

# MULTIPERIOD INSURANCE SUPERVISION: TOP-DOWN MODELS<sup>†</sup>

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ABSTRACT. We describe a top-down procedure for the supervisory accounting of insurance companies with special emphasis on market impacts. The technical tools are a multiperiod risk assessment, a market consistent best estimate and an eligible asset. First, to avoid supervisory arbitrage by financial market instruments, the risk assessment is bounded by a market consistent best estimate. Applied to the risk bearing capital, i.e. asset value minus best estimate of obligations, the risk assessment immediately gives the free capital which has to be positive for acceptability.

Next, optimal hedging of the obligation process by suitable asset portfolios yields the supervisory provision as the minimal initial value of a portfolio acceptable with respect to the given obligations. The problem to attain this minimal value leads to the definition of an optimal replicating portfolio.

A further task of supervision is the determination of the “Fremd”-capital in the supervisory balance sheet. This is formalized by the cost-of-capital method, i.e. a fictitious standardized transfer of the obligations to new investors on the market. The regulated price of such a transfer leads to the technical provision and the risk margin as “Fremd”-capital items.

Finally, the additional financial risks within the insurance’s real asset portfolio are taken care of by the solvency capital requirement defined as the minimal acceptable “Eigen”-capital for a given business plan. It measures the adequacy or inadequacy of the trading risks incorporated in the portfolio with respect to the obligation risks. An optimal replicating portfolio is characterized by a minimal solvency capital requirement.

Solvency II and the Swiss Solvency Test (SST) are defined as bottom-up models. In the forthcoming paper Eisele and Artzner (2011), we shall show how bottom-up and top-down models can be made congruent.

## INTRODUCTION

In this paper we give a construction of a multiperiod supervisory accounting, based on a time consistent family of risk assessment functionals, in order to formalize top-down models permitting to define notions like those presented in Solvency II and the Swiss Solvency Test (SST) which are bottom-up models. In particular, the notion of best estimates of obligations is used and natural derivations of technical provision, risk margin, and solvency capital requirement are given out of the risk assessment functionals. A cornerstone in this development is the concept of supervisory provision of obligations as the minimal asset value of a portfolio, acceptable with the existing obligations. This new concept of supervisory accounting, which has been introduced in the one-period case by Artzner and Eisele (2010 a) but does not show up in bottom-up models, provides the definition of an optimal replicating portfolio. The challenge consists in the interesting connection of the top-down constructions with those of Solvency II and SST to be presented in Eisele and Artzner (2011).

Three technical tools are used for our construction: an eligible asset, a market consistent best estimate and a multiperiod risk assessment. In addition an assumption on the cost-of-capital ratio is used.

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*Key words and phrases.* Cost of capital, “Fremd”- versus “Eigen”-capital (own funds), optimal replicating portfolio, regulated market, risk margin, solvency capital requirement, time consistency.

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The eligible asset represents a specific benchmark investment which from the supervisor's point of view is risk-free. For simplicity, we use the eligible asset also as a unit of account and as a numeraire.

It is now well agreed upon to use a market risk neutral probability for a best estimate of obligation cash-flows since it provides a market consistent evaluation. The third tool, a multiperiod risk assessment, is a time consistent family of cash-invariant, lower semi-continuous functionals on the space of discrete time processes. We do not assume any convexity or coherence property.

Any supervision has to face the existence of a financial market. From the very beginning, the influence of zero-cost portfolios onto the supervisory risk assessment must be controlled in order to avoid supervisory arbitrage possibilities. Interweaving market and supervisory assessments, we suppose that the supervisory risk assessment is bounded by the best estimate. A comparable condition is to be found already in Artzner et al. (1999), Section 4.3.

Further market impacts on supervision include financial risks due to the asset portfolio, the opportunity to find new investors ready to agree to risky situations in a regulated market, and, thirdly, the hedging of financial risks explicit or hidden in the obligations. Supervisory accounting deals with these three effects in reversed order: Eliminating optimally financial risks within the obligations leads to supervisory provision and to an optimal replicating portfolio. The possibility of finding new investors is the base of the transfer principle with the cost-of-capital method, which will define the "Fremd"-capital of the supervisory balance sheet (Subsection 5.2). Finally, the inadequacy between financial risks due to the company's asset strategy and the underwriting risks is taken care of by the solvency capital requirement representing the "gebundene Eigen"-capital<sup>1</sup> (see Definition 6.2 and Artzner and Eisele (2010 a)).

An insurance company business plan is given by its cash-flow process of obligations and a self-financing asset portfolio (see condition (3.3)). It is understood that obligations are exogenously given while there is room for rebalancing of the portfolio. The investment of a net asset position in instruments other than the eligible asset creates "trading-risk exposures" which can be regarded as control variables of the asset strategy. Applying the risk assessment to the process of the risk bearing capital, i.e. the process of asset values minus the best estimate of the remaining obligations, defines the company's free capital. A company is acceptable if its free capital is positive.

The first layer in the supervisory building is the supervisory provision. As presented in Artzner and Eisele (2010 a and b) for the one-period case, supervisory provision generated by obligations is the minimal initial asset value of a business plan providing a positive free capital (see Definition 4.1). It predates considerations of any asset portfolio choice and is "market consistent" according to the formalization of Cheridito et al. (2008). The supervisory margin, defined as the difference between supervisory provision and best estimate of the obligations, is (up to signs) the result of the market consistent majorant of the risk assessment (see Theorem 2.1) applied to the martingale part of the process of best estimates.

If an asset portfolio exists which yields an acceptable business plan and whose "not-free" part of the initial asset value equals the supervisory provision, it is called an "optimal replicating portfolio" (see Definition 4.2).

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<sup>1</sup>required own funds.

As already seen in the one-period case (see Artzner and Eisele (2010 a)), optimal replicating portfolios do not always exist. An additional condition on the risk assessment, called “market prudence”, is sufficient to guaranty the existence of optimal replicating portfolios.

Besides the decision of acceptability or not, a second task of supervisory accounting is to analyze the conditions of a possible transfer of the obligations, and thereby to define the “Fremd”-capital. For this fictitious transfer to a new, so far empty, company, it is assumed that the initial asset value of the new company is equal to the supervisory provision and a fortiori its portfolio must be optimal replicating for acceptability. Since the supervisory provision is in general considerably larger than the best estimate of obligations, it follows that there remain non-negligible benefits in the end. This “option on final benefits” under a minimal business plan is sold in a regulated manner to the new company whose investment forms the “Eigen”-capital part of the supervisory provision. The remaining part paid by the old company in order to get rid of its obligations is “Fremd”-capital in the new company and defines the so-called “technical provision”.

The third task of supervision is to take care of the company’s trading-risk exposures by determining the “gebundene Eigen”-capital. This solvency capital requirement, as it is called, is defined in analogy to the supervisory provision and is minimal for the optimal replicating portfolio where it defines exactly the “Eigen”-capital part of the supervisory provision. For a general business plan, the excess of the solvency capital requirement above its minimal value for an optimal replicating portfolio is the relative solvency capital requirement. This measures the inadequacy of the company’s actual trading risk with respect to the underwriting risks: In fact, relative solvency capital requirement is zero if and only if the portfolio is an optimal replicating one.

## 1. A FINANCIAL MARKET

### 1.1 Probabilistic framework.

Let the time space be given by  $\mathbb{T} := \{0, 1, \dots, T\}$  for some  $T \in \mathbb{N}$ . Random variables and stochastic processes are defined on a finite filtered probability space  $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$  with  $\Omega$  the support of  $\mathbb{P}$ . Let  $L_t$  be the set of  $\mathcal{F}_t$ -measurable random variables and  $L = \prod_{t \in \mathbb{T}} L_t$  the set of adapted stochastic processes in the time space  $\mathbb{T}$ .  $L$  is a closed subspace of  $\mathbb{R}^{\mathbb{T} \times \Omega}$ . By  $L_t^+$  (resp.  $L^+$ ) we denote the space of all non-negative random variable  $X_t \geq 0$  in  $L_t$  (resp. of non-negative processes  $X \geq 0$  in  $L$ ).  $\bar{L}_t^+$  is the set of  $\mathcal{F}_t$ -measurable variables with values in  $[0, \infty]$ . The spaces  $L_t$  (resp.  $L$ ) represent random variables  $X_t$  (resp. processes  $X$ ) of financial amounts  $X_t$  at date  $t$  of date  $t$  money.

The following obvious short-hand writing will be useful: We write  $\mathbb{1}_{\leq t}$  instead of  $\mathbb{1}_{\{u, u \leq t\}}$ , similarly for expressions like  $\mathbb{1}_{=t}$  or  $\mathbb{1}_{< t}$ . Moreover, we combine this with stochastic processes and random variables: for example, if  $X \in L$  and  $Y_{t+1} \in L_{t+1}$  then  $X \mathbb{1}_{\leq t} + Y_{t+1} \mathbb{1}_{> t}$  is the process with values  $X_u$  at times  $u \leq t$  and the constant value  $Y_{t+1}$  at all times  $u > t$ .

