Linear forms in the logarithms of algebraic numbers close to 1 and applications to Diophantine equations

Yann Bugeaud (Strasbourg)

À T. N. Shorey, pour son soixantième anniversaire

Abstract. In 1974, Shorey established the first sharp lower bound for linear forms in the logarithms of real algebraic numbers close to 1. His estimate was subsequently slightly refined and applied to various Diophantine equations. In many cases, it yields very spectacular statements. We survey several of its applications and we discuss some new ones.

1. Introduction

First, let us begin with explaining what is meant by a *linear form in logarithms*. Let $n \geq 1$ be an integer. For $i = 1, \ldots, n$, let x_i/y_i be a non-zero rational number, b_i be a positive integer, and set $A_i := \max\{|x_i|, |y_i|, 3\}$. Set $B := \max\{b_1, \ldots, b_n, 3\}$. We consider the quantity

$$\Lambda := \left(\frac{x_1}{y_1}\right)^{b_1} \cdots \left(\frac{x_n}{y_n}\right)^{b_n} - 1, \tag{1}$$

which occurs naturally in many Diophantine questions. It is often quite easy to prove that Λ is non-zero and to derive a very small upper bound for $|\Lambda|$. Thus, to reach some conclusion, it remains to establish a good lower bound for $|\Lambda|$.

Although (1) is far from being linear, nor contains logarithms, it is often called a *linear form in logarithms*. The reason is the following. Without any restriction, since $|\Lambda|$ is very small, we may assume that $|\Lambda| \leq 1/2$. Then, a *linear form in logarithms* occurs, namely:

$$|\Lambda| \geq \frac{|\log(1+\Lambda)|}{2} = \frac{1}{2}\left|b_1 \log \frac{x_1}{y_1} + \cdots + b_n \log \frac{x_n}{y_n}\right|.$$

Now, we assume that Λ is non-zero. First, let us state a trivial lower bound for $|\Lambda|$. By estimating straightforwardly the denominator of (1), we get

$$\log|\Lambda| \geq -\sum_{i=1}^{n} b_i \log|y_i| \geq -B \sum_{i=1}^{n} \log A_i.$$

2000 *Mathematics Subject Classification :* 11J86, 11D61.

The dependence on the A_i is very satisfactory, unlike the dependence on B. However, to solve many Diophantine questions, we need a better dependence on B, even if the dependence on the A_i is not the best possible. In 1966, Alan Baker [2] was the first to establish such a result and, after many subsequent refinements by several authors (see [39] for a comprehensive bibliography and the state of the art of the theory in 2000), Matveev [19] proved that, under the above hypotheses, we have

$$\log|\Lambda| \geq -30^{n+4}\,(n+1)^6\,(\log A_1)\cdots(\log A_n)\,(\log B). \tag{2}$$

More generally, we can state analogous lower bounds when the x_i/y_i are replaced by non-zero algebraic numbers α_i, the real numbers A_i being then expressed in terms of the Weil height of α_i. However, to avoid an additional technical difficulty, we confine our attention to the rational case.

We have just seen how, roughly speaking, we are able to bound $\log|\Lambda|$ from below by a numerical constant times $\log B$ times the product of the $\log A_i$. The dependence on B is essentially best possible, but, conjecturally, it is believed that a lower estimate of the form

$$\log|\Lambda| \geq -c_1(n)\,(\log A_1 + \ldots + \log A_n)(\log B), \tag{3}$$

or even of the form

$$\log|\Lambda| \geq -c_2(n)\,(\log A_1 + \ldots + \log A_n + \log B),$$

should hold, with suitable numerical constants $c_1(n)$ and $c_2(n)$ depending only on n. This is related to the *abc*-conjecture, see Philippon [24] and Baker [4,5].

