



ELSEVIER

Available online at [www.sciencedirect.com](http://www.sciencedirect.com)

Insurance: Mathematics and Economics ■■■■■ ■■■■■

[www.elsevier.com/locate/ime](http://www.elsevier.com/locate/ime)

# Recursions for multivariate compound phase variables

Karl-Theodor Eisele\*

Université Louis Pasteur Strasbourg, Pôle Européen de Gestion et d'Économie, 61, av. de la Forêt-Noire, F-67085 Strasbourg, France

Received May 2006; received in revised form December 2006; accepted 19 December 2006

## Abstract

We show how to generalize the result given in [Eisele, K.-Th., 2006. Recursions for compound phase distributions. *Insurance: Math. Econom.* 38, 149–156] to the multivariate case, i.e. we find a Panjer-like recursion principle for the distribution of a multivariate compound phase variable. Recursion formulas and procedures for the bivariate case are given in detail. We give a possible application for agricultural risks and calculate concrete examples via a VB-program.

© 2007 Elsevier B.V. All rights reserved.

IME Classification: IM11

MSC: 60E05; 62P05

Keywords: Discrete multivariate phase variables; Recursions for multivariate compound variables

## 0. Introduction

Variables of phase-type – in what follows simply called phase variables – are very nice, easy to handle tools in actuarial modeling and calculations (e.g. Bladt (2004) and Rolski et al. (2000)). With nearly the same setting, discrete and continuous phase variables can be treated, and in many cases they allow for explicit formulas, for instance for the ruin probability (see Asmussen (2000), Asmussen and Rolski (1991) or Schmidli (2005)). In Assaf et al. (1984), multivariate phase distributions were probably investigated for the first time. We believe that these advantages of uni- or multivariate phase variables will become even more important in the future (see e.g. Cai and Li (2005)).

Phase variables are almost equivalent to variables having rational generating functions (for the continuous case, see Asmussen (2000), p. 241). We used this fact in Eisele (2006) to show a recursion principle of Panjer's type for univariate compound phase variables. In a different context, Hipp showed in Hipp (2006) that Panjer's recursion is reduced to one of a local depth if the severity distributions are of phase type.

For a long time, Panjer's recursion principle has been generalized to various multivariate cases (see Hesselager (1996), Sundt (1999), Vernic (1999), Sundt (2000) and the forthcoming extensive book by Sundt and Vernic (in press)). Using the ideas in Eisele (2006), it is therefore obvious to look for recursion formulas for multivariate compound phase variables. In fact, the generalizations turn out to be straightforward. Though, at first sight, the recursion formulas seem to be complicated, a general VB-program is presented to calculate the common distribution of a bivariate compound phase variable. In Section 5, we describe a possible application for this recursion principle in the context of an insurance with agricultural risks.

## 1. Discrete multivariate phase variables

The basis of discrete phase variables is a discrete homogeneous Markov chain  $(X(t))_{t \in \mathbb{N}_0}$ ,  $\mathbb{N}_0 = \{0, 1, \dots\}$ , with state space  $\tilde{D} = \{0, 1, 2, \dots, d\}$ , a transition matrix  $\tilde{P} = (P_{jk})_{0 \leq j, k \leq d}$  and a starting distribution  $\tilde{\pi} = (\pi_0, \pi_1, \dots, \pi_d) = (\pi_0, \pi)$ . The extra state 0 is assumed to be an absorbing and attracting point: i.e.

- (i)  $\tilde{P}_{0j} = 0$  for all  $j \in D := \{1, 2, \dots, d\}$  (absorption at 0),
- (ii) there exists a power  $\tau$  of the transition matrix  $\tilde{P}$  such that  $\tilde{P}_{j0}^\tau > 0$  for all  $j \in D$  (attraction of 0).

\* Tel.: +33 390 24 21 08; fax: +33 390 24 21 07.

E-mail address: [eisele@math.u-strasbg.fr](mailto:eisele@math.u-strasbg.fr).

In the univariate case, a phase variable  $T$  is defined as the survival time of the Markov process  $X(t)$  in the transient part  $D$  of the state space:

$$T = \min\{t \in \mathbb{N}_0, X(t) \notin D\}. \tag{1.1}$$

Since we have  $X(t) = 0$  for  $t \rightarrow \infty$  almost surely, the phase variable  $T$  is finite. We can replace the singleton  $\{0\}$  by an absorbing subset  $A$  of  $\tilde{D}$ , where we call a subset  $A \subseteq \tilde{D}$  absorbing if

$$\mathbb{P}(X(t) \notin A \text{ for some } t \geq 0 | X(0) \in A) = 0. \tag{1.2}$$

The complement  $C = \tilde{D} \setminus A$  of an absorbing set is called inaccessible. Of course,  $D$  is inaccessible. Moreover, since  $0$  is absorbing and attractive, we must have that  $0 \in A$  for any absorbing subset  $A$ ; hence

$$C \subseteq D$$

for any inaccessible set  $C$ . The obvious generalization of (1.1) is

$$T_C = \min\{t > 0, X(t) \notin C\}. \tag{1.3}$$

To define now a multivariate discrete phase variable, we simply take a sequence  $\mathcal{C} = \{C_1, \dots, C_m\}$  of inaccessible subsets  $(C_\nu)_{\nu=1, \dots, m}$  of  $D$  to get the multivariate phase variable

$$\mathbf{T} = (T_1, \dots, T_m) = (T_{C_1}, \dots, T_{C_m}). \tag{1.4}$$

Since  $0 \in \bigcap_{\nu=1}^m \tilde{D} \setminus C_\nu$  and the elements of the last set are irrelevant for the phase variables  $T_\nu$ , we may assume in what follows without loss of generality that

$$\tilde{D} = \bigcup_{\nu=1}^m C_\nu \cup \{0\}. \tag{1.5}$$

The following notation will be useful in what follows: For  $C \subseteq D$ , let  $\Delta^C \in \mathbb{R}^{d \times d}$  be defined by

$$(\Delta^C)_{jk} = \begin{cases} 1 & \text{if } j = k \in C \\ 0 & \text{else,} \end{cases} \tag{1.6}$$

and

$$\eta = (1, \dots, 1)^{\text{tr}} \in \mathbb{R}^d. \tag{1.7}$$

Let

$$P = (\tilde{P}_{jk})_{1 \leq j, k \leq d} \tag{1.8}$$

be the sub-matrix of  $\tilde{P}$  suppressing the first line and the first column. Therefore

$$\tilde{P} = \begin{pmatrix} 1 & 0 \dots 0 \\ (Id - P)\eta & P \end{pmatrix}, \tag{1.9}$$

and for  $t \in \mathbb{N}_0$

$$\tilde{P}^t = \begin{pmatrix} 1 & 0 \dots 0 \\ (Id - P^t)\eta & P^t \end{pmatrix}. \tag{1.10}$$

## 2. Distributions of discrete bivariate phase variables

In the following, we shall concentrate on the bivariate case. With minor modifications, the results hold also in the multivariate case; however, the formulas become rather complicated to write down.