### 1.2 Modeling a financial market.

A financial market is given by a family of asset processes  $(S^i)_{i=0, \dots, d}$  in  $L$  whose values of date  $t$  money are the random variables  $S_t^i \in L_t$ . Their current market prices are  $S_0^i$ .

We take  $S^0$  as “numéraire”, meaning simply that  $S_0^0 = 1$  and  $S_t^0(\omega) > 0$  for all  $\omega \in \Omega$  and  $t \in \mathbb{T}$ .

We suppose that the set of trading-risk neutral probabilities is not empty:  $\mathcal{M} \neq \emptyset$  where

$$\mathcal{M} := \left\{ \mathbb{Q} \mid \mathbb{E}_{\mathbb{Q}} \left[ S_{t+1}^i \frac{S_t^0}{S_{t+1}^0} \mid \mathcal{F}_t \right] = S_t^i, i = 1, \dots, d, 0 \leq t < T \right\}. \quad (1.1)$$

The set of flows of zero-cost portfolios in the period  $[t-1, t]$ ,  $t > 0$  is

$$N_t := \left\{ D_t = \sum_{i=0}^d S_t^i \cdot \xi_{t-1}^i \mid \sum_{i=0}^d S_{t-1}^i \cdot \xi_{t-1}^i = 0, \xi_{t-1}^i \in L_{t-1} \right\}. \quad (1.2)$$

We call  $D_t \in N_t$  a **trading-risk exposure** at  $t$  and  $N := \{0\} \times \prod_{t=1}^T N_t$  is the set of cash-flows processes of trading-risk exposures.

### 1.3 The eligible asset.

An “eligible asset”  $r \in L$  is a financial asset (or the value of a portfolio) which from the supervisor’s point of view provides a benchmark of a (relatively) risk-free investment. In particular,  $r_t$  is strictly positive and we normalize the eligible asset to  $r_0 = 1$ . There is no mathematical restriction in treating numéraire and eligible asset as equal:  $r = S^0$  (see Artzner and Eisele (2010 a)).

For  $s, t \in \mathbb{T}$ , we set  $r_{s,t} := r_t/r_s$ . The **saving process**  $r(s)$  in the eligible asset  $r$ , starting at  $s \in \mathbb{T}$ , is given by

$$r(s) := \frac{1}{r_s} r \mathbb{1}_{\geq s}. \quad (1.3)$$

Now we can introduce for a process  $X$  its increment process  $\Delta X$  by  $\Delta X_t := X_t - X_{t-1} \cdot r_{t-1,t}$  for  $t \in \mathbb{T}$ , with the convention  $\Delta X_0 = 0$ . The following relation between the value process  $X \in L$  and the increment process  $\Delta X$  holds:

$$X \mathbb{1}_{\geq t} = X_t \cdot r(t) + \sum_{u=t+1}^T \Delta X_u \cdot r(u). \quad (1.4)$$

Let  $\mathbb{Q}$  be a probability measure on  $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{T}})$ . A process  $X \in L$  is called a **discounted  $\mathbb{Q}$ -martingale** (resp. a **discounted  $\mathbb{Q}$ -supermartingale**) if and only if for all  $t < T$

$$\mathbb{E}_{\mathbb{Q}}[\Delta X_{t+1}/r_{t,t+1} \mid \mathcal{F}_t] = 0 \quad (\text{resp. } \leq 0). \quad (1.5)$$

By  $S_{\mathbb{Q}}$  we denote the space of all  $\mathbb{Q}$ -supermartingales.  $S_{\mathbb{Q}}$  is a closed cone in  $L$ . The uniform Doob decomposition (see Föllmer and Schied (2004), Theorem 9.20) says that

$$S_{\mathcal{M}} := \bigcap_{\mathbb{Q} \in \mathcal{M}} S_{\mathbb{Q}} = \{X, \exists D \in N, B \in L^+, B_0 = 0 : \Delta X = D - B\}. \quad (1.6)$$

## 2. GENERAL RISK ASSESSMENT

### 2.1 Time consistent risk assessment.

**Definition 2.1.** Given an eligible asset  $r$ , we understand by a (multiperiod) **risk assessment**  $\Psi = (\Psi_t)_{t \in \mathbb{T}}$  a family of functionals  $\Psi_t : L \rightarrow L_t$  with the following properties: For all  $X, X^{(n)}, Y \in L$ ,  $t \in \mathbb{T}$ ,  $Y_t \in L_t$ , and  $A \in \mathcal{F}_t$

- (i) *Final assessment:*  $\Psi_T = 0$ ,
- (ii) *Localization:*  $\Psi_t(X \cdot \mathbb{1}_{>t} \cdot \mathbb{1}_A) = \Psi_t(X \cdot \mathbb{1}_{>t}) \cdot \mathbb{1}_A$ ,
- (iii) *Monotonicity:* if  $X \leq Y$  then  $\Psi_t(X) \leq \Psi_t(Y)$ ,
- (iv) *Dependence on future values:* If  $X \mathbb{1}_{>t} = Y \mathbb{1}_{>t}$  then  $\Psi_t(X) = \Psi_t(Y)$ ,
- (v) *Cash invariance:*  $\Psi_t(X + Y_t \cdot r(t)) = \Psi_t(X) + Y_t$ ,

- (vi) *Lower semi-continuity:*  
If  $X^{(n)} \nearrow X$  in  $L$  for  $n \rightarrow \infty$ , then  $\Psi_t(X^{(n)}) \nearrow \Psi_t(X)$  for all  $t \in \mathbb{T}$ ,
- (vii) *Time consistency:*  
For all  $0 \leq s \leq t < T$   $\Psi_s(X) = \Psi_s(X \mathbb{1}_{\leq t} + \Psi_t(X) \cdot r(t) \mathbb{1}_{>t})$ .

*Remarks 2.1.*

- (i) Definition 2.1. does not contain any convexity or coherence condition.
- (ii) Since in this paper we regard only time consistent multiperiod risk assessments, we do not refer to this property explicitly. Similarly, we suppress the reference to the eligible asset  $r$ .
- (iii) The mathematical formulations of cash invariance and time consistency imply that the variables  $X_t$  represent cumulative amounts of money.

The following consequence of time consistency and cash invariance of a risk assessment  $\Psi$  will be useful:

**Lemma 2.1.** For  $s \leq t \leq T$  and  $Y \in L$ , we have

$$\Psi_s\left(\sum_{u \in \mathbb{T}} Y_u \cdot r(u)\right) = \Psi_s\left(\sum_{u \leq t} Y_u \cdot r(u) + \Psi_t\left(\sum_{u > t} Y_u \cdot r(u)\right) \cdot r(t) \mathbb{1}_{>t}\right). \quad (2.1)$$

## 2.2 Market consistent functionals.

Cheridito et al. (2008) give a formal definition of the much spoken about concept of “market consistency”, along the following lines:

**Definition 2.2.** Let  $S \subset L$  with  $S + S_{\mathcal{M}} = S$ . A family  $\Phi = (\Phi_t)_{t \in \mathbb{T}}$  of functionals  $\Phi_t : L \rightarrow L_t$  is called **market consistent** on  $S$  if for all  $X \in S$ ,  $t < u \leq T$ , and  $D \in N$

$$\Phi_t(X) = \Phi_t(X + D_u \cdot r(u)). \quad (2.2)$$

A simple example of market consistent functionals on  $L$  is given by the following definition:

**Definition 2.3.** *Best estimate*

Let  $\mathbb{Q} \in \mathcal{M}$ . Then for  $X \in L$   $\Psi_{\mathbb{Q},t}(X) := 0$  and

$$\Psi_{\mathbb{Q},t}(X) := \mathbb{E}_{\mathbb{Q}}[X_T/r_{t,T} | \mathcal{F}_t] \quad \text{for } t < T \quad (2.3)$$

is called **best estimate** of  $X$ .

We notice that  $\Psi_{\mathbb{Q},t}(X) \leq X_t$  for all  $X \in S_{\mathbb{Q}}$  and  $0 \leq t < T$ . It can be shown that any linear market consistent risk assessment  $\Psi$  with the following (discounted) time-shift property: for all  $t, u \in \mathbb{T}$ ,  $Y_u \in L_u$

$$\Psi_t(Y_u \cdot r(u)) = \Psi_t(Y_u \cdot r_{u,T} \cdot \mathbb{1}_{=T}) \quad (2.4)$$

is of the form (2.3) with some  $\mathbb{Q} \in \mathcal{M}$ .

## 2.3 Market arbitrage of a risk assessment.

**Definition 2.4.** Let  $S \subset L$  with  $S + S_{\mathcal{M}} = S$ , and  $\Phi = (\Phi_t)_{t \in \mathbb{T}}$  a family of functionals  $\Phi_t : L \rightarrow L_t$ .