The main purpose of the present survey is to point out that estimate (3) holds under some conditions on the rational numbers x_i/y_i: basically, when all of them are very close to 1. This was first proved by Shorey [30]. Although very restrictive, these assumptions are satisfied in many cases, and we display examples in Sections 3 to 7. Section 2 is devoted to some theoretical statements and to a discussion on the improvement of (3) under the above additional assumption.

2. A theoretical contribution of Shorey

The first result in the direction of (3) was obtained in 1974 by Shorey [30]. Lemma 1 from [30] can be stated as follows (we copy it below with a slight change in the notation). According to [30], the size of a rational number a/b written in its lowest form is, by definition, equal to $|b| + |a/b|$.

Theorem 1 (Shorey). *Let a_1, a_2 and b be rational numbers. Assume that a_1 and a_2 are multiplicatively independent. Assume that the size of a_1, a_2, b respectively do not exceed S_1, S_1 and $(\log S_1)^A = S$, where A is an arbitrary positive constant, respectively. Assume that $|\log a_1|$ and $|\log a_2|$ do not exceed $\exp(-(\log S_1)^\theta)$, where θ is an arbitrary (but fixed) constant satisfying $0 < \theta < 1$. Then*

$$|b\log a_1 - \log a_2| > C\,\exp(-(\log S_1)^{3-2\theta+\varepsilon}),$$

where $\varepsilon > 0$ is an arbitrary fixed constant and C is an effectively computable positive constant depending only on ε, θ and A.

The main point in the above estimate is the exponent $3 - 2\theta + \varepsilon$, which is less than 2 if θ is large enough, that is, if the rational numbers a_1 and a_2 are close enough to 1.

Using ideas from [30], Shorey [29] improved upon a previous result obtained jointly with Ramachandra [25] and established Theorem 2 below. Recall that Sylvester was the first to prove that any product of k consecutive integers greater than k has always a prime divisor exceeding k.

Theorem 2. *Let k be a fixed positive integer and let $n_1, n_2, \ldots$ be all positive integers that have at least one prime factor exceeding k, arranged in increasing order. Set $f(k) = \max\{n_{i+1} - n_i : i \geq 1\}$. Then we have*

$$f(k) = O\biggl(\frac{k(\log\log\log k)}{(\log k)(\log\log k)} \biggr)\cdot$$

Say differently, there exists an absolute positive constant c_3 such that any product of $k(\log k)^2$ consecutive integers greater than k has always a prime divisor exceeding

$$c_3\, k(\log k)\, \frac{\log\log k}{\log\log\log k}\cdot$$

Thirty years after its publication, Theorem 2 has not been improved.

Shortly thereafter, Ramachandra, Shorey and Tijdeman [26,27] applied a version of Theorem 1 to investigate Grimm's problem on factorisation of blocks of consecutive integers.

Most of the lower bounds for linear forms in logarithms that can be found in [39] involve an extra parameter E. It appeared first in [22] and, according to [39, p. 366], the idea of introducing it arose from the work of Shorey [30] quoted above. This parameter is very useful when the x_i/y_i are close to 1. To give an example of an estimate involving E, we display a consequence of Corollaire 3 of Laurent, Mignotte and Nesterenko [18].

Our notation is the following. Let x_1/y_1 and x_2/y_2 be multiplicatively independent rational numbers, both > 1. Let b_1 and b_2 be positive rational integers and consider the linear form

$$\Lambda = b_2 \log(x_2/y_2) - b_1 \log(x_1/y_1).$$

Let A_1 and A_2 be real numbers such that

$$\log A_i \geq \max\{\log x_i, 1\}, \quad (i = 1, 2).$$

Theorem 3. *Keep the above notation. Let $E \geq 3$ be a real number such that*

$$E \leq 1 + \min\biggl\{ \frac{\log A_1}{\log(x_1/y_1)}, \frac{\log A_2}{\log(x_2/y_2)} \biggr\},$$

and set

$$\log B = \max\left\{\log\left(\frac{b_1}{\log A_1} + \frac{b_2}{\log A_2}\right) + \log\log E + 0.47, 10\log E\right\}.$$