Let  $(\tilde{D}, \tilde{P}, \tilde{\pi}, \mathcal{C})$  with  $\mathcal{C} = \{C_1, C_2\}$  be the characteristics of a discrete bivariate phase variable.

**Theorem 2.1.** We have

(i) 
$$\mathbb{P}(T_1 = T_2 = 0) = \pi_0, \tag{2.1}$$

and for  $\nu = 1$  or  $2$ ,

(ii) 
$$\begin{aligned} \mathbb{P}(T_\nu = 0, T_{3-\nu} = t) &= \pi \Delta^{D \setminus C_\nu} P^{t-1} (Id - P)\eta \\ \mathbb{P}(T_\nu = 0, T_{3-\nu} > t) &= \pi \Delta^{D \setminus C_\nu} P^t \eta, \end{aligned} \tag{2.2}$$

(iii)  $0 < t_1 < t_2 \in \mathbb{N}$

$$\begin{aligned} \mathbb{P}(T_\nu = t_1, T_{3-\nu} = t_2) &= \pi P^{t_1-1} \Delta^{C_\nu} P \Delta^{C_{3-\nu} \setminus C_\nu} P^{(t_2-t_1-1)} (Id - P)\eta \\ \mathbb{P}(T_\nu > t_1, T_{3-\nu} > t_2) &= \pi P^{t_1} \Delta^{C_\nu} P^{t_2-t_1} \Delta^{C_{3-\nu}} \eta, \end{aligned} \tag{2.3}$$

(iv)  $0 < t \in \mathbb{N}$

$$\begin{aligned} \mathbb{P}(T_1 = T_2 = t) &= \pi P^{t-1} \Delta^{C_1 \cap C_2} (Id - P)\eta \\ \mathbb{P}(T_1 = T_2) &= \pi_0 + \pi (Id - P)^{-1} \Delta^{C_1 \cap C_2} (Id - P)\eta. \end{aligned} \tag{2.4}$$

**Proof.** (i) is evident. (ii) Setting  $\nu = 1$ , we get for  $t > 0$  with (1.5)

$$\begin{aligned} \mathbb{P}(T_1 = 0, T_2 = t) &= \sum_{\substack{j \in D \setminus C_1 \\ k \in C_2}} \mathbb{P}(X(0) = j) \mathbb{P}(X(t-1) \\ &= k | X(0) = j) \mathbb{P}(X(t) = 0 | X(t-1) = k) \\ &= \pi \Delta^{D \setminus C_1} P^{t-1} \Delta^{C_2} (Id - P)\eta \\ &= \pi \Delta^{D \setminus C_1} P^{t-1} (Id - P)\eta. \end{aligned}$$

The second equation of (ii) follows similarly.

(iii) With  $t_0 = 0$ , we get

$$\begin{aligned} \mathbb{P}(T_1 = t_1, T_2 = t_2) &= \sum_{\substack{j_0 \in D \\ k_\nu \in C_\nu, \nu=1,2 \\ j_1 \in D \setminus C_1, j_2=0}} \mathbb{P}(X(0) = j_0) \\ &\cdot \prod_{\nu=1}^2 [\mathbb{P}(X(t_\nu - 1) = k_\nu | X(t_{\nu-1}) = j_{\nu-1}) \mathbb{P}(X(t_\nu) \\ &= j_\nu | X(t_\nu - 1) = k_\nu)] \\ &= \pi P^{t_1-1} \Delta^{C_1} P \Delta^{D \setminus C_1} P^{t_2-t_1-1} \Delta^{C_2} (Id - P)\eta \\ &= \pi P^{t_1-1} \Delta^{C_1} P \Delta^{C_2 \setminus C_1} P^{t_2-t_1-1} (Id - P)\eta, \end{aligned}$$

the second equation of (iii) being similar.

(iv) Here, we find

$$\begin{aligned} \mathbb{P}(T_1 = T_2 = t) &= \sum_{\substack{j \in D \\ k \in C_1 \cap C_2}} \mathbb{P}(X(0) = j) \mathbb{P}(X(t-1) \\ &= k | X(0) = j) \mathbb{P}(X(t) = 0 | X(t-1) = k) \\ &= \pi P^{t-1} \Delta^{C_1 \cap C_2} (Id - P)\eta, \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(T_1 = T_2 > 0) &= \sum_{t \geq 1} \pi P^{t-1} \Delta^{C_1 \cap C_2} (Id - P) \eta \\ &= \pi (Id - P)^{-1} \Delta^{C_1 \cap C_2} (Id - P) \eta. \quad \square \end{aligned}$$

**3. The generating function**

For the discrete bivariate phase variable  $\mathbf{T} = (T_1, T_2)$  with inaccessible sets  $\{C_1, C_2\}$ , we want to calculate the *generating function*

$$\Phi_{\mathbf{T}}(z_1, z_2) = \mathbb{E} \left( z_1^{T_1} z_2^{T_2} \right). \tag{3.1}$$