- (i) We say that  $\Phi$  is **bounded by identity** on  $S$  if for all  $X \in S$  and  $0 \leq t < T$

$$\Phi_t(X) \leq X_t. \quad (2.5)$$

- (ii) We say that  $\Phi$  allows **arbitrage** on  $S$  if for some  $X \in S$  and some  $t$ ,  $0 \leq t < T$ ,

$$\text{ess.sup} \Phi_t\left(X + \sum_{u=t+1}^T D_u \cdot r(u)\right) = +\infty \quad (2.6)$$

where the *ess.sup* is taken over all  $D \in N$ .

**Proposition 2.1.** Let  $\Phi$  be a family of functionals bounded on  $S \subset L$  by a positive affine transformation of identity: there exist  $b \in \mathbb{R}$ ,  $c > 0$  with

$$\Phi_t(X) \leq b + c \cdot X_t \quad (2.7)$$

for all  $X \in S$  and  $t \in \mathbb{T}$ . Then  $\Phi$  does not allow arbitrage on  $S$ .

*Proof.* Let  $X \in S$ ,  $t < T$  and  $D \in N$ . Then

$$\begin{aligned} \Phi_t\left(X + \sum_{u=t+1}^T D_u \cdot r(u)\right) &\leq b + c \cdot \left(X + \sum_{u=t+1}^T D_u \cdot r(u)\right)_t \\ &= b + c \cdot X_t < +\infty. \end{aligned} \quad \square$$

The following result, whose proof is left to the reader, will be used later:

**Lemma 2.2.** Let  $\Psi$  be a risk assessment bounded on  $S_{\mathcal{M}}$  by (a positive multiple of) identity, i.e. (2.7) with  $b = 0$ . Then for  $0 \leq s < t \leq T$ ,  $Y \in L$ , and  $Z_{t-1} \in L_{t-1}$

$$\begin{aligned} \Psi_s\left(\sum_{u<t} Y_u \cdot r(u) + Z_{t-1} \cdot r(t-1) \mathbb{1}_{\geq t}\right) & \quad (2.8) \\ &= \sup_{D \in N} \Psi_s\left(\sum_{u<t} Y_u \cdot r(u) + (D_t + Z_{t-1} \cdot r_{t-1,t}) \cdot r(t) + \sum_{u \geq t+1} D_u \cdot r(u)\right). \end{aligned}$$

#### 2.4 The market consistent majorant.

**Theorem 2.1.** Let  $\mathbb{Q} \in \mathcal{M}$  and  $\Psi$  be a risk assessment bounded by identity on  $S_{\mathbb{Q}}$ . We define a family  $\Psi^* = (\Psi_t^*)_{t \in \mathbb{T}}$  of functionals  $\Psi_t^* : S_{\mathbb{Q}} \rightarrow L_t$  by  $\Psi_T^* = 0$  and for  $t < T$  by

$$\Psi_t^*(X) := \text{ess.sup}_{D \in N} \Psi_t\left(X + \sum_{u=t+1}^T D_u \cdot r(u)\right), \quad (2.9)$$

$X \in S_{\mathbb{Q}}$ . Then  $\Psi^*$  is the **market consistent majorant** of  $\Psi$  on  $S_{\mathbb{Q}}$ : i.e.

- (i)  $\Psi^*$  is a market consistent risk assessment on  $S_{\mathbb{Q}}$ .
- (ii) For all  $t \in \mathbb{T}$  and all  $X \in S_{\mathbb{Q}}$ :  $\Psi_t^*(X) \geq \Psi_t(X)$ .
- (iii) For any market consistent risk assessment  $\Phi = (\Phi_t)_{t \in \mathbb{T}}$  on  $S_{\mathbb{Q}}$  with  $\Phi_t(X) \geq \Psi_t(X)$  for all  $t$  and  $X \in S_{\mathbb{Q}}$  we get  $\Phi_t(X) \geq \Psi_t^*(X)$  for all  $t$  and  $X \in S_{\mathbb{Q}}$ .

*Proof.* Let  $t < T$ . Using the boundedness of  $\Psi$  by identity on  $S_{\mathbb{Q}}$ , we first note that for  $X \in S_{\mathbb{Q}}$  the right-hand side of (2.9) is bounded by  $X_t$ .

We start the proof of (i) by showing that for  $X \in S_{\mathbb{Q}}$  fixed the set

$$\mathcal{G} := \left\{ \left( \Psi_t\left(X + \sum_{u>t} D_u \cdot r(u)\right) \right)_{t \in \mathbb{T}}, D \in N \right\}$$

is upward directed. Indeed, if  $D^{(1)}, D^{(2)} \in N$ , we define  $D' \in N$  recursively, starting with  $t = T$

$$\begin{aligned} A_{t-1} &:= \left\{ \omega, \Psi_{t-1}\left(X + D_t^{(1)} \cdot r(t) + \sum_{u>t} D'_u \cdot r(u)\right)(\omega) \right. \\ &\quad \left. \geq \Psi_{t-1}\left(X + D_t^{(2)} \cdot r(t) + \sum_{u>t} D'_u \cdot r(u)\right)(\omega) \right\} \end{aligned}$$

and  $D'_t := D_t^{(1)} \mathbb{1}_{A_{t-1}} + D_t^{(2)} \mathbb{1}_{\Omega \setminus A_{t-1}} \in N_t$ . With the time consistency, localization, and monotonicity properties of  $\Psi$ , one verifies recursively that for  $i = 1, 2$  and  $t \in \mathbb{T}$

$$\Psi_t\left(X + \sum_{u>t} D'_u \cdot r(u)\right) \geq \Psi_t\left(X + \sum_{u>t} D_u^{(i)} \cdot r(u)\right).$$

Thus  $\mathcal{G}$  is upward directed. A  $|\mathbb{T}|$ -dimensional version of Proposition VI-1-1 in Neveu (1972) shows that there exists a sequence  $D^{(n)} \in N$  with  $(\Psi_t(X + \sum_{u=t+1}^T D_u^{(n)} \cdot r(u)))_{t \in \mathbb{T}}$  as an increasing sequence in  $L$  such that p.s. for all  $t \in \mathbb{T}$

$$\Psi_t \left( X + \sum_{u>T} D_u^{(n)} \cdot r(u) \right) \xrightarrow[n \rightarrow \infty]{} \Psi_t^*(X) \quad (2.10)$$

Next, we remark that locality, monotonicity, dependence on future values, cash invariance as well as market consistency follow directly from (2.9) and the corresponding properties of  $\Psi_t$ .

To show the lower semi-continuity of  $\Psi_t^*$  on  $S_{\mathbb{Q}}$ , let  $S_{\mathbb{Q}} \ni X^{(n)} \nearrow X \in S_{\mathbb{Q}}$  for  $n \rightarrow \infty$ . The lower semi-continuity of  $\Psi_t$  implies then

$$\begin{aligned} \lim_{n \rightarrow \infty} \Psi_t^*(X^{(n)}) &= \sup_n \operatorname{ess.\,sup}_D \Psi_t \left( X^{(n)} + \sum_{u>T} D_u \cdot r(u) \right) \\ &= \operatorname{ess.\,sup}_D \sup_n \Psi_t \left( X^{(n)} + \sum_{u>T} D_u \cdot r(u) \right) \\ &= \operatorname{ess.\,sup}_D \Psi_t \left( X + \sum_{u>T} D_u \cdot r(u) \right) = \Psi_t^*(X). \end{aligned}$$

To show time consistency, let  $0 \leq s \leq t < T$  and  $X \in S_{\mathbb{Q}}$ . Then by (2.1) and (2.9)

$$\begin{aligned} \Psi_s^*(X) - X_s &= \operatorname{ess.\,sup}_D \Psi_s \left( \sum_{s<u} (D_u + \Delta X_u) \cdot r(u) \right) \\ &\stackrel{(2.1), (2.10)}{=} \sup_{D^{(n)}} \Psi_s \left( \sum_{s<u \leq t} (D_u^{(n)} + \Delta X_u) \cdot r(u) \right. \\ &\quad \left. + \Psi_t \left( \sum_{t<u} (D_u^{(n)} + \Delta X_u) \cdot r(u) \right) \cdot r(t) \mathbb{1}_{>t} \right) \\ &= \sup_{D^{(n)}} \Psi_s \left( \sum_{s<u \leq t} (D_u^{(n)} + \Delta X_u) \cdot r(u) \right. \\ &\quad \left. + \sup_{D^{(n)}} \Psi_t \left( \sum_{t<u} (D_u^{(n)} + \Delta X_u) \cdot r(u) \right) \cdot r(t) \mathbb{1}_{>t} \right) \\ &= \operatorname{ess.\,sup}_D \Psi_s \left( \sum_{s<u \leq t} (D_u + \Delta X_u) \cdot r(u) + (\Psi_t^*(X) - X_t) \cdot r(t) \mathbb{1}_{>t} \right) \\ &\stackrel{(2.8)}{=} \operatorname{ess.\,sup}_D \Psi_s \left( \sum_{s<u \leq t} (D_u + \Delta X_u) \cdot r(u) \right. \\ &\quad \left. + (D_{t+1} + (\Psi_t^*(X) - X_t) \cdot r_{t-1,t}) \cdot r(t+1) + \sum_{t+1<u} D_u \cdot r(u) \right) \\ &= \Psi_s^* \left( X \mathbb{1}_{\leq t} + \Psi_t^*(X) \cdot r(t) \mathbb{1}_{>t} \right) - X_s. \end{aligned} \quad (2.11)$$

