

SENSITIVITY AND DEVANEY'S CHAOS IN UNIFORM SPACES

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ABSTRACT. We give sufficient conditions for sensitivity of continuous group actions on uniform spaces.

1. INTRODUCTION

Sensitivity is an important property of chaotic dynamical systems which is related to what was popularized as the “butterfly effect” in the 1970s. When the phase space is equipped with a metric, the definition for group actions goes as follows. One says that an action of a group G on a metric space (X, d) is *sensitive* if there exists a constant $\varepsilon > 0$ such that, for every point $x \in X$ and every neighborhood N of x , there exist a point $y \in N$ and an element $g \in G$ such that $d(gx, gy) \geq \varepsilon$ (cf. [1], [3], [4], [5], [6], [7]). The goal of this note is to give a definition of sensitivity in the case when the phase space X is only equipped with an uniform structure instead of a metric and to extend to this more general setting two classical results providing sufficient conditions for sensitivity.

The theory of uniform spaces was developed in [8]. A uniform space is a set X equipped with a uniform structure, i.e., a set \mathcal{U} consisting of subsets of the Cartesian square $X \times X$. These subsets are called the *entourages* of the uniform space and are required to verify a certain list of axioms (see Section 2). Heuristically, two points $x, y \in X$ are “close” when the pair (x, y) belongs to a “small” entourage. Sensitivity for group actions on uniform spaces may be defined as follows.

Definition 1.1. One says that the action of a group G on a uniform space (X, \mathcal{U}) has *sensitive dependence on initial conditions*, or more briefly that the action is *sensitive*, if there is an entourage $U \in \mathcal{U}$ such that, for all $x \in X$ and every neighborhood N of x , there exist a point $y \in N$ and an element $g \in G$ such that $(gx, gy) \notin U$. Such an entourage U is then called a *sensitivity entourage* for the dynamical system (X, G) .

2000 *Mathematics Subject Classification.* 37D45, 37B05, 54E15.

Key words and phrases. dynamical system, uniform space, chaos, Devaney’s definition of chaos, density of periodic points, topologically transitive, topologically mixing, sensitive.

When the uniform structure comes from a metric, this definition is equivalent to the one given above.

Note that every uniform space admitting a sensitive action must be *perfect*, i.e., without isolated points. In particular, every Hausdorff uniform space admitting a sensitive action is infinite.

Sensitivity is a weak local version of expansivity. We recall that the action of a group G on a uniform space (X, \mathcal{U}) is called *expansive* if there is an entourage $U \in \mathcal{U}$ such that, for all distinct points $x, y \in X$, there exists an element $g \in G$ such that $(gx, gy) \notin U$. Such an entourage U is then called an *expansivity entourage* for the dynamical system (X, G) . It is clear that if the action of a group G on a perfect uniform space is expansive then this action is also sensitive (any expansivity entourage is a sensitivity entourage).

Recall the following standard definitions. Let X be a set equipped with an action of a group G . The orbit of a point $x \in X$ is the subset $Gx = \{gx : g \in G\} \subset X$ and its *stabilizer* is the subgroup $\text{Stab}_G(x) = \{g \in G : gx = x\} \subset G$. A point $x \in X$ is called *periodic* if its orbit is finite. Equivalently, x is periodic if and only if its stabilizer is of finite index in G . Suppose now that X is a topological space. One says that the action of G on X is *continuous* if, for each $g \in G$, the map $x \mapsto gx$ is continuous on X . The action of G on X is called *topologically transitive* if, given any two non-empty open subsets V and W of X , there exists an element $g \in G$ such that gV meets W . Our main result is the following.

Theorem 1.2. *Let X be an infinite Hausdorff uniform space equipped with a continuous and topologically transitive action of a group G . Suppose in addition that X admits a dense set of periodic points. Then the action of G on X is sensitive.*

In the case when X is a metric space, the preceding result was first obtained in [1] and [7] for $G = \mathbb{Z}$, and in [6] for an arbitrary group G .

We say that the action of a group G on a uniform space X is *chaotic* in the sense of Devaney if it is topologically transitive and periodic points are dense in X (cf. [3]). With this definition, Theorem 1.2 may be rephrased by saying that any continuous action of a group G on an infinite Hausdorff uniform space X which is chaotic in the sense of Devaney is also sensitive.

An action of a group G on a topological space X is called *topologically mixing* if, given any two non-empty open subsets V and W of X , there exists a finite subset $K \subset G$ such that gV meets W for all $g \in G \setminus K$. It trivially follows from this definition that every action of a finite group on a topological space is topologically mixing and that every topologically mixing action of an infinite group on a topological space is topologically transitive. We shall also establish the following result

which gives another sufficient condition for sensitivity of group actions on uniform spaces.