First, we get

**Theorem 3.1.**

$$\begin{aligned} \Phi_{\mathbf{T}}(z_1, z_2) &= \pi_0 + \pi \left[ \sum_{v=1,2} \Delta^{C_v \setminus C_{3-v}} (Id z_v^{-1} - P)^{-1} \right. \\ &\quad + (Id(z_1 z_2)^{-1} - P)^{-1} \Delta^{C_1 \cap C_2} + (Id(z_1 z_2)^{-1} - P)^{-1} \\ &\quad \times \left. \sum_{v=1,2} \left( \Delta^{C_v} P \Delta^{C_{3-v} \setminus C_v} (Id z_{3-v}^{-1} - P)^{-1} \right) \right] \\ &\quad (Id - P) \eta \end{aligned} \tag{3.2}$$

and for  $v = 1, 2$  the marginal generating function

$$\begin{aligned} \Phi_{T_v}(z) &= \mathbb{E}(z^{T_v}) = (\pi_0 + \pi \Delta^{C_{3-v} \setminus C_v} \eta) \\ &\quad + \pi (Id z^{-1} - P)^{-1} (Id - P) \Delta^{C_v} \eta. \end{aligned} \tag{3.3}$$

**Proof.** We divide the sum of the generating function into the following partial sums:

$$\begin{aligned} \Phi_{\mathbf{T}}(z_1, z_2) &= \pi_0 + \sum_{\substack{0 \leq j, k \\ j+k \geq 1}} \mathbb{P}(T_1 = j, T_2 = k) z_1^j z_2^k \\ &= \pi_0 + S_{(0=T_1 < T_2)}(z_1, z_2) + S_{(0=T_2 < T_1)}(z_1, z_2) \\ &\quad + S_{(0 < T_1 = T_2)}(z_1, z_2) \\ &\quad + S_{(0 < T_1 < T_2)}(z_1, z_2) + S_{(0 < T_2 < T_1)}(z_1, z_2). \end{aligned} \tag{3.4}$$

From (2.2), we get

$$\begin{aligned} S_{(0=T_1 < T_2)}(z_1, z_2) &= \sum_{0 < t} \pi \Delta^{C_2 \setminus C_1} P^{t-1} z_2^t (Id - P) \eta \\ &= \pi \Delta^{C_2 \setminus C_1} (Id z_2^{-1} - P)^{-1} (Id - P) \eta. \end{aligned} \tag{3.5}$$

Similarly, from (2.4)

$$\begin{aligned} S_{(0 < T_1 = T_2)}(z_1, z_2) &= \sum_{0 < t} \pi P^{t-1} (z_1 z_2)^t \Delta^{C_1 \cap C_2} (Id - P) \eta \\ &= \pi (Id(z_1 z_2)^{-1} - P)^{-1} \Delta^{C_1 \cap C_2} (Id - P) \eta \end{aligned} \tag{3.6}$$

and finally from (2.3)

$$\begin{aligned} S_{(0 < T_1 < T_2)}(z_1, z_2) &= \sum_{0 < t_1 < t_2} \pi P^{t_1-1} (z_1 z_2)^{t_1} \Delta^{C_1} P \\ &\quad \times \Delta^{C_2 \setminus C_1} P^{t_2-t_1-1} z_2^{t_2-t_1} (Id - P) \eta \\ &= \pi (Id(z_1 z_2)^{-1} - P)^{-1} \Delta^{C_1} P \\ &\quad \times \Delta^{C_2 \setminus C_1} (Id z_2^{-1} - P)^{-1} (Id - P) \eta. \end{aligned} \tag{3.7}$$

These expressions give us (3.2). There are two ways to get (3.3): either directly from the definition, or – slightly more complicatedly – by setting  $v = 1, z = z_1$  and  $z_2 = 1$  in (3.2).  $\square$

Now, we introduce the characteristic polynomial of  $P$ :

$$\zeta(z) = \det(Idz - P) = b_0 z^d + b_1 z^{d-1} + \dots + b_d \text{ with } \tag{3.8}$$

$$b_0 = 1 \text{ and for } i = 1, \dots, d \tag{3.9}$$

$$b_i = (-1)^i \sum_{1 \leq k_1 < k_2 < \dots < k_{d-i} \leq d} \det(P(k_1, k_2, \dots, k_{d-i})), \tag{3.10}$$

where  $z \in \mathbb{C}$  and  $P(k_1, k_2, \dots, k_{d-i})$  denotes the sub-matrix of  $P$ , where we have deleted the  $k_1$ th up to the  $k_{d-i}$ th lines and columns. In particular, we have

$$b_1 = -\text{trace}(P) \text{ and } b_d = (-1)^d \det(P).$$

For the calculation of the coefficients of the polynomial, we recall *Newton's identity*:

**Proposition 3.2.** *Let*

$$q_k = \text{trace}(P^k) \tag{3.11}$$

for  $k = 1, \dots, d$ . Then

$$\begin{aligned} q_k + b_1 q_{k-1} + \dots + b_{k-1} q_1 + k b_k &= 0 \\ \text{for } 1 \leq k \leq d \text{ and} \end{aligned} \tag{3.12}$$

$$q_k + b_1 q_{k-1} + \dots + b_d q_{k-d} = 0 \text{ for } k > d. \tag{3.13}$$

See [Cohn \(1993\)](#) for a proof.

**Remark 3.3.** If we write (3.12) like

$$b_k = -\frac{1}{k} \sum_{l=0}^{k-1} b_l q_{k-l}, \tag{3.14}$$

then the coefficients of the characteristic polynomial satisfy a recursion principle quite similar to Panjer's one. Indeed, the proofs are also similar.

We also define

$$\bar{\zeta}(z) = \zeta(z^{-1}) z^d = 1 + b_1 z + \dots + b_d z^d. \tag{3.15}$$

With these notations, we get

**Theorem 3.4.** *The generating function (3.1) is rational in  $\bar{z} = (z_1, z_2)$ ; more precisely,*

$$\Phi_{\mathbf{T}}(z_1, z_2) = \pi_0 + \frac{1}{\bar{\zeta}(z_1 z_2) \bar{\zeta}(z_1) \bar{\zeta}(z_2)} \sum_{\substack{0 \leq j, k \leq 2d \\ 1 \leq j+k}} a_{jk} z_1^j z_2^k, \tag{3.16}$$

where  $\bar{\zeta}$  is given in (3.15).