Part (ii) of the theorem is obvious. For part (iii), let  $\Phi$  be a market consistent risk assessment with  $\Phi_t(X) \geq \Psi_t(X)$  for all  $t$  and  $X \in S_{\mathbb{Q}}$ . Then

$$\begin{aligned} \Phi_t(X) &= X_t + \Phi_t \left( \sum_{u>t} (\Delta X_u) \cdot r(u) \right) = X_t + \sup_{D \in N} \Phi_t \left( \sum_{u>t} (D_u + \Delta X_u) \cdot r(u) \right) \\ &\geq X_t + \sup_{D \in N} \Psi_t \left( \sum_{u>t} (D_u + \Delta X_u) \cdot r(u) \right) = \Psi_t^*(X). \quad \square \end{aligned}$$

As an immediate consequence of Theorem 2.1 we get

**Corollary 2.1.** *If  $\Psi$  is a risk assessment bounded by identity on  $S_{\mathbb{Q}}$  with  $\mathbb{Q} \in \mathcal{M}$ , then so is its market consistent majorant  $\Psi^*$  on  $S_{\mathbb{Q}}$ .*

The time consistency of the market consistent majorant in the form (2.11) together with Lemma 2.2 allows for its recursive calculation:

**Proposition 2.2.** *Let  $\Psi$  as in Theorem 2.1. Then the market consistent majorant satisfies for all  $X \in S_{\mathbb{Q}}$ :  $\Psi_T^*(X) = 0$  and for  $t < T$*

$$\Psi_t^*(X) = \operatorname{ess.\,sup}_{D_{t+1} \in N_{t+1}} \Psi_t \left( X \mathbb{1}_{\leq t+1} + D_{t+1} \cdot r(t+1) + \Psi_{t+1}^*(X) \cdot r(t+1) \mathbb{1}_{> t+1} \right). \quad (2.12)$$

We will now apply the best estimate  $\Psi_{\mathbb{Q}}$  and the general risk assessment  $\Psi$  to the special situation of an insurance company.

### 3. SUPERVISORY RISK ASSESSMENT OF AN INSURANCE COMPANY

In the rest of the paper, we let  $\Psi$  be a risk assessment bounded by identity on  $S_{\mathbb{Q}}$  with  $\mathbb{Q} \in \mathcal{M}$  kept fixed:

$$\Psi(X)_t \leq X_t \quad \text{for all } X \in S_{\mathbb{Q}} \text{ and } t < T. \quad (3.1)$$

#### 3.1 The business plan of an insurance company.

By  $Z \in L$  we design the process of obligation cash-flows  $Z_t$ ,  $t \in \mathbb{T}$ , where  $Z_t$  is the obligation that company has to pay at time  $t$  to its policy-holders according to the contracts signed at date 0,  $Z_0 = 0$ . There will be no more obligations after risk horizon  $T$ .

*Remark 3.1.* The obligation cash-flows  $Z_t$  are supposed to be exogenously given, i.e. they do not depend on the company's asset portfolio. This condition excludes the cases of participation in benefits which depend on the outcome of the asset portfolio.

Let  $C \in L$  be the process of current asset values  $C_t$  of the company after payment of the obligation  $Z_t$ . The company's bookkeeping equation has the form

$$C_t = C_{t-1} \cdot r_{t-1,t} - Z_t + D_t, \quad t > 0,$$

or equivalently

$$\Delta C = D - Z, \quad (3.2)$$

where  $D_t$  is the difference of the company's current asset value before payment of the obligation  $Z_t$  from a pure investment of  $C_{t-1}$  in the eligible asset. We assume for the current asset value process  $C$  the **self-financing condition** (w.r.t.  $Z$ )

$$D = \Delta C + Z \in N. \quad (3.3)$$

The couple  $(C, Z)$  satisfying (3.3) is the company's **business plan**. For  $0 < t \leq T$ , we call  $D_t = \Delta C_t + Z_t$  the **trading-risk exposure** at  $t$  of the business plan.

Since (3.2) is invariant under the transformation of  $C$  to  $C + c \cdot r$ ,  $c \in \mathbb{R}$ , we get the following form of business plans with the same trading-risk exposures:

**Proposition 3.1.** *Let  $Z \in L$  and  $D \in N$  be cash-flow processes of obligations and of trading-risk exposures, respectively. Then*

$$\mathcal{C}_t(D) = \left\{ C_t \cdot r(t) + \sum_{u=t+1}^T (D_u - Z_u) \cdot r(u) \mid C_t \in L_t \right\} \quad (3.4)$$

*is the set of self-financing (w.r. to  $Z$ ) asset value processes in  $(t, T]$  having  $D \mathbb{1}_{>t}$  as trading-risk exposures.*

### 3.2 Best estimate of obligations.

The **process of cumulated past obligations** is defined by:

$$Z_t^\circ := \sum_{u=0}^t Z_u \cdot r_{u,t}. \quad (3.5)$$

The process  $Z^\circ$  is retrospective. Its prospective counterpart is given by the following definition where we use the fixed  $\mathbb{Q} \in \mathcal{M}$ :

#### Definition 3.1.

(i) The process of **estimated obligations**  $\bar{Z} = (\bar{Z}_t)_{t \in \mathbb{T}}$  is

$$\bar{Z}_t := \Psi_{\mathbb{Q},t} \left( \sum_{u>t} Z_u \cdot r(u) \right) = \mathbb{E}_{\mathbb{Q}} \left[ \sum_{u>t} Z_u / r_{t,u} \mid \mathcal{F}_t \right]. \quad (3.6)$$

(ii)  $C - \bar{Z}$  is the **process of risk bearing capital**.

#### Proposition 3.2.

(i) The process  $\tilde{Z}$  defined by  $\tilde{Z} := \bar{Z} + Z^\circ$  has the increment process

$$\Delta \tilde{Z}_t = \mathbb{E}_{\mathbb{Q}} \left[ \sum_{u=t}^T Z_u / r_{t,u} \mid \mathcal{F}_t \right] - \mathbb{E}_{\mathbb{Q}} \left[ \sum_{u=t}^T Z_u / r_{t-1,u} \mid \mathcal{F}_{t-1} \right] \cdot r_{t-1,t} \quad (3.7)$$

which implies that  $\tilde{Z}$  is a **discounted  $\mathbb{Q}$ -martingale**, i.e.

$$\mathbb{E}_{\mathbb{Q}} \left[ \Delta \tilde{Z}_t / r_{t-1,t} \mid \mathcal{F}_{t-1} \right] = 0. \quad (3.8)$$

(ii) Moreover,  $\Delta(C - \bar{Z}) = D - \Delta \tilde{Z}$  and  $C - \bar{Z}$  is also a **discounted  $\mathbb{Q}$ -martingale**:  $C - \bar{Z} \in S_{\mathbb{Q}}$ .

The proof of Proposition 3.2 is straight forward.

*Remark 3.3.* Naturally, the mathematical tools of Proposition 3.2 (i) are similar to those used in stochastic “reserving”. For example, (3.7) corresponds to the “claims development result” (see Definition 2.4 in Merz and Wüthrich (2008)). Note that we use the condition  $\mathbb{Q} \in \mathcal{M}$  to get the second part of Proposition 3.2.

### 3.3 Acceptability and free capital.

We take  $\Psi$  as the supervisor’s risk assessment applied to the risk bearing capital  $C - \bar{Z}$ . This means that a business plan  $(C, Z)$  satisfies the solvency condition at time  $t \in \mathbb{T}$  if and only if

$$\Psi_t(C - \bar{Z}) \geq 0. \quad (3.9)$$

In this case, we call  $(C, Z)$  a  **$t$ -acceptable plan**. The notion of “free capital” in the next definition is justified by Proposition 3.3.

**Definition 3.2.** The **free capital process** of a business plan  $(C, Z)$  is  $F(C, Z) = (F_t(C, Z))_{t \in \mathbb{T}}$  with

$$F_t(C, Z) := \Psi_t(C - \bar{Z}). \quad (3.10)$$

**Proposition 3.3.** Let  $(C, Z)$  be a business plan. Then

$$C - F_t(C, Z) \cdot r(t) \quad (3.11)$$

is  **$t$ -acceptable**. Moreover, its current asset value  $C_t - F_t(C, Z)$  is minimal among all  **$t$ -acceptable business plans** with the same trading-risk exposures  $(D_u)_{u>t}$  as  $(C, Z)$ .