Assuming that $E \le \min\{A_1^{3/2}, A_2^{3/2}\}$, *we have*

$$\log\Lambda \ge -35.1\,(\log A_1)(\log A_2)(\log B)^2(\log E)^{-3}. \tag{4}$$

When x_1/y_1 and x_2/y_2 are *very* close to 1, the factor $(\log E)^{-3}$ allows us, roughly speaking, to replace the product $(\log A_1)(\log A_2)$ occurring in the 'classical estimate' (2) by the sum $(\log A_1)+(\log A_2)$. Indeed, assume for instance that we have $|(x_i/y_i)-1| \le x_i^{-1/2}$ for $i = 1, 2$. Then, we get $\log(x_i/y_i) \le x_i^{-1/2}$ for $i = 1, 2$ and, if $x_1 \ge x_2 \ge 5$, we see than we can choose $\log E = (\log x_2)/2 = (\log A_2)/2$ in Theorem 3. By (4) this gives

$$\log\Lambda \ge -71\,(\log A_1 + \log A_2)\,(10 + \log(b_1+b_2))^2.$$

To be even more precise, we display a useful consequence of Theorem 3. It deals with a particular situation that occurs in many applications, as we shall see throughout the next sections. We assume that $b_2 = 1$, $x_1 \ge 3$, $3 \le x_2 < 2y_2$ and that $|\Lambda|$ is very small. It follows that $x_2/y_2 < x_1/y_1$, thus the parameter E satisfies

$$\log E \le -\log\log(x_2/y_2).$$

Define ε by

$$\frac{x_2}{y_2} = 1 + x_2^{\varepsilon-1}.$$

Then we have $0 < \varepsilon < 1$ and $-\log\log(x_2/y_2) \ge (1-\varepsilon)\log x_2$. Taking into account the assumption $E \le \min\{A_1^{3/2}, A_2^{3/2}\}$ in Theorem 3, we set

$$\log E = \min\{\log x_1, (1-\varepsilon)\log x_2\}.$$

Corollary 1. *Under the above assumption, we have*

$$\log\Lambda \ge -35.1\,\frac{(\log x_1)(\log x_2)}{\min\{\log x_1, (1-\varepsilon)\log x_2\}}\,(10+\log b_1)^2.$$

As we shall see in the sequel, this crucial improvement upon the 'classical' estimate (2) turns out to have many spectacular applications.

3. An emblematic application

To my opinion, the most striking application of Theorem 3 deals with a multiparametric family of Thue equations.

Let a, b and c be positive integers. The Thue equation

$$ax^n - by^n = c \tag{5}$$

has been studied by many authors, including Thue, Siegel and Evertse. Basically, they were able to show that (5) has at most one positive solution if n is sufficiently large (in an effective way) in terms of a, b and c.

As an application of sharp bounds for linear forms in two logarithms, Mignotte [20] was able to prove the following uniform result.

Theorem 4. *Let b and n be positive integers. If $n \geq 600$, then the only solution in positive integers of the Thue equation*

$$(b+1)x^n - by^n = 1 \tag{6}$$

is given by $x = y = 1$.

The idea of the proof is very easy to explain. For simplicity, we assume that $b \geq 2$ and set $\varepsilon := (\log(3/2))/\log 3$. One immediately gets from (6) that

$$\Lambda := \left| \left(1 + \frac{1}{b}\right)\left(\frac{x}{y}\right)^n - 1 \right| \leq \frac{1}{by^n}. \tag{7}$$

We infer from $(b+1)b^n < b(b+1)^n$ that if (6) has a positive solution (x, y) other than $(1, 1)$, then $y \geq b+1$. Consequently, Corollary 1 gives us the lower bound

$$\log \Lambda \geq -\frac{36}{1-\varepsilon} (\log y) (10 + \log n)^2.$$

Combined with (7), we obtain that

$$n \log y \leq \frac{36}{1-\varepsilon} (\log y) (10 + \log n)^2.$$

This yields an absolute upper bound for n.