**Proof.** Let us look at the expressions  $(Id z^{-1} - P)^{-1}$  in (3.5)–(3.7). By Cramer's rule, we get with certain coefficients  $a_2(j, k), \dots, a_d(j, k)$ :

$$\begin{aligned} ((Id z^{-1} - P)^{-1})_{jk} &= (-1)^{j+k} \\ &\quad \times \frac{1}{\bar{\zeta}(z^{-1})} \left( z^{-(d-1)} + a_2(j, k) z^{-(d-2)} + \dots + a_d(j, k) \right) \end{aligned}$$

$$= (-1)^{j+k} \frac{1}{\zeta(z)} \left( z + a_2(j, k)z^2 + \dots + a_d(j, k)z^d \right).$$

Using this equation in the formulas (3.5)–(3.7), we can rewrite them with coefficients  $\alpha_i, \beta_i, \gamma_i$  for  $i = 1, \dots, d$  and  $\delta_{jk}, \eta_{jk}$  with  $1 \leq j, k \leq 2d$  as

$$\begin{aligned} S_{(0=T_1 < T_2)}(z_1, z_2) &= \frac{1}{\zeta(z_2)} \left( \alpha_1 z_2 + \alpha_2 z_2^2 + \dots + \alpha_d z_2^d \right) \\ S_{(0=T_2 < T_1)}(z_1, z_2) &= \frac{1}{\zeta(z_1)} \left( \beta_1 z_1 + \beta_2 z_1^2 + \dots + \beta_d z_1^d \right) \\ S_{(0 < T_1 = T_2)}(z_1, z_2) &= \frac{1}{\zeta(z_1 z_2)} \left( \gamma_1 (z_1 z_2) + \gamma_2 (z_1 z_2)^2 + \dots + \gamma_d (z_1 z_2)^d \right) \\ S_{(0 < T_1 < T_2)}(z_1, z_2) &= \frac{1}{\zeta(z_1 z_2) \bar{\zeta}(z_2)} \sum_{1 \leq j, k \leq d} \delta_{jk} z_1^j z_2^{j+k} \\ S_{(0 < T_2 < T_1)}(z_1, z_2) &= \frac{1}{\zeta(z_1 z_2) \bar{\zeta}(z_1)} \sum_{1 \leq j, k \leq d} \eta_{jk} z_1^{j+k} z_2^k. \end{aligned}$$

Joining these formulas over the common denominator  $\frac{1}{\zeta(z_1 z_2) \bar{\zeta}(z_1) \bar{\zeta}(z_2)}$ , we get (3.16).  $\square$

We rewrite the common denominator as

**Proposition 3.5.**

$$\bar{\zeta}(z_1 z_2) \bar{\zeta}(z_1) \bar{\zeta}(z_2) = \sum_{0 \leq j, k \leq 2d} b_{jk} z_1^j z_2^k. \tag{3.17}$$

With  $b_0 = 1$  and  $b_i$  from (3.8), the coefficients in (3.17) are

$$b_{jk} = b_{kj} = \sum_{i=\max(0, j-d, k-d)}^{\min(d, j, k)} b_i b_{j-i} b_{k-i} \tag{3.18}$$

for  $0 \leq j, k \leq 2d$ . Here,  $\sum_{i=p}^q (\dots) = 0$  if  $p > q$ . In particular,  $b_{00} = 1$ .

**Proof.** We have

$$\begin{aligned} \bar{\zeta}(z_1 z_2) \bar{\zeta}(z_1) \bar{\zeta}(z_2) &= (1 + b_1(z_1 z_2) + \dots + b_d(z_1 z_2)^d) \\ &\quad \times (1 + b_1 z_1 + \dots + b_d z_1^d) (1 + b_1 z_2 + \dots + b_d z_2^d) \\ &= \sum_{0 \leq j, k \leq 2d} \left( \sum_{i=\max(0, j-d, k-d)}^{\min(d, j, k)} b_i b_{j-i} b_{k-i} \right) z_1^j z_2^k. \quad \square \end{aligned}$$

Having calculated the coefficients  $b_{jk}$  of the denominator, we are left to determine those of the numerator from (3.16) since we want to use them in our recursion formula for bivariate compound phase variables.

**Theorem 3.6.** For  $0 \leq j, k \leq 2d, 1 \leq j + k$

$$a_{jk} = \sum_{\substack{0 \leq l \leq j \\ 0 \leq m \leq k \\ l+m < j+k}} b_{lm} \mathbb{P}(T_1 = j - l, T_2 = k - m), \tag{3.19}$$

in particular

$$\begin{aligned} a_{10} &= \mathbb{P}(T_1 = 1, T_2 = 0) = \pi \Delta^{C_1 \setminus C_2} (Id - P) \eta \\ a_{01} &= \mathbb{P}(T_1 = 0, T_2 = 1) = \pi \Delta^{C_2 \setminus C_1} (Id - P) \eta, \end{aligned} \tag{3.20}$$

where we get the last equations from Theorem 2.1.

**Proof.** Since  $b_{00} = 1$ , we get from (3.4)

$$\Phi_{\mathbf{T}}(z_1, z_2) = \pi_0 + \frac{1}{\sum_{0 \leq j, k \leq 2d} b_{jk} z_1^j z_2^k} \sum_{\substack{0 \leq j, k \leq 2d \\ 1 \leq j+k}} a_{jk} z_1^j z_2^k.$$

Hence,

$$\begin{aligned} \Phi_{\mathbf{T}}(z_1, z_2) - \pi_0 &= \sum_{\substack{0 \leq j, k \leq 2d \\ 1 \leq j+k}} a_{jk} z_1^j z_2^k \\ &= \sum_{\substack{0 \leq j, k \leq 2d \\ 1 \leq j+k}} b_{jk} z_1^j z_2^k (\pi_0 - \Phi_{\mathbf{T}}(z_1, z_2)) \\ &= - \sum_{\substack{0 \leq j_1, k_1 \leq 2d \\ 1 \leq j_1+k_1}} \sum_{\substack{0 \leq j_2, k_2 < \infty \\ 1 \leq j_2+k_2}} b_{j_1 k_1} \mathbb{P}(T_1 = j_2, T_2 = k_2) z_1^{j_1+j_2} \\ &\quad \times z_2^{k_1+k_2}. \end{aligned}$$