Theorem 1.3. *Every topologically mixing continuous action of an infinite group on an infinite Hausdorff uniform space is sensitive.*

For group actions on metric spaces, the preceding result was established in [5, Proposition 7.2.14].

Symbolic dynamics provides many interesting examples of sensitive actions on non-metrizable uniform spaces. Indeed, let A be a set having more than one element and let G be an infinite group. Consider the set A^G consisting of all maps $x: G \rightarrow A$. The *shift* on A^G is the action of G on A^G defined by $gx(h) = x(g^{-1}h)$ for all $g, h \in G$ and $x \in A^G$. We equip A^G with its *prodiscrete* uniform structure. This is the uniform structure admitting as a base of entourages the sets

$$W(\Omega) = \{(x, y) \in A^G \times A^G : x|_\Omega = y|_\Omega\},$$

where Ω runs over all finite subsets of G (see [2]). Then the shift action on A^G is continuous and expansive (the set $W(\{1_G\})$ is an expansivity entourage). As A^G is perfect, the shift on A^G is sensitive. Moreover, it is topologically mixing and hence topologically transitive. It is well known that periodic points are dense in A^G if and only if the group G is residually finite (see for example [2, Theorem 2.7.1]). Thus, the shift on A^G is chaotic in the sense of Devaney if and only if G is residually finite. If G is uncountable then A^G is not metrizable (not even first countable).

2. BACKGROUND MATERIAL ON UNIFORM SPACES

Let X be a set.

We denote by Δ_X the diagonal in $X \times X$, that is the set $\Delta_X = \{(x, x) : x \in X\}$.

The *inverse* $\overset{-1}{U}$ of a subset $U \subset X \times X$ is the subset of $X \times X$ defined by $\overset{-1}{U} = \{(x, y) : (y, x) \in U\}$. One says that U is *symmetric* if $\overset{-1}{U} = U$.

Note that $U \cap \overset{-1}{U}$ is symmetric for any $U \subset X \times X$.

We define the *composite* $U \circ V$ of two subsets U and V of $X \times X$ by $U \circ V = \{(x, y) : \text{there exists } z \in X \text{ such that } (x, z) \in U \text{ and } (z, y) \in V\} \subset X \times X$.

Definition 2.1. Let X be a set. A *uniform structure* on X is a non-empty set \mathcal{U} of subsets of $X \times X$ satisfying the following conditions:

- (UN-1) if $U \in \mathcal{U}$, then $\Delta_X \subset U$;
- (UN-2) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$;
- (UN-3) if $U \in \mathcal{U}$ and $V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;
- (UN-4) if $U \in \mathcal{U}$, then $\overset{-1}{U} \in \mathcal{U}$;
- (UN-5) if $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

The elements of \mathcal{U} are then called the *entourages* of the uniform structure and the set X is called a *uniform space*.

Note that conditions (UN-3), (UN-4), and (UN-5) imply that, for any entourage U there exists a symmetric entourage V such that $V \circ V \subset U$.

Let X be a set and let $U \subset X \times X$. Given a point $x \in X$, we define the subset $U[x] \subset X$ by $U[x] = \{y \in X : (x, y) \in U\}$.

If X is a uniform space, there is an induced topology on X characterized by the fact that the neighborhoods of an arbitrary point $x \in X$ consist of the sets $U[x]$, where U runs over all entourages of X . This topology is Hausdorff if and only if the intersection of all the entourages of X is reduced to the diagonal Δ_X .

If (X, d) is a metric space, there is a natural uniform structure on X whose entourages are the sets $U \subset X \times X$ satisfying the following condition: there exists a real number $\varepsilon > 0$ such that U contains all pairs $(x, y) \in X \times X$ such that $d(x, y) < \varepsilon$. The topology associated with this uniform structure is then the same as the topology induced by the metric.

3. PROOFS

Lemma 3.1. *Let A and B be two disjoint finite subsets of a Hausdorff uniform space X . Then there exists an entourage W of X such that $A \times B$ does not meet W .*

Proof. As X is Hausdorff, we can find, for all $a \in A$ and $b \in B$, an entourage $V_{a,b}$ of X such that $(a, b) \notin V_{a,b}$. Then the entourage

$$W = \bigcap_{(a,b) \in A \times B} V_{a,b}$$

has the required property. \square

Proof of Theorem 1.2. We first claim that we can find an entourage V of X such that, for all $x \in X$, there exists a finite orbit $C \subset X$ that does not meet $V[x]$. Indeed, the hypotheses that X is Hausdorff, infinite, and contains a dense set of periodic points, imply that we can find two disjoint finite orbits A and B in X . By Lemma 3.1, there is an entourage W of X such that $A \times B$ does not meet W . Now, if we take a symmetric entourage V satisfying $V \circ V \subset W$, then there is no $x \in X$ such that A and B both meet $V[x]$. This proves the claim.