Theorem 4 was subsequently improved by Bennett and de Weger [8] in 1998. Three years later appeared a remarkable paper of Bennett [6], who managed to solve completely the remaining few hundreds of Thue equations (5) with $c = 1$.

Theorem 5. *Let a, b and n be integers with $a > b \geq 1$ and $n \geq 3$. Then, the equation*

$$|ax^n - by^n| = 1$$

has at most one solution in positive integers x and y.

The beautiful proof of Theorem 5 combines many different tools from Diophantine approximation.

4. Several more applications

Sharp estimates for linear forms in the logarithms of algebraic numbers close to 1 combine well with the irrationality measures obtained by Baker [1,3] using the hypergeometric method. This was first observed and applied by Shorey [32,34] in order to get the following result.

Theorem 6. *Let $m \geq 0$ and $k \geq 2$ be integers. Let $d_1, \dots, d_t$ with $t \geq 2$ be distinct integers in the interval $[1, k]$. For integers $\ell \geq 2$, $y > 0$ and $b > 0$ with b free of prime factors $> k$, consider the equation*

$$(m + d_1) \dots (m + d_t) = by^\ell. \tag{8}$$

Equation (8) with

$$\ell \geq 3,\ m > k^\ell,\ t \geq 47k/56$$

implies that k is bounded by an effectively computable number.

To establish Theorem 6, Shorey first used an argument of Erdős to derive from (8) a suitable linear form in two logarithms of rational numbers close to 1. He applied a previous version of Theorem 3 to bound ℓ from above. Then, for fixed ℓ, results from [1] yield an upper bound for k.

The combination of both methods was subsequently used in many papers, for example in [36], where Shorey and Nesterenko obtained further results on Equation (8), including the following.

Theorem 7. *Keep the assumption of Theorem 6. Equation (8) with*

$$\ell \geq 7,\ m > k^\ell,\ t \geq (4k)/\ell$$

implies that k is bounded from above by an effectively computable constant depending only on ℓ.

The irrationality measures used in the proof of Theorem 7 were derived from the Theorem of Baker [3]. A precise statement of the auxiliary result from [36] (see also [17]) is given as Lemma 1 below.

Lemma 1. *Let A, B, K and n be positive integers such that $A > B, K < n, n \geq 3$ and $\omega = (B/A)^{1/n}$ is not a rational number. For $0 < \phi < 1$, put*

$$\delta = 1 + \frac{2 - \phi}{K}, \quad s = \frac{\delta}{1 - \phi}$$

and

$$u_1 = 40^{n(K+1)(s+1)/(Ks-1)}, \quad u_2^{-1} = K 2^{K+s+1} 40^{n(K+1)}.$$

Assume that

$$A(A - B)^{-\delta} u_1^{-1} > 1. \tag{9}$$

Then

$$\left| \omega - \frac{p}{q} \right| > \frac{u_2}{A q^{K(s+1)}}$$

for all integers p and q with $q > 0$.

The crucial point in Lemma 1 is the condition (9). A similar inequality, but with $\delta = 2$, needs to be satisfied in an earlier result of Baker [1] (this was used in the proof of Theorem 6). The main improvement in Lemma 1 above is that we can take $\delta < 2$, and even δ quite close to 1. This is crucial for most of the applications to be quoted in this section, including for the following one, obtained by Shorey [35].

Theorem 8. *Consider the equation*

$$m(m+d)\dots(m+(k-1)d) = by^\ell, \tag{10}$$

in positive integers b, d, k, ℓ, m *and* y *satisfying* $\gcd(m,d)=1$, $k>2$, $\ell \ge 2$, *and* b *free of prime factors* $>k$. *There exists an effectively computable absolute positive constant* c_4 *such that Equation (10) with* $\ell \ge 7$ *implies that*

$$m \ge k^{c_4 \log\log k}.$$

We quote two additional statements on classical Diophantine equations and we postpone a detailled application to the next section. We begin with a result of Hirata-Kohno and Shorey [17] on the Nagell–Ljunggren equation.