Using the set of indices

$$\begin{aligned} \Xi_{jk} &= \{(l, m); 0 \leq l \leq j \wedge 2d, 0 \leq m \leq k \wedge 2d, \\ &\quad 1 \leq l + m < j + k\}, \end{aligned}$$

the last expression can be written as

$$- \sum_{\substack{0 \leq j, k < \infty \\ 2 \leq j+k}} \sum_{(l, m) \in \Xi_{jk}} b_{lm} \mathbb{P}(T_1 = j - l, T_2 = k - m) z_1^j z_2^k.$$

Therefore,

$$\begin{aligned} &\sum_{\substack{0 \leq j, k \leq 2d \\ 1 \leq j+k}} (\mathbb{P}(T_1 = j, T_2 = k) - a_{jk}) z_1^j z_2^k \\ &= - \sum_{\substack{0 \leq j, k < \infty \\ 2 \leq j+k}} \sum_{(l, m) \in \Xi_{jk}} b_{lm} \mathbb{P}(T_1 = j - l, T_2 = k - m) z_1^j z_2^k, \end{aligned}$$

which gives us

$$\begin{aligned} a_{10} &= \mathbb{P}(T_1 = 1, T_2 = 0) \\ a_{01} &= \mathbb{P}(T_1 = 0, T_2 = 1), \end{aligned}$$

and for  $2 \leq j + k$

$$\begin{aligned} a_{jk} &= \mathbb{P}(T_1 = j, T_2 = k) \\ &\quad + \sum_{(l, m) \in \Xi_{jk}} b_{lm} \mathbb{P}(T_1 = j - l, T_2 = k - m). \end{aligned}$$

This is identical to (3.19), since  $b_{00} = 1$ .  $\square$

**Example 3.7.** In the most simple case  $d = 1$  and  $P = p$  ( $0, 1$ ),  $\pi = \pi_1 = 1, C_1 = \{1\}$  and  $C_2 = \emptyset$ , we have

$$b_0 = 1 \quad \text{and} \quad b_1 = -p.$$

From (3.18) and (3.19), we get

$$B = (b_{jk})_{0 \leq j, k \leq 2} = \begin{pmatrix} 1 & -p & 0 \\ -p & -p(1-p) & p^2 \\ 0 & p^2 & -p^3 \end{pmatrix} \quad \text{and} \tag{3.21}$$

$$\begin{aligned} A &= (a_{jk})_{0 \leq j, k \leq 2} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 1-p & -p(1-p) & 0 \\ 0 & -p(1-p) & p^2(1-p) \end{pmatrix}. \end{aligned} \tag{3.22}$$

4. Recursion for bivariate compound phase variables

Let us look at the bivariate compound phase variables

$$S_1 = \sum_{t=1}^{T_1} X_t, \quad S_2 = \sum_{t=1}^{T_2} Y_t, \tag{4.1}$$

where

- $\mathbf{T} = (T_1, T_2)$  is a discrete bivariate phase variable,
- $(X_t)_{t \geq 1}$  and  $(Y_t)_{t \geq 1}$  are two families of iid variables with distributions  $\mathbb{P}_X$  and  $\mathbb{P}_Y$ , respectively,
- $\mathbf{T}, (X_t)_{t \geq 1}$  and  $(Y_t)_{t \geq 1}$  are mutually independent.

For the common distribution of  $\mathbf{S} = (S_1, S_2)$ , we are going to give a Panjer-like recursion, where we restrict ourselves here to the equidistant case (see also the Remark 4.2), i.e.

$$X_t, Y_t \in \mathbb{N}. \tag{4.2}$$

**Theorem 4.1** (The Bivariate Recursion Formula). Using the coefficients  $a_{jk}$  and  $b_{jk}$  from Theorem 3.4, Proposition 3.5, and Theorem 3.6, we have

$$\mathbb{P}(S_1 = S_2 = 0) = \pi_0 \tag{4.3}$$

and for  $j, k \geq 0, 1 \leq j+k$

$$\begin{aligned} \mathbb{P}(S_1 = j, S_2 = k) &= \sum_{\substack{0 \leq l \leq j \wedge 2d \\ 0 \leq m \leq k \wedge 2d \\ 1 \leq l+m}} \left[ a_{lm} \mathbb{P}_{X^{*l}}(j) \mathbb{P}_{Y^{*m}}(k) - b_{lm} \right. \\ &\quad \left. \times \sum_{\substack{l \leq p \leq j \\ m \leq q \leq k \\ p+q < j+k}} \mathbb{P}_{X^{*l}}(p) \mathbb{P}_{Y^{*m}}(q) \mathbb{P}(S_1 = j-p, S_2 = k-q) \right]. \end{aligned} \tag{4.4}$$

**Proof.** By (3.16), we have

$$\begin{aligned} \Phi_{\mathbf{S}}(z_1, z_2) &= \Phi_{\mathbf{T}}(\Phi_X(z_1), \Phi_Y(z_2)) \\ &= \pi_0 + \frac{\sum_{1 \leq l+m}^{2d} a_{lm} \Phi_X^l(z_1) \Phi_Y^m(z_2)}{1 + \sum_{1 \leq l+m}^{2d} b_{lm} \Phi_X^l(z_1) \Phi_Y^m(z_2)}, \end{aligned}$$

hence,

$$\begin{aligned} \Phi_{\mathbf{S}}(z_1, z_2) - \pi_0 &= \sum_{1 \leq m+l}^{2d} a_{lm} \Phi_X^m(z_1) \Phi_Y^l(z_2) \\ &\quad - \sum_{1 \leq l+m}^{2d} b_{lm} \Phi_X^l(z_1) \Phi_Y^m(z_2) (\Phi_{\mathbf{S}}(z_1, z_2) - \pi_0) \end{aligned}$$

or

$$\sum_{1 \leq j+k} z_1^j z_2^k \left[ \mathbb{P}(S_1 = j, S_2 = k) - \sum_{1 \leq l+m}^{2d} a_{lm} \mathbb{P}_{X^{*l}}(j) \mathbb{P}_{Y^{*m}}(k) \right]$$

$$\begin{aligned} &= - \sum_{\substack{0 \leq l \leq 2d \\ 0 \leq m \leq 2d \\ 1 \leq l+m}} b_{lm} \Phi_{X^{*l}}(z_1) \Phi_{Y^{*m}}(z_2) (\Phi_{\mathbf{S}}(z_1, z_2) - \pi_0) \\ &= - \sum_{1 \leq j+k} z_1^j z_2^k \left[ \sum_{\substack{0 \leq l \leq j \wedge 2d \\ 0 \leq m \leq k \wedge 2d \\ 1 \leq l+m}} \sum_{\substack{l \leq p \leq j \\ m \leq q \leq k \\ p+q < j+k}} b_{lm} \mathbb{P}_{X^{*l}}(p) \right. \\ &\quad \left. \times \mathbb{P}_{Y^{*m}}(q) \mathbb{P}(S_1 = j-p, S_2 = k-q) \right]. \end{aligned}$$

Comparison of coefficients gives (4.4).  $\square$

**Remark 4.2.** In the case where the distributions of  $(X_t)$  and  $(Y_t)$  have densities on  $(0, \infty)$ , we can derive a continuous version of (4.4), similar to Proposition 3.4. in Eisele (2006).