*Proof.* By Proposition 3.1 and cash-invariance, we have that  $t$ -acceptability of  $(C + c \cdot r(t), Z)$  is equivalent to

$$\Psi_t(C + c \cdot r(t) - \bar{Z}) = F_t(C, Z) + c \geq 0,$$

hence equivalent to  $c \geq -F_t(C, Z)$ . And  $c^* = -F_t(C, Z)$  is the minimal value.  $\square$

#### 4. SUPERVISORY PROVISION AND THE OPTIMAL REPLICATING PORTFOLIO

##### 4.1 Supervisory provision and supervisory margin.

As in Artzner and Eisele (2010 b), we derive the definition of supervisory provision out of the free capital functional  $\Psi$ :

###### Definition 4.1.

(i) *The process of supervisory provisions*  $SP(Z) = SP_\Psi(Z) = (SP_{\Psi,t}(Z))_{t \in \mathbb{T}}$  for a cash-flow process of obligations  $Z \in L$  is given by

$$SP_{\Psi,t}(Z) := \text{ess.inf} \left\{ \tilde{C}_t \mid \exists \text{ a business plan } (\tilde{C}, Z) : \Psi_t(\tilde{C} - \bar{Z}) \geq 0 \right\}. \quad (4.1)$$

(ii) *We define the process of supervisory margin*  $SM$  as

$$SM_t(Z) = SP_t(Z) - \bar{Z}_t. \quad (4.2)$$

It follows from the definition that supervisory provision and margin are market consistent. They have the following representation:

**Proposition 4.1.** *We have for*  $t \in \mathbb{T}$

$$SP_t(Z) = \bar{Z}_t - \text{ess.sup}_{D \in N} \Psi_t \left( \sum_{u=t+1}^T (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right) \geq \bar{Z}_t, \quad (4.3)$$

$$SM_t(Z) = - \text{ess.sup}_{D \in N} \Psi_t \left( \sum_{u=t+1}^T (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right) \geq 0. \quad (4.4)$$

*Proof.* Let  $(C, Z)$  be a  $t$ -acceptable plan with trading-risk exposures  $(D_u)_{1 \leq u \leq T}$ . By Proposition 3.2 and cash invariance we get

$$0 \leq \Psi_t(C - \bar{Z}) = C_t - \bar{Z}_t + \Psi_t \left( \sum_{u>t} (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right) \quad \text{or}$$

$$C_t \geq \bar{Z}_t - \Psi_t \left( \sum_{u>t} (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right). \quad (4.5)$$

To take the infimum in (4.4) is equivalent in (4.5) to the infimum over  $D \in N$  and to put  $C_t$  equal to the right-hand side. Since for  $X \in S_{\mathbb{Q}}$  we have  $\Psi_t(X) \leq X_t$ , or by cash invariance

$$\Psi_t \left( \sum_{u>t} \Delta \tilde{X}_u \cdot r(u) \right) \leq 0,$$

we get the inequality in (4.3). (4.4) follows immediately from (4.3).  $\square$

*Remark 4.1.* The proof of Proposition 4.1 shows that the ess.inf-operator in (4.1) acts twofold: on the current asset value  $C_t$  and on the process of trading-risk exposures  $D$ . The first problem has a trivial solution, namely the right hand side of (4.5): If the exposures  $D$  are kept fixed, the minimal current asset value for acceptability is

$$\bar{Z}_t - \Psi_t \left( \sum_{u>t} (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right).$$

The minimality problem with respect to the exposures  $D$ , independently of the current asset value  $C_t$ , gives rise to the definition of an optimal replicating portfolio in the next subsection.

From (4.4) and the definition of the market consistent majorant we derive immediately the following result:

**Proposition 4.2.**

$$SM_t(Z) = -\Psi_t^* \left( - \sum_{u=t+1}^T \Delta \tilde{Z} \cdot r(u) \right). \quad (4.6)$$

The recursion formula of the market consistent majorant  $\Psi^*$  in Proposition 2.2 gives the following recursion procedures for the supervisory margin and supervisory provision  $SP$ :

**Proposition 4.3.** *For the supervisory margin we have  $SM_T(Z) = 0$  and for  $0 \leq t < T$*

$$SM_t(Z) = - \operatorname{ess.\,sup}_{D_{t+1} \in N_{t+1}} \Psi_t \left( (D_{t+1} - \Delta \tilde{Z}_{t+1}) \cdot r(t+1) - SM_{t+1}(Z) \cdot r(t+1) \mathbb{1}_{>t+1} \right), \quad (4.7)$$

and for the supervisory provision  $SP_T(Z) = 0$ , for  $0 \leq t < T$

$$SP_t(Z) = \bar{Z}_t - \operatorname{ess.\,sup}_{D_{t+1} \in N_{t+1}} \Psi_t \left( (D_{t+1} - \Delta \tilde{Z}_{t+1}) \cdot r(t+1) + (\bar{Z}_{t+1} - SP_{t+1}(Z)) \cdot r(t+1) \mathbb{1}_{>t+1} \right). \quad (4.8)$$

*Proof.* By Proposition 2.2 we get for  $t < T$

$$\begin{aligned} SM_t(Z) &= - \operatorname{ess.\,sup}_{D_{t+1}} \Psi_t \left( D_{t+1} \cdot r(t+1) - \Delta \tilde{Z}_{t+1} \mathbb{1}_{=t+1} \right. \\ &\quad \left. + \Psi_{t+1}^* \left( - \sum_{u>t+1} \Delta \tilde{Z}_u \cdot r(u) \right) \cdot r(t+1) \mathbb{1}_{>t+1} \right) \\ &= - \operatorname{ess.\,sup}_{D_{t+1}} \Psi_t \left( (D_{t+1} - \Delta \tilde{Z}_{t+1}) \cdot r(t+1) \right. \\ &\quad \left. - SM_{t+1}(Z) \cdot r(t+1) \mathbb{1}_{>t+1} \right). \end{aligned}$$

(4.8) follows immediately from (4.7).  $\square$

## 4.2 Optimal replicating portfolios.

We now treat the minimality problem (4.1) with respect to the process of trading-risk exposures  $D$  keeping the current asset value  $C_t$  fixed. By Proposition 3.3  $(C - F_t(C, Z) \cdot r(t), Z)$  is  $t$ -acceptable with minimal current asset value  $C_t - F_t(C, Z)$ . Minimality with respect to the argument  $D$  is therefore characterized by  $C_t - F_t(C, Z) = \bar{Z}_t - \Psi_t(\sum_{u>t} (D_u - \Delta \tilde{Z}_u) \cdot r(u)) = SP_t(Z)$  (see also Remark 4.1.):

**Definition 4.2.** *Optimal replicating portfolio for supervisory provision.*

For  $t \in \mathbb{T}$ , a current asset value process  $C^* = (C_u^*)_{u \in \mathbb{T}}$  is called a  **$t$ -optimal replicating portfolio** for the obligation process  $Z = (Z_u)_{u \in \mathbb{T}}$  if and only if  $C^*$  satisfies the self-financing condition (3.3) and

$$C_t^* - F_t(C^*, Z) = SP_t(Z). \quad (4.9)$$

It has been shown in Artzner and Eisele (2010 a) that an optimal replicating portfolio does not always exist. To guarantee its existence, the following additional condition on the risk assessment is sufficient.

**Definition 4.3.** *Market prudent risk assessment.*

The risk assessment  $\Psi$  is called **market prudent** if for all  $0 \leq t < T$  and all  $D_{t+1} \in N_{t+1}, D_{t+1} \neq 0$ , we have

$$\Psi_t(D_{t+1} \cdot r(t+1)) < 0 \quad \text{with positive probability.} \quad (4.10)$$

**Proposition 4.4.** *Let  $\Psi$  be a market prudent risk assessment which moreover is homogeneous and continuous, i.e. for all  $t \in \mathbb{T}$ ,  $\lambda > 0$ ,  $X^{(n)}, X \in L$  with  $\lim_{n \rightarrow \infty} X^{(n)} = X$*

$$\Psi_t(\lambda X) = \lambda \Psi_t(X) \quad \text{and} \quad (4.11)$$

$$\lim_{n \rightarrow \infty} \Psi_t(X^{(n)}) = \Psi_t(X). \quad (4.12)$$

Then for every obligation process  $Z \in L$ ,  $t \in \mathbb{T}$  and value  $C_t$ , there exists a self-financing  $t$ -optimal replicating portfolio  $C^*$  with  $C_t^* = C_t$  which moreover is  $u$ -optimal replicating for all  $u \geq t$ .

*Remark 4.2.* In the context of Bellman's dynamic optimization, the process of trading risk exposures  $D^* \in N$  correspond to control variables of a **subproblem-perfect optimal strategy** (to be compared with Fudenberg and Levine (1983)).