Theorem 9. *Let* $z>1$ *be an integer and* $q \ge 3$ *be a prime number. Then, the Diophantine equation*

$$\frac{x^n-1}{x-1} = y^p \quad \text{in integers } x>1,\ y>1,\ n>2,\ p\ge 2 \tag{11}$$

with $x=z^q$ *and* $p > 2(q-1)(2q-3)$ *implies that* $\max\{x,y,n,p\}$ *is bounded by an effectively computable number depending only on* q.

Sharp estimates for linear forms in the logarithms of algebraic numbers close to 1 have also been used by Saradha and Shorey [28], as they investigated (11) when the base x is a square. They established that (11) with $x=z^2$ implies that $z \le 31$ and $z \notin \{2,3,4,8,9,16,25,27\}$. Actually, Bugeaud, Mignotte, Roy and Shorey [13] proved shortly thereafter that (11) has no solution whenever x is a square. The same result was obtained, independently and by a different method, by Bennett [6].

Next result, on the Goormaghtigh equation, was proved by Bugeaud and Shorey [15].

Theorem 10. *Let* $\theta>1$. *Equation*

$$\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1} \qquad \text{in integers } x>1,\ y>1,\ m>2,\ n>2 \text{ with } y>x \text{ and } \frac{m-1}{n-1} \le \theta$$

implies that $\gcd\,(m-1,n-1) \le 743(\theta+1/2)$. *If, furthermore, we have* $\gcd\,(m-1,n-1) \ge 4\theta+7$, *then* $\max\,(x,y,m,n)$ *is bounded by an effectively computable number depending only on* θ.

We refer to Section 7 for an extension of the first assertion of Theorem 10.

Further applications of the combined application of Corollary 1 and Lemma 1 were given by Bugeaud and Dujella [11] and by Bugeaud [10]. For instance, it is proved in [11] that if $a<b<c<d$ and k denote positive integers such that $ac+1$, $ad+1$, $bc+1$ and $bd+1$ are perfect k-th powers, then k is at most equal to 176. Furthermore, for any k with $11 \le k \le 176$, there are only finitely many quadruples with the above property.

5. A further illustration of the power of Theorem 3

Using tools developed for the resolution of the Fermat equation, Darmon and Merel [16] proved that the Diophantine equation $x^k + y^k = 2z^k$ has no solution with $k \geq 3$ and $|x| \neq |y|$. In other words, they established that, for $k \geq 3$, there do not exist positive integers a and b such that the three numbers a, $a+b$ and $a+2b$ are all perfect k-th powers.

We would like to consider the multiplicative analogue to this problem and address the following

Question. Do there exist positive integers a, $b > 1$ and $k \geq 3$ such that the three numbers $a+1$, $ab+1$ and ab^2+1 are all perfect k-th powers ?

Quite surprisingly, the theory of linear forms in logarithms yields a very satisfactory contribution to this problem, studied in [10].

Theorem 11. *Let a, $b \geq 2$ and $k \geq 3$ be integers such that $a+1$, $ab+1$ and ab^2+1 are all perfect k-th powers. Then, we have $k \leq 149$. Furthermore, if $k \geq 8$, then there are only finitely many integer pairs (a,b) of positive integers with $b \geq 2$, such that $a+1$, $ab+1$ and ab^2+1 are all perfect k-th powers.*

The rest of this section is devoted to the proof of Theorem 11.