Let V be an entourage of X satisfying the conditions of the preceding claim and let U be a symmetric entourage of X such that

$$(3.1) \quad U \circ U \circ U \circ U \subset V.$$

Let us show that U is a sensitivity entourage for the action of G on X . Let $x \in X$ and let N be a neighborhood of X .

As periodic points are dense in X , we can find a periodic point $p \in X$ such that

$$(3.2) \quad p \in N \cap U[x].$$

Denote by H the stabilizer of p in G and let $T \subset G$ be a complete set of representatives for the left cosets of H in G , so that we have $G = \coprod_{t \in T} tH$.

As V satisfies our first claim, we can find a finite orbit C such that

$$(3.3) \quad C \cap V[x] = \emptyset.$$

Choose an arbitrary point $q \in C$. Then observe that the set

$$I = \bigcap_{t \in T} tU[t^{-1}q]$$

is a neighborhood of q since it is a finite intersection of neighborhoods of q .

As the action of G on X is topologically transitive, we can find a point $z \in N \cap U[x]$ and an element $g_0 \in G$ such that $g_0z \in I$. The element g_0 can be uniquely written in the form $g_0 = t_0h_0$, where $t_0 \in T$ is the representative of the class g_0H and $h_0 \in H$. We have

$$(3.4) \quad h_0z = t_0^{-1}g_0z \in t_0^{-1}I \subset t_0^{-1}(t_0U[t_0^{-1}q]) = U[t_0^{-1}q],$$

so that

$$(3.5) \quad (h_0z, t_0^{-1}q) \in U.$$

We now claim that we always have

$$(3.6) \quad (h_0x, p) \notin U \quad \text{or} \quad (h_0x, h_0z) \notin U.$$

Indeed, suppose on the contrary that (h_0x, p) and (h_0x, h_0z) both belong to U . This would imply $(p, h_0z) \in U \circ U$, and hence $(x, t_0^{-1}q) \in U \circ U \circ U \circ U$ since $(x, p) \in U$ by (3.2) and $(h_0z, t_0^{-1}q) \in U$ by (3.4). This would contradict (3.3) because $U \circ U \circ U \circ U \subset V$ by (3.1) and $t_0^{-1}q \in C$.

Observe now that $h_0p = p$ since $h_0 \in H$. Thus, we deduce from (3.6) that we can always find a point $y \in N$ and an element $g \in G$ such that $(gx, gy) \notin U$. Indeed, we can take $g = h_0$ and either $y = p$ or $y = z$. \square

Proof of Theorem 1.3. Let X be an infinite Hausdorff uniform space equipped with a continuous and topologically mixing action of an infinite group G . Let x_1 and x_2 be two distinct points in X . Since X is Hausdorff we can find an entourage V of X such that

$$(3.7) \quad (x_1, x_2) \notin V.$$

Let U be a symmetric entourage of X such that

$$(3.8) \quad U \circ U \circ U \circ U \subset V.$$

Let us show that U is a sensitivity entourage for the action of G on X . Let $x \in X$ and let N be a neighborhood of X . As the action of G on X is topologically mixing, we can find, for $i = 1, 2$, a finite set $F_i \subset G$ such that we have

$$g(N \cap U[x]) \cap U[x_i] \neq \emptyset$$

for all $g \in G \setminus F_i$. Since G is infinite, the set $G \setminus (F_1 \cup F_2)$ is not empty. Choose an element $g \in G \setminus (F_1 \cup F_2)$. Then we can find $y_1, y_2 \in N \cap U[x]$ such that

$$(3.9) \quad gy_1 \in U[x_1] \quad \text{and} \quad gy_2 \in U[x_2].$$

From (3.9), (3.8), and (3.7) we deduce that

$$(gy_1, gy_2) \notin U \circ U$$

and therefore

$$(3.10) \quad (gx, gy_1) \notin U \quad \text{or} \quad (gx, gy_2) \notin U.$$

It follows from (3.10) that we can always find a point $y \in N$ (namely, $y = y_1$ or $y = y_2$) and an element $g \in G$ such that $(gx, gy) \notin U$.

This shows that U is a sensitivity entourage for the action of G on X . \square

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