*Proof.* (see also Artzner and Eisele (2010 a) in the one-period case)

Since (4.9) is trivial for  $t = T$ , let  $t < T$ . We first prove:

For all  $0 \leq s < T$ , there exists a trading risk exposure  $D_{s+1}^* \in N_{s+1}$  such that the  $\text{ess.sup}$  in (4.8) for  $s$  is attained in  $D_{s+1}^*$ .

We set  $X = -\Delta \tilde{Z}_{s+1} \cdot r(s+1) + (\bar{Z}_{s+1} - SP_{s+1}(Z)) \cdot r(t+1) \mathbb{1}_{>t+1}$ . As in (2.10), let  $D^{(n)} \in N_{s+1}$  be a sequence with

$$\Psi_s \left( D^{(n)} \cdot r(s+1) + X \right) \nearrow \text{ess.sup}_{D \in N_{s+1}} \Psi_s (D \cdot r(s+1) + X) \leq \max_{s < u \leq T} \|X_u\|_\infty.$$

If  $\|D^{(n)}\|_\infty$  stays bounded, then — for a subsequence —  $D^{(n)} \rightarrow D^* \in N_{s+1}$  and  $\Psi_s (D^* \cdot r(s+1) + X) = \text{ess.sup}_D \Psi_s (D \cdot r(s+1) + X)$ . On the other hand if

$\|D^{(n)}\|_\infty \nearrow +\infty$ , then — again for a subsequence —  $D^{(n)} / \|D^{(n)}\|_\infty \rightarrow \bar{D} \in N_{s+1} \setminus \{0\}$  and by (4.11), (4.12) we get

$$\Psi_s (\bar{D} \cdot r(s+1)) = \lim_n \frac{1}{\|D^{(n)}\|_\infty} \Psi_s \left( D^{(n)} \cdot r(s+1) + X \right) = 0$$

a.e., contradicting the market prudence of  $\Psi_s$ .

Now we prove by induction: For any current asset value  $C_t$  there exists a self-financing portfolio  $(C_u^*)_{t \leq u \leq T}$  with  $C_t^* = C_t$  and equality (4.9) for all  $u$ ,  $t \leq u \leq T$ . For  $t = T$  there is nothing to prove. Let  $t < T$ . Given  $C_t = C_t^*$ , we set  $C_{t+1}^* := C_t^* \cdot r_{t,t+1} + D_{t+1}^* - Z_{t+1}$  with

$$SP_t(Z) = \bar{Z}_t - \Psi_t \left( (D_{t+1}^* - \Delta \tilde{Z}_{t+1}) \cdot r(t+1) + (\bar{Z}_{t+1} - SP_{t+1}(Z)) \cdot r(t+1) \mathbb{1}_{>t+1} \right).$$

By induction, there exists a self-financing portfolio  $(C_u^*)_{t+1 \leq u \leq T}$  starting at the above defined  $C_{t+1}^*$  and satisfying (4.9) for all  $u$ ,  $t+1 \leq u \leq T$ . We have to prove (4.9) for  $t$ : With Proposition 3.2 (ii) we find  $C_t^* - \bar{Z}_t \cdot r_{t,t+1} + D_{t+1}^* - \Delta \tilde{Z}_{t+1} = C_{t+1}^* - \bar{Z}_{t+1}$  and therefore

$$C_t^* - SP_t(Z) = C_t^* - \bar{Z}_t + \Psi_t \left( (D_{t+1}^* - \Delta \tilde{Z}_{t+1}) \cdot r(t+1) + (\bar{Z}_{t+1} - SP_{t+1}(Z)) \cdot r(t+1) \mathbb{1}_{>t+1} \right)$$

$$\begin{aligned}
&= \Psi_t \left( (C_{t+1}^* - \bar{Z}_{t+1}) \mathbb{1}_{=t+1} + (C_{t+1}^* - SP(Z)_{t+1}) \cdot r(t+1) \mathbb{1}_{>t+1} \right) \\
&= \Psi_t \left( (C_{t+1}^* - \bar{Z}_{t+1}) \mathbb{1}_{=t+1} + \Psi_{t+1}(C^* - \bar{Z}) \cdot r(t+1) \mathbb{1}_{>t+1} \right) \\
&= \Psi_t(C^* - \bar{Z}) = F_t(C^*, Z).
\end{aligned}$$

This proves (4.9) for  $t$ .  $\square$

### 4.3 Acceptability by rebalancing.

By definition, the supervisory provision represents the minimal asset value for a possible acceptability. However, an initial asset value greater than the supervisory provision does not eo ipso imply acceptability. This gives rise to the following definition.

**Definition 4.4.** *A business plan  $(C, Z)$  is called  $t$ -solvable if*

$$C_t \geq SP_t(Z). \quad (4.13)$$

As an immediate consequence of Proposition 3.3 we get

#### Proposition 4.5.

- (i) *A  $t$ -acceptable business plan is  $t$ -solvable.*
- (ii) *Let  $(C, Z)$  be a business plan with  $C_t = SP_t(Z)$ . Then  $(C, Z)$  is  $t$ -acceptable if and only if  $C$  is  $t$ -optimal replicating.*

*Proof.* (i) If  $(C, Z)$  is  $t$ -acceptable, i.e.  $F_t(C, Z) \geq 0$ , then Proposition 3.3 implies

$$C_t \geq C_t - F_t(C, Z) \geq SP_t(Z). \quad (4.14)$$

Hence,  $(C, Z)$  is  $t$ -solvable.

(ii) If in addition to (4.14)  $C_t = SP_t(Z)$ , then  $C_t - F_t(C, Z) = SP_t(Z)$  and  $C$  is  $t$ -optimal replicating. Conversely, the last two equalities imply  $F_t(C, Z) = 0$ , i.e.  $t$ -acceptability.  $\square$

The contrary of Proposition 4.5 (i) is not true. However, we realize the following fact:

#### Proposition 4.6. *Acceptability by rebalancing of the portfolio.*

*Let  $(C, Z)$  be a  $t$ -solvable business plan:  $C_t \geq SP_t(Z)$ . Then — under the conditions of Proposition 4.4 — there exists a  $t$ -acceptable business plan  $C^*$  with the same asset value at  $t$ :  $C_t^* = C_t$ .*

*Proof.* If  $C^*$  is the optimal replicating portfolio according to Proposition 4.4 with  $C_t^* = C_t$ , then (4.9) implies

$$C_t - F_t(C, Z) = SP_t(Z) \leq C_t.$$

Hence,  $F_t(C, Z) \geq 0$  and  $(C^*, Z)$  is a  $t$ -acceptable business plan.  $\square$

## 5. TRANSFER OF OBLIGATIONS AND “FREM D”-CAPITAL

We now turn to the second task of supervisory accounting: to analyze the conditions of a possible transfer of the obligations and thereby determine the “Fremd”-capital on the passive side of the supervisory balance sheet. The following reflections serve as guidelines to this analysis. In Subsection 5.1 and the beginning of Subsection 5.2 the considerations are formulated for  $t = 0$ , but the reader can easily generalize them to any  $t < T$ .

In Artzner and Eisele (2010 b), we have erroneously excluded this transfer possibility to protect the policyholder.

### 5.1 A regulated transfer of an insurance company.

We saw that a solvable insurance company can reach acceptability in a self-financing way by rebalancing its portfolio towards an optimal replicating one. This is no longer possible if the company is not solvable, i.e.  $C_0 < SP_0(Z)$ .

However, under some conditions the obligations vis-à-vis the policy-holders can still be kept by a **regulated** transfer to a new company. To illustrate this procedure we argue as follows: One can find investors who are willing

1. to take over the obligations  $(Z_t)_{t>0}$  together with a cash amount  $A$  from the old company and
2. to invest an amount  $B$  of own funds in the new company.

In the new company, the amount  $A$  coming from the old company is definitely “Fremd”-capital, while  $B$  is “Eigen”-capital. The incentive to do this business lies in the promise to get the insurance leftover  $(\hat{C}_T)^+$  at the end of the risk horizon. In financial terms,  $(\hat{C}_T)^+$  can be regarded as an “insurance option”.

Of course, the new company can only get acceptability if its initial asset value  $\hat{C}_0$  satisfies

$$\hat{C}_0 = A + B \geq SP_0(Z). \quad (5.1)$$

On the other hand, the new investors will certainly impose the amount  $A$  paid by the old company to be greater than the best estimate  $\bar{Z}_0$  of the obligations:

$$A > \bar{Z}_0. \quad (5.2)$$

Otherwise, the investment will lead (in the mean) to a deficit. This corresponds to the result in classical risk theory where premiums without a strictly positive safety loading will lead the company — in the long run — to ruin with probability one. We define

$$q_0 := \frac{A - \bar{Z}_0}{B} \in [0, +\infty] \quad (5.3)$$

as the transfer ratio, where we include the extreme cases:  $q_0 = 0$  meaning that  $A = \bar{Z}_0$ ,  $B = SP_0 - \bar{Z}_0$  and  $q_0 = +\infty$  meaning that  $A = SP_0$ ,  $B = 0$ . Note that for the old company the condition  $C_0 \geq A$  is necessary to realize the transfer. The old company is ruined if its initial asset value  $C_0$  is less than its “exit value”  $A$ .