We are looking for triples of positive rational integers (a,b,k) such that $a+1$, $ab+1$ and ab^2+1 are distinct perfect k-th powers. First, observe that b must be large in terms of a and k. Indeed, since $b > 1$, we have $(a+1)(ab^2+1) > (ab+1)^2$ and both numbers are perfect k-th powers. It follows that

$$(a+1)(ab^2+1) \geq \big((ab+1)^{2/k} + 1\big)^k.$$

Consequently, we get

$$a^2b^2 + 1 + a(b^2+1) \geq (ab+1)^2 + k(ab+1)^{2(k-1)/k},$$

and

$$ab^2 \geq k(ab)^{2(k-1)/k}.$$

This immediately implies a strong gap principle, namely that

$$b \geq (k^k a^{k-2})^{1/2}. \tag{12}$$

Now, let x, y and z be positive rational integers such that

$$a+1 = x^k, \quad ab+1 = y^k, \quad ab^2+1 = z^k.$$

Set

$$\alpha_1 = \frac{x_1}{y_1} = \frac{(a+1)^{1/k}(ab^2+1)^{1/k}}{(ab+1)^{2/k}} = \frac{xz}{y^2} \quad \text{and} \quad \alpha_2 = \frac{x_2}{y_2} = \frac{a+1}{a}$$

and consider

$$\Lambda = |\log \alpha_2 - k \log \alpha_1|.$$

Clearly, with the notation of Corollary 1, we can take

$$\log x_2 = \log(a+1), \quad \log x_1 = \log(xz).$$

We may assume that $k \geq 50$, thus, $a \geq 2^{49}$. Consequently, we get

$$\log \Lambda \geq -40(\log(xz))\,(10+\log k)^2 \geq -120\,\frac{\log b}{k}\,(10+\log k)^2,$$

since $(xz)^k \leq b^3$, by (12). On the other hand, a short calculation yields

$$\left| \frac{a+1}{a} - \left(\frac{xz}{y^2}\right)^k \right| \leq \frac{1}{b}, \tag{13}$$

thus, we have

$$\log \Lambda \leq -\frac{\log b}{12}.$$

Combining both estimates, we get an absolute upper bound for k.

Using the main result of Mignotte [21] instead of Corollary 1, we get after some computation that $k \leq 149$, as announced.

We derive the last assertion of the lemma from a (straightforward corollary of a) theorem of Baker [1].

Lemma 2. *Let $k \geq 3$ be an integer and $\varepsilon > 0$ be a real number. There exist effectively computable positive constants c_5 and c_6 depending only on k and on ε such that*

$$\left| \left(1+\frac{1}{a}\right)^{1/k} - \frac{p}{q} \right| > \frac{c_5}{a\,q^{2+\varepsilon}}$$

holds for any integer $a > c_6$ and any rational number p/q.

Assume now that k is fixed with $3 \leq k \leq 149$. The notation $\ll_k$ used below means that the implicit constant depends only on k. It follows from (13) and Lemma 2 with $\varepsilon = 0.1$ that we have

$$b \ll_k a(a^2b^2)^{2.1/k},$$

that is, by (12),

$$b^{k(k-2)} \ll_k a^{k(k-2)}(a^2b^2)^{2.1(k-2)} \ll_k b^{4.2(k-2)}\,b^{8.4+2k}$$

if $a \gg_k 1$. Hence, we get a contradiction for $b \gg_k 1$ as soon as $k \geq 8$. Now, assume that a and k are fixed. The integer $(ab+1)(ab^2+1)$ is then a k-th power. Since the polynomial $(aX+1)(aX^2+1)$ has distinct roots, this can happen only for finitely many values of b. This proves the second assertion of Theorem 11.

During the Conference, I learned from Mike Bennett that he considerably improved upon Theorem 11 above [7].