5. Applications and examples

One possible application of dependent compound variables may be the following:

Let us consider an insurance company specialized in agricultural risks. For one line of business, it holds contracts in two neighbouring regions  $A$  and  $B$ . Quite a number of casualties in these regions have common causes, like thunderstorms or flood disasters hitting both regions equally, while other damages are independent for every region. Therefore, it seems to be reasonable to model the numbers of casualties  $T_A$  and  $T_B$  for region  $A$ , resp.  $B$ , as

$$\begin{aligned} T_A &= T_1 + T_2 \quad \text{and} \\ T_B &= T_1 + T_3, \end{aligned} \tag{5.1}$$

where  $T_1, T_2$  and  $T_3$  are numbers of casualties due to independent causes. For example,  $(T_i)_{i=1,2,3}$  may be independent negative binomial variables with natural numbers as shape parameters, in which case the pair  $(T_A, T_B)$  is a bivariate phase variable. If the amounts of casualties are described by two independent families of iid variables  $(X_{A,j})_{j \geq 1}$  and  $(X_{B,j})_{j \geq 1}$ , then the total losses of the insurance company are

$$S_A = \sum_{j=1}^{T_A} X_{A,j} \quad \text{and} \quad S_B = \sum_{j=1}^{T_B} X_{B,j}. \tag{5.2}$$

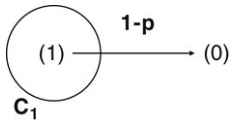
The common distribution of  $S_A$  and  $S_B$  can be calculated by a recursion principle similar to Theorem 4.1. In particular, if we assume for simplicity the variables  $X_{A,j}$  and  $X_{B,j}$  having values in equidistant lattices  $\mathbb{N}_A = h_A \cdot \mathbb{N}$ , resp.  $\mathbb{N}_B = h_B \cdot \mathbb{N}$  with some positive step widths  $h_A$  and  $h_B$ , then formula (4.4) gives the common probability  $\mathbb{P}(S_A = h_A \cdot j, S_B = h_B \cdot k)$  for  $j, k \geq 0$ . See also Example 5.2.

Another example for the dependence structure of discrete phase variables would be

$$\begin{aligned} T_A &= T_1 + \min(T_2, T_3) \quad \text{and} \\ T_B &= T_1 + \min(T_2, T_4) \end{aligned} \tag{5.3}$$

with  $T_i$ ,  $i = 1, \dots, 4$ , independent discrete phase variables. While it is known that continuous bivariate phase variables have in general copulas or tail-copulas of Marshall–Olkin type (see Hipp (2005)), not much is known in the discrete case. For more concrete examples, we start with the easiest case:

**Example 5.1.** Let  $d = 1$ ,  $P = p \in (0, 1)$ ,  $\pi = \pi_1 = 1$ ,  $C_1 = \{1\}$ , and  $C_2 = \emptyset$ . We have the following simple situation:



Then  $b_0 = 1$  and  $b_1 = -p$ .

From (3.18) and (3.19), we get

$$B = (b_{jk})_{0 \leq j, k \leq 2} = \begin{pmatrix} 1 & -p & 0 \\ -p & -p(1-p) & p^2 \\ 0 & p^2 & -p^3 \end{pmatrix} \text{ and } (5.4)$$

$$A = (a_{jk})_{0 \leq j, k \leq 2} = \begin{pmatrix} 0 & 0 & 0 \\ 1-p & -p(1-p) & 0 \\ 0 & -p(1-p) & p^2(1-p) \end{pmatrix}. (5.5)$$

The common distribution of  $\mathbf{T} = (T_1, T_2)$  is of course the geometric distribution (starting from 1) for  $T_1$ , while  $T_2 = 0$  almost surely:

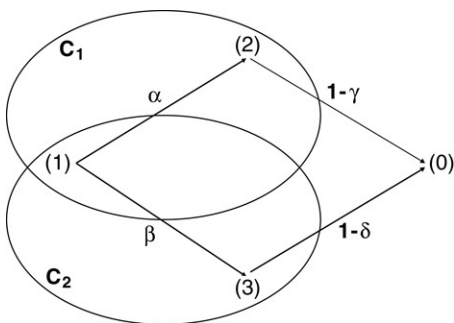
$$\mathbb{P}(T_1 = j, T_2 = k) = \mathbf{1}_{\{0\}}(k) \mathbf{1}_{\mathbb{N}}(j) (1-p)p^{j-1}. (5.6)$$

We get this result also from (4.4), if we put  $X = Y = 1$  with probability 1.

**Example 5.2.** We put  $d = 3$ ,  $C_1 = \{1, 2\}$ ,  $C_2 = \{1, 3\}$ ,  $\pi = (1, 0, 0)$  and with parameters  $\alpha, \beta, \gamma, \delta \in (0, 1)$ , such that  $0 < \alpha + \beta \leq 1$ ,

$$P = \begin{pmatrix} 1-\alpha-\beta & \alpha & \beta \\ 0 & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix}. (5.7)$$

The Markov chain can be characterized by the following diagram:



We remark that this diagram almost corresponds to the situation described by (5.1) if we let  $T_1, T_2$  and  $T_3$  be independent geometric variables with parameters  $1 - \alpha - \beta, \gamma$  and  $\delta$ ,

respectively. Here, however, we would have  $\mathbb{P}(T_2 = 0, T_3 = 0) = 0$ .