In Solvency II and SST, the difference  $A - \bar{Z}_0$  would be interpreted as the “cost of capital” to attract the new capital  $B$ . The ratio  $q_0$  means then an additional return on this investment over the one of the eligible asset, to compensate for the riskier investment in the new insurance company: it is called the **cost-of-capital ratio**.

*Remark 5.1.* Neither is the amount  $A$  the (liquid) market price for the obligation cash-flows process  $Z$ , nor is  $B$  such a market price for the non-hedgeable “insurance option”  $(\hat{C}_T)^+$ . Both can only be regarded as prices in a regulated market: The two companies have been or will be submitted to supervision.

### 5.2 Technical provision and risk margin.

The procedure described in the previous subsection serves to define the “Fremd”-capital in supervisory provision. Since only a fictitious transfer is analyzed the procedure must be standardized. This means that

1. special properties like diversification effects in the new company cannot be taken into account: The new company must be initially empty.
2. from the supervisor’s point of view, it is enough that the new company is minimally acceptable: This means that the new company applies an

optimal replicating portfolio  $C^*$ . Moreover, its free capital  $F_0(C^*, Z)$  can be assumed to be zero since it is not necessary for acceptability:

$$C_0^* = SP_0(Z) \quad \text{and} \quad F_0(C^*, Z) = 0. \quad (5.4)$$

Coming back to the situation  $t < T$ , we specify a sequence of transfer ratios  $(q_t)_{0 \leq t \leq T}$ :

**Assumption 5.1.** *Within a fictitious transfer to a minimal insurance company satisfying (5.4) for  $0 \leq t < T$ , the ratio*

$$q_t \in \bar{L}_t^+ \quad (5.5)$$

*is fixed by the supervisor. Since for  $t = T$  we have  $\bar{Z}_T = SP_T(Z) = 0$  and  $q_T$  can not be defined by (5.3), we set*

$$q_T = 0. \quad (5.6)$$

Combining (5.1), (5.3), and (5.4), we can quantify the “Fremd”-capital  $A$  of subsection 5.1.:

**Definition 5.1.** *Technical provision and risk margin.*

- (i) *The “Fremd”-capital due to obligations is called **technical provision**  $TP(Z) = (TP_t(Z))_{t \in \mathbb{T}}$  and defined by:*

$$TP_t(Z) := \frac{q_t}{1 + q_t} SP_t(Z) + \frac{1}{1 + q_t} \bar{Z}_t. \quad (5.7)$$

- (ii) *The **risk margin**  $RM(Z) = (RM_t(Z))_{t \in \mathbb{T}}$  is the “Fremd”-capital part of the supervisory margin  $SM(Z)$ :*

$$RM_t(Z) := TP_t(Z) - \bar{Z}_t = \frac{q_t}{1 + q_t} SM_t(Z). \quad (5.8)$$

*Remark 5.2.* By (4.3) and (5.7), we have for  $t \in \mathbb{T}$

$$\bar{Z}_t \leq TP_t(Z) \leq SP_t(Z), \quad (5.9)$$

where the extreme cases  $TP_t(Z) = \bar{Z}_t$ , resp.  $TP_t(Z) = SP_t(Z)$ , correspond to  $q_t = 0$ , resp.  $q_t = +\infty$ . In the later case, the transfer of obligations and rebalancing of the portfolio lead to the same business plan.

## 6. SOLVENCY CAPITAL REQUIREMENT AND ACCEPTABILITY

In the last Section we have determined the company’s “Fremd”-capital. Its “Eigen”-capital is the difference of the asset value and the “Fremd”-capital. We now investigate that part of the “Eigen”-capital required to provide acceptability. Let’s first treat the case of a company with an optimal replicating portfolio.

### 6.1 “Eigen”-capital in the supervisory provision.

**Definition 6.1.** *We call the “Eigen”-capital part in the supervisory provision the **absolute solvency capital requirement**  $SCR_t^*(Z)$ :*

$$SCR_t^*(Z) := SP_t(Z) - TP_t(Z) = \frac{1}{1 + q_t} SM_t(Z). \quad (6.1)$$

In the next subsection, we will see that  $SCR_t^*(Z)$  is the special case of a general definition of solvency capital requirement, which here is applied to a business plan with an optimal replicating portfolio. Equations (5.8) and (6.1) yield

$$RM_t(Z) = q_t \cdot SCR_t^*(Z) \quad \text{and} \quad (6.2)$$

$$SM_t(Z) = RM_t(Z) + SCR_t^*(Z). \quad (6.3)$$

*Remark 6.1.* The right-hand side of (6.3) combines “Fremd”- and “Eigen”-capital items. This explains why  $SM_0$  and a fortiori  $SP_0$  does not show up in the supervisory balance sheet.

Further note that the  $SCR_0^*$  depends only on the obligation process  $Z$  and cannot be diminished by the choice of the asset portfolio, whence the adjective “absolute”.

*Remark 6.2.* At first sight, relation (6.2) seems to correspond to the definition of the risk margin and the market value margin, in Solvency II or SST, respectively. However, it differs clearly from formulas like

$$“RM_0 = 6\% \cdot \sum_t SCR_t” \quad (*)$$

in Solvency II and SST. A formula like  $(*)$  raises the question whether “ $SCR_t$ ” are flows or stock values (see Eisele and Artzner (2011) for more details).

## 6.2 The general solvency capital requirement.

In analogy to Definition 4.1 of the supervisory provision, we are looking for the minimal “Eigen”-capital with respect to a given process of trading-risk exposures  $D \in N$ :

**Definition 6.2.** *The general solvency capital requirement.*

*Given the cash-flow processes of obligations  $Z$  and of trading-risk exposures  $D \in N$ , the minimal “Eigen”-capital of a business plan with  $D$  as trading-risk exposures to be  $t$ -acceptable is the **solvency capital requirement (SCR)**:*

$$SCR_t(D, Z) := \text{ess.inf} \left\{ \tilde{C}_t - TP_t(Z) \mid \text{there exists a business plan } (\tilde{C}, Z) \text{ with} \right. \\ \left. D = \Delta \tilde{C} + Z \text{ and } \Psi_t(\tilde{C} - \bar{Z}) \geq 0 \right\}. \quad (6.4)$$

The solvency capital requirement  $SCR_t(D, Z)$  determines the part of “Eigen”-capital subject to regulatory supervision. This part may therefore be also called “gebundenes Eigen”-capital in contrast to the free capital  $F_t(C, Z)$ .

The solvency capital requirement can be determined as follows:

**Proposition 6.1.**

$$SCR_t(D, Z) = - \left( RM_t(Z) + \Psi_t \left( \sum_{u=t+1}^T (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right) \right). \quad (6.5)$$

*Proof.* By Proposition 3.1 and the equality  $D - Z = \Delta C$

$$C_\gamma = \gamma \cdot r(t) + \sum_{u>t} \Delta C_u \cdot r(u), \quad (6.6)$$

$\gamma \in \mathbb{R}$ , are the self-financing processes of current asset values with trading-risk exposures  $D$ . By Proposition 3.3, among them the minimal asset value for a  $t$ -acceptable business plan  $(C_\gamma, Z)$  is

$$\begin{aligned} \gamma^* &= C_{\gamma,t} - F_t(C_\gamma, Z) = \gamma - \Psi_t \left( \gamma \cdot r(t) + \sum_{u>t} \Delta C_u \cdot r(u) - \bar{Z} \right) \\ &= \bar{Z}_t - \Psi_t \left( \sum_{u>t} (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right). \end{aligned} \quad (6.7)$$

Therefore, using (5.7)

$$\begin{aligned} SCR_t(D, Z) &= \bar{Z}_t - \Psi_t \left( \sum_{u>t} (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right) - TP_t(Z) \\ &= -RM_t(Z) - \Psi_t \left( \sum_{u=t+1}^T (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right). \end{aligned}$$

is the minimal ‘‘Eigen’’-capital with respect to trading-risk exposures  $D$ .  $\square$

The comparison with Definition 6.1 gives:

**Proposition 6.2.**

(i) With  $SCR^*(Z)$  given by (6.1) we have

$$SCR_t(D, Z) = SCR_t^*(Z) \quad (6.8)$$

if and only if  $D$  is the process of trading-risk exposures of a  $t$ -optimal replicating portfolio.

