_0^\infty e^{ry} g_S(y)\,\mathrm{d}y$$

satisfies for $r \leq 0$

$$\Psi_S(r) = \Phi_T(\Psi_Y(r)).$$

Using (2.9), we get

$$\Psi_S(r) = \pi_0 + \frac{\sum_{j=1}^{d} a_j \Psi_Y(r)^j}{1 + \sum_{j=1}^{d} b_j \Psi_Y(r)^j},$$

such that

$$\Psi_S(r) - \pi_0 = \sum_{j=1}^{d} a_j \Psi_Y(r)^j - \sum_{j=1}^{d} b_j \Psi_Y(r)^j (\Psi_S(r) - \pi_0),$$

which is equivalent to

$$\int_0^\infty e^{ry} g_S(y)\,\mathrm{d}y = \sum_{j=1}^{d} a_j \int_0^\infty e^{ry} g_Y^{*j}(y)\,\mathrm{d}y - \sum_{j=1}^{d} b_j \int_0^\infty e^{ry} \left(\int_0^y g_Y^{*j}(y-u) g_S(u)\,\mathrm{d}u \right) \mathrm{d}y.$$

Since this holds for all $r \leq 0$, we get (3.8). □

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