The characteristic polynomial of  $P$  has the coefficients

$$b_0 = 1, \quad b_1 = -(1 - \alpha - \beta + \gamma + \delta), \\ b_2 = (1 - \alpha - \beta)(\gamma + \delta) + \gamma\delta, \\ \text{and } b_3 = -(1 - \alpha - \beta)\gamma\delta.$$

This gives us the symmetric matrix (we only write the upper triangle) as given in Box I.

Passing to numerical data with  $\alpha = \beta = 0.2$  and  $\gamma = \delta = 0.3$ , we find  $B =$

1	-1,200	0,450	-0,054	0	0	0
-1,200	0,240000	0,900000	-0,475200	0,064800	0	0
0,450	0,900000	-1,075500	0,083700	0,124740	-0,024300	0
-0,054	-0,475200	0,083700	0,353916	-0,149040	0,004860	0,002916
0	0,064800	0,124740	-0,149040	0,009866	0,018225	-0,003499
0	0	-0,024300	0,004860	0,018225	-0,009623	0,001312
0	0	0	0,002916	-0,003499	0,001312	-0,000157

for the matrix of probabilities

$$M = (\mathbb{P}(T_1 = j, T_2 = k))_{0 \leq j, k \leq 2d} =$$

0	0	0	0	0	0	0
0	0	0,140000	0,042000	0,012600	0,003780	0,001134
0	0,140000	0	0,084000	0,025200	0,007560	0,002268
0	0,042000	0,084000	0	0,050400	0,015120	0,004536
0	0,012600	0,025200	0,050400	0	0,030240	0,009072
0	0,003780	0,007560	0,015120	0,030240	0	0,018144
0	0,001134	0,002268	0,004536	0,009072	0,018144	0

and for  $A =$

0	0	0	0	0	0	0
0	0	0,140000	-0,126000	0,025200	0,000000	0,000000
0	0,140000	-0,336000	0,130200	0,037800	-0,015120	0,000000
0	-0,126000	0,130200	0,088200	-0,097776	0,011340	0,002268
0	0,025200	0,037800	-0,097776	0,035078	0,008392	-0,003402
0	0,000000	-0,015120	0,011340	0,008392	-0,008573	0,001633
0	0,000000	0,000000	0,002268	-0,003402	0,001633	-0,000245

Finally, if for  $X$  and  $Y$  we use the distributions

$$\mathbb{P}(X = j) = \begin{cases} 1/3 & \text{for } 1 \leq j \leq 3 \\ 0 & \text{else} \end{cases} \text{ and } \\ \mathbb{P}(Y = j) = \begin{cases} 1/5 & \text{for } 1 \leq j \leq 5 \\ 0 & \text{else,} \end{cases}$$

the common probabilities  $S = (\mathbb{P}(S_1 = j, S_2 = k))$  (here restricted to  $0 \leq j, k \leq 11$ ) are given in Table 1.

### 6. The multivariate case

It is not very difficult to see that the procedure given in Section 4 also holds in the multivariate case, where

$$\mathbf{T} = (T_1, \dots, T_m) = (T_{C_1}, \dots, T_{C_m}) (6.1)$$

for inaccessible subsets  $C_1, \dots, C_m \subseteq D$ . The crucial point is the rationality of the generating function

$$\Phi_{\mathbf{T}}(z_1, \dots, z_m) = \mathbb{E}(z_1^{T_1} \dots z_m^{T_m}). (6.2)$$

$$B = \begin{pmatrix} 1 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ b_1 + b_1^2 & b_1^2 + b_1 b_2 & b_1 b_2 + b_1 b_3 & b_1 b_3 & 0 & 0 & 0 \\ b_2 + b_1^3 + b_2^2 & b_2 b_1 + b_1^2 b_2 + b_2 b_3 & b_2^2 + b_1^2 b_3 & b_2 b_3 & 0 & 0 & 0 \\ b_3 + b_2 b_1^2 + b_1 b_2^2 + b_3^2 & b_3 b_1 + b_2^2 b_1 + b_1 b_2 b_3 & b_3 b_2 + b_1 b_2 b_3 & b_3 b_3 & 0 & 0 & 0 \\ b_3 b_1^2 + b_3^2 + b_1 b_3^2 & b_3 b_1^2 + b_3^2 + b_1 b_3^2 & b_3 b_2 + b_1 b_2 b_3 & b_3 b_3 & 0 & 0 & 0 \\ b_1 b_2 b_3 + b_2^2 b_3 & b_1 b_2 b_3 + b_2^2 b_3 & b_1 b_2 b_3 + b_2^2 b_3 & b_1 b_2 b_3 & 0 & 0 & 0 \\ b_3 b_2^2 + b_2 b_3^2 & b_3 b_2^2 + b_2 b_3^2 & b_3 b_2^2 + b_2 b_3^2 & b_3 b_2^2 + b_2 b_3^2 & 0 & 0 & 0 \\ b_1 b_3^2 & b_1 b_3^2 & b_1 b_3^2 & b_1 b_3^2 & 0 & 0 & 0 \\ b_2 b_3^2 & b_2 b_3^2 & b_2 b_3^2 & b_2 b_3^2 & 0 & 0 & 0 \\ b_3^3 & b_3^3 & b_3^3 & b_3^3 & 0 & 0 & 0 \end{pmatrix}.$$

Box I.