(ii) For a general process of trading risk exposures  $D$  we have

$$SCR_t(D, Z) \geq SCR_t^*(Z) \geq 0. \quad (6.9)$$

*Proof.* (i) Let  $C^*$  be an optimal replicating portfolio of a  $t$ -acceptable business plan with minimal asset value  $C_t^*$  and let  $D^* = \Delta C^* - Z$  be its trading-risk exposures. Then  $F_t(C^*, Z) = 0$  and by (4.9)  $C_t^* = C_t^* - F_t(C^*, Z) = SP_t(Z)$ . Therefore by (6.1)

$$SCR_t(D^*, Z) = C_t^* - TP_t(Z) = SP_t(Z) - TP_t(Z) = SCR_t^*(Z).$$

Conversely, let  $C$  be a self-financing process of asset values with trading-risk exposures  $D$ . Then (6.7) and (6.8) imply

$$\begin{aligned} C_t - F_t(C, Z) &= \bar{Z}_t - \Psi_t \left( \sum_{u>t} (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right) \\ &= SCR_t(D, Z) + TP_t(Z) = SCR_t^*(Z) + TP_t(Z) = SP_t(Z) \quad (6.10) \end{aligned}$$

and by (4.4)  $C$  is an optimal replicating portfolio. (ii) We take  $C$  as in (6.6) with  $C_t = C_t^*$  from (6.7). Since  $C$  is the asset value process of a  $t$ -acceptable and hence  $t$ -solvable plan,  $C_t^* \geq SP_t(Z)$  and hence by (4.4)

$$\begin{aligned} SCR_t(D, Z) &= C_t^* - TP_t(Z) \geq SP_t(Z) - TP_t(Z) \\ &= SCR_t^*(Z) = \frac{1}{1 + q_t} SM_t(Z) \geq 0. \quad \square \end{aligned}$$

**Definition 6.3.** For a process  $D$  of trading-risk exposures, we call the difference between  $SCR_t(D, Z)$  and the absolute solvency capital requirement  $SCR_t^*(Z)$  the **relative solvency capital requirement**:

$$SCR_t^r(D, Z) := SCR_t(D, Z) - SCR_t^*(Z). \quad (6.11)$$

Proposition 6.1 together with (6.3) gives

$$SCR_t^r(D, Z) = - \left( SM_t(Z) + \Psi_t \left( \sum_{u=t+1}^T (D_u - \Delta \tilde{Z}_u) \cdot r(u) \right) \right). \quad (6.12)$$

Recalling the formula of the supervisory margin in (4.4) we see that the relative solvency capital requirement measures the inadequacy of the company’s actual trading risk with respect to the underwriting risks. It is a risk assessment of the asset-liability management (see also Artzner and Eisele (2010 a), Section 4.1).

The different items of supervisory accounting defined so far yield:

**Proposition 6.3.** *The Supervisory Accounting Equality.*

Let  $(C, Z)$  be the business plan of an insurance company with the process  $D = \Delta C + Z$  of trading risk exposures. Then for  $t \in \mathbb{T}$

$$\begin{aligned} C_t &= TP_t(Z) + SCR_t(D, Z) + F_t(C, Z) \\ &= \bar{Z}_t + RM_t(Z) + SCR_t^*(Z) + SCR_t^r(D, Z) + F_t(C, Z) \end{aligned} \quad (6.13)$$

*Proof.* Equation (6.13) is equivalent to the first two equalities in (6.10) which hold generally.  $\square$

## 7. THE COHERENT ONE-PERIOD CASE

In Artzner and Eisele (2010 a), in this section shortly cited as AEa, we investigated the case  $T = 1$  with the coherent risk assessment  $\Phi_{\mathcal{P}_0}$  defined for  $X = (X_0, X_1) \in L$  by

$$\Phi_{\mathcal{P}_0}(X) := \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E} \left[ X_1 / r_1 \right]. \quad (7.1)$$

where  $\mathcal{P}$  was a closed convex set of probabilities on  $(\Omega, \mathcal{F})$ . Setting  $\Phi_{\mathcal{P}} = (\Phi_{\mathcal{P}_0}, \Phi_{\mathcal{P}_1})$  with  $\Phi_{\mathcal{P}_1} \equiv 0$ , we get a time consistent risk assessment on  $L$ .

In AEa, the non-supervisory-arbitrage (NSA) was given by  $\mathcal{P} \cap \mathcal{M} \neq \emptyset$ . We shall show that this is equivalent to the negation of Definition 2.4 (ii) with  $S = S_{\mathbb{Q}}$  for some  $\mathbb{Q} \in \mathcal{M}$ .

Indeed, suppose  $\mathbb{Q} \in \mathcal{P} \cap \mathcal{M}$ . Then for  $X = (X_0, X_1) \in S_{\mathbb{Q}}$  we get

$$\begin{aligned} \sup_{D_1 \in N_1} \Phi_{\mathcal{P}_0}(X + D_1 \cdot r(1)) &\leq \sup_{D_1 \in N_1} \mathbb{E}_{\mathbb{Q}} \left[ (X_1 + D_1) / r_1 \right] \leq \mathbb{E}_{\mathbb{Q}} \left[ X_1 / r_1 \right] \\ &\leq X_0 < +\infty. \end{aligned} \quad (7.2)$$

and  $\Phi_{\mathcal{P}}$  does not allow arbitrage on  $S_{\mathbb{Q}}$ .

Conversely, let  $\mathcal{P} \cap \mathcal{M} = \emptyset$ . Since both  $\mathcal{P}$  and  $\mathcal{M}$  are closed convex sets, there exists  $X_1 \in L_1$  and  $\beta \in \mathbb{R}$  such that

$$\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} [X_1 / r_1] < \beta < \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [X_1 / r_1]. \quad (7.3)$$

It follows that  $X = (0, X_1 - \beta \cdot r_1) \in S_{\mathcal{M}}$ . By the uniform Doob decomposition (1.6), we get  $X_1 - \beta \cdot r_1 = D_1 - B_1$  with  $D_1 \in N_1$  and  $B_1 \geq 0$ . Now (7.3) implies  $0 < \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [(D_1 - B_1) / r_1] \leq \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} [D_1 / r_1] = \Phi_{\mathcal{P}_0}(D_1 \cdot r(1))$  such that for all  $X \in L$

$$\lim_{\lambda \rightarrow \infty} \Phi_{\mathcal{P}_0}(X + \lambda \cdot D_1 \cdot r(1)) \geq \Phi_{\mathcal{P}_0}(X) + \lim_{\lambda \rightarrow \infty} \lambda \cdot \Phi_{\mathcal{P}_0}(D_1 \cdot r(1)) = +\infty.$$

This shows that  $\Phi_{\mathcal{P}}$  allows arbitrage on  $S_{\mathbb{Q}}$  for all  $\mathbb{Q} \in \mathcal{M}$ . For  $\Phi_{\mathcal{P}}$ , NSA in AEa and the non-arbitrage condition of Definition 2.4 are equivalent.

It is worthwhile noticing that by (2.2) in AEa and (2.9) above

$$\Phi_{\mathcal{P} \cap \mathcal{M} \ 0}(X) = \sup_{D_1 \in N_1} \Phi_{\mathcal{P}_0}(X + D_1 \cdot r(1)) = \Phi_{\mathcal{P}_0}^*(X) \quad (7.4)$$

is the market consistent majorant of  $\Phi_{\mathcal{P}}$ .

A mere comparison of Definition 1.2 in AEa and Definition 4.1 above shows that  $L_0(Z_1) = SP_0(Z)$ . For the cost-of-capital ratios in (5.5) we take  $q_0 = +\infty, q_1 = 0$ . It follows from (5.7) that  $TP_0(Z) = SP_0(Z) = L_0(Z_1)$  and  $SCR_0^*(Z) = 0$ . Finally, the two Solvency Accounting Equalities (4.5) in AEa and (6.13) above show that we have equality between (4.1) in AEa and (6.2) above:  $M_0(C_1) = SCR_0(D, Z) = SCR_0^r(D, Z)$ .

## CONCLUSION

A top-down construction of an accounting procedure for insurance companies is presented. Following the philosophy of Solvency II, we put special emphasis on the interweaving of supervision with market impacts: First, the danger of supervisory arbitrage by financial instruments is encountered by a best estimate bound. We next define the supervisory provision and along with it the optimal replicating portfolio using financial assets as hedging tools. The possibility to find new investors is the basis for the transfer principle which by the cost-of-capital ratio determines the “Fremd”-capital in the balance sheet. Finally, the solvency capital requirement defined as the minimal “Eigen”-capital necessary for acceptability takes care of the additional financial risk included in the choice of the company’s asset portfolio.

These steps are again reflected in the four categories of supervisor’s consecutive actions:

- (i) acceptance, if  $C_0 \geq TP_0(Z) + SCR_0(D, Z)$ ,
- (ii) rebalancing of the portfolio, if  $C_0 \geq TP_0(Z) + SCR_0^*(Z)$ ,
- (iii) transfer to a new company, if  $C_0 \geq TP_0(Z)$ ,
- (iv) closing of the company, if  $C_0 < TP_0(Z)$ .

The bridge from our top-down presentation to Solvency II or SST will be built in Eisele and Artzner (2011).

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