Table 1

0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0,001867	0,003845	0,005943	0,008166	0,010523	0,009287	0,007866	0,006240	0,004392	0,002298		
0	0,003111	0,004978	0,007031	0,009282	0,011743	0,011315	0,010501	0,009376	0,007912	0,006075	0,003831		
0	0,006533	0,008524	0,010777	0,013309	0,016142	0,012761	0,012275	0,011371	0,010001	0,008109	0,005638		
0	0,010298	0,010684	0,011298	0,012164	0,013306	0,004450	0,005448	0,006116	0,006394	0,006219	0,005520		
0	0,008216	0,009014	0,009982	0,011146	0,012536	0,005967	0,006311	0,006472	0,006382	0,005964	0,005133		
0	0,005616	0,006618	0,007747	0,009029	0,010494	0,006557	0,006482	0,006315	0,005991	0,005437	0,004570		
0	0,002413	0,003378	0,004431	0,005592	0,006885	0,005926	0,005640	0,005317	0,004912	0,004369	0,003620		
0	0,001625	0,002274	0,003035	0,003921	0,004952	0,004524	0,004612	0,004589	0,004427	0,004087	0,003522		
0	0,000965	0,001351	0,001842	0,002451	0,003192	0,003120	0,003414	0,003593	0,003633	0,003505	0,003167		
0	0,000500	0,000700	0,000981	0,001353	0,001831	0,001932	0,002275	0,002542	0,002709	0,002749	0,002627		
0	0,000309	0,000433	0,000606	0,000840	0,001149	0,001239	0,001498	0,001737	0,001932	0,002054	0,002069		

In fact, using the modified characteristic polynomial  $\bar{\zeta}(z)$  from (3.15), we find that  $\Phi_{\mathbf{T}}$  has the following form:

**Proposition 6.1.**

$$\begin{aligned} \Phi_{\mathbf{T}}(z_1, \dots, z_m) &= \pi_0 + \frac{1}{\prod_{v=1}^m \prod_{1 \leq j_1 < \dots < j_v \leq m} \bar{\zeta}(z_{j_1} \dots z_{j_v})} \\ &\times \sum_{\substack{0 \leq l_1 \dots l_m \leq 2^{m-1}d \\ 1 \leq l_1 + \dots + l_m}} a_{l_1 \dots l_m} z_1^{l_1} \dots z_m^{l_m} \\ &= \pi_0 + \frac{\sum_{\substack{0 \leq l_1 \dots l_m \leq 2^{m-1}d \\ 1 \leq l_1 + \dots + l_m}} a_{l_1 \dots l_m} z_1^{l_1} \dots z_m^{l_m}}{1 + \sum_{\substack{0 \leq l_1 \dots l_m \leq 2^{m-1}d \\ 1 \leq l_1 + \dots + l_m}} b_{l_1 \dots l_m} z_1^{l_1} \dots z_m^{l_m}}. \end{aligned} \tag{6.3}$$

Once we have the rationality of the generating function, it is now awkward, but straightforward, to get the general recursion principle for multivariate compound phase variables,

$$\mathbf{S} = \left( S_1 = \sum_{i=1}^{T_1} X_{1i}, \dots, S_m = \sum_{i=1}^{T_m} X_{mi} \right), \tag{6.4}$$

with  $(X_{vi})_{i \geq 1}$  iid random variables with values in  $\mathbb{N}$ ,  $1 \leq v \leq m$ , and mutually independent families  $(X_{1i})_i, \dots, (X_{mi})_i, \mathbf{T}$ .

**Theorem 6.2.** We have

$$\mathbb{P}(S_1 = \dots = S_m = 0) = \pi_0 \tag{6.5}$$

and for  $j_1, \dots, j_m \geq 0, \sum_{v=1}^m j_v \geq 1$

$$\begin{aligned} \mathbb{P}(S_v = j_v, 1 \leq v \leq m) &= \sum_{\substack{0 \leq k_v \leq j_v \wedge 2^{m-1}d \\ 1 \leq \sum k_v \\ 1 \leq v \leq m}} a_{k_1 \dots k_m} \prod_{v=1}^m \mathbb{P}_{X_v^{*k_v}}(j_v) - b_{k_1 \dots k_m} \\ &\times \sum_{\substack{k_v \leq l_v \leq j_v \\ \sum k_v \leq \sum j_v \\ 1 \leq v \leq m}} \prod_{v=1}^m \mathbb{P}_{X_v^{*k_v}}(l_v) \mathbb{P}(S_v = j_v - l_v, 1 \leq v \leq m) \end{aligned} \tag{6.6}$$

We omit proofs in this section since they are direct generalizations of those given in Section 4.

**Acknowledgements**

This work has partly been executed while the author was visiting the China Institute for Actuarial Science at the Central University of Finance and Economics at Beijing, China. Its financial support is gratefully acknowledged. Also, the author is grateful to Frédérique Henge who helped in working out a VB-program and to one of the referees for valuable suggestions.

**References**

Asmussen, S., 2000. Ruin Probabilities. World Scientific.  
 Asmussen, S., Rolski, T., 1991. Computational methods in risk theory: A matrix algorithmic approach. Insurance: Mathematics and Economics 10, 259–274.

- Assaf, D., Landberg, N., Savits, T., Shaked, M., 1984. Multivariate phase-type distributions. *Operations Research* 32, 688–702.
- Bladt, M., 2004. Phase method. In: *Encyclopedia of Actuarial Science*. J. Wiley, pp. 1283–1288.
- Cai, J., Li, H., 2005. Multivariate risk model of phase type. *Insurance: Mathematics and Economics* 36, 137–152.
- Cohn, P.M., 1993. *Algebra*, vol. 1. J. Wiley.
- Eisele, K.-Th., 2006. Recursions for compound phase distributions. *Insurance: Mathematics and Economics* 38, 149–156.
- Gerber, H.U., Shiu, E.S.W., 2005. The time value of ruin in a Sparre Andersen. *North American Actuarial Journal* 9, 49–84.
- Hesselager, O., 1996. Recursions for certain bivariate counting distributions and their compound distributions. *Astin Bulletin* 26, 35–52.
- Hipp, C., 2005. Phasetype distributions and their applications. Strasbourg lecture.
- Hipp, C., 2006. Speedy convolution algorithms and Panjer recursions for phase-type distributions. *Insurance: Mathematics and Economics* 38, 176–188.
- Rolski, T., Schmidli, H., Schmidt, V., Teugels, J., 2000. *Stochastic Processes for Insurance and Finance*. J. Wiley.
- Schmidli, H., 2005. Discussion of Gerber and Shiu (2005). *North American Actuarial Journal* 9, 69–70.
- Sundt, B., 1999. On multivariate Panjer recursions. *Astin Bulletin* 29, 29–45.
- Sundt, B., 2000. On multivariate Vernic recursions. *Astin Bulletin* 30, 111–122.
- Sundt, B., Vernic, R., Recursions for convolutions and compound distributions with actuarial applications (in press).
- Vernic, V., 1999. Recursive evaluation of some bivariate compound distributions. *Astin Bulletin* 29, 315–325.