Theorem 12. *There do not exist positive integers a, $b > 1$ and $k \geq 4$ such that the three numbers $a+1$, $ab+1$ and ab^2+1 are all perfect k-th powers.*

The proof of Theorem 12 for $k \geq 5$ rests on a deep refinement of Lemma 2 above and does not use any estimate from the theory of linear forms in logarithms. The case $k = 4$ requires a combination of various techniques, including the theory of Frey curves and associated Galois representations. It seems that no result is known for the case $k = 3$, which corresponds to the Diophantine equation $(x^3-1)(z^3-1) = (y^3-1)^2$. According to Noam Elkies, the usual heuristics suggest that the Diophantine equation $(x^3-1)(z^3-1) = t^2$ should have only finitely many solutions in integers x, z and 1 distinct; see

`http://lsze.cosam.calpoly.edu/wcntc/problems2001.pdf`

and Section 7 from [7].

6. Perfect powers in Lucas sequences

We begin this section by recalling the definition of a *Lucas sequence*.

Definition 1. *A Lucas pair is a pair (α, β) of algebraic integers such that $\alpha+\beta$ and $\alpha\beta$ are non-zero rational integers and α/β is not a root of unity. For a given Lucas pair (α, β), the corresponding Lucas sequence is defined by*

$$u_n = u_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad (n = 0, 1, 2, \ldots).$$

For example, the Fibonacci sequence is the Lucas sequence corresponding to the Lucas pair $((1+\sqrt{5})/2, (1-\sqrt{5})/2)$.

Shorey and Stewart [37] and, independently, Pethő [23] established that there are only finitely many perfect powers in any given Lucas sequence. Very recently, Bugeaud, Mignotte and Siksek [14] were able to confirm that 1, 8 and 144 are the only positive perfect powers in the Fibonacci sequence. Their work has motivated subsequent papers on the determination of all the perfect powers in a given Lucas sequence, including [12].

Throughout this section and the next one, we consider only *real* Lucas pairs (α, β) for which $\alpha > |\beta|$. Our aim is to point out an application of Corollary 1 to the equation

$$u_n = y^p \tag{14}$$

in integers $n \geq 1$, y and prime exponent $p \geq 2$. For any p, it has the trivial solution $u_1 = 1^p$, and also sometimes the trivial solution $u_{-1} = \pm 1^p$ (exactly when $\alpha\beta = \pm 1$). The existence of solutions for any p makes its complete resolution very difficult, since one can no longer hope to solve it by only using congruences to suitable moduli.

As pointed out in [12], one important step is to prove, for large n, that

$$n \equiv 1 \pmod p \quad \text{or} \quad n \equiv \pm 1 \pmod p,$$

respectively. This was a difficult part of the proof for the Fibonacci case, but a rather easy part in the example considered in [12].

The purpose of this section is to give, modulo a suitable assumption on the pair (α, β), an absolute upper bound for p for all solutions (n, y, p) to (14) with $n \equiv 1 \pmod p$.

Theorem 13. *Let η be a positive real number. Let (α, β) be a Lucas pair of real algebraic integers with $\alpha \geq |\beta|^{1+\eta}$. There exists an effectively computable constant $P_1(\eta)$, depending only on η, such that all solutions of*

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = y^p$$

with $n \equiv 1 \pmod{p}$ and $|y| \geq 2$ satisfy $p \leq P_1(\eta)$.

Under the assumption of Theorem 13, Shorey and Stewart [38] established that (14) has no solution (n, y, p) with $|y| \geq 2$ and p greater than some effectively computable constant depending only on η, α and the greatest prime factor of $\alpha - \beta$.

Proof of Theorem 13. For simplicity, we sketch the proof when α and β are positive integers. Without any restriction, we assume that α and $|\beta|$ are sufficiently large, and that $\eta \leq 1/2$. Furthermore, the constants c_7, c_8 and c_9 occurring below are absolute and effectively computable. Write

$$n = \nu p + 1.$$

Then, Equation (14) becomes

$$(\alpha^\nu)^p \, \alpha - \beta^{\nu p+1} = (\alpha - \beta)\, y^p,$$

that is

$$\Lambda := \frac{\alpha}{\alpha - \beta} \left(\frac{\alpha^\nu}{y}\right)^p - 1 = \frac{\beta^{\nu p+1}}{(\alpha - \beta)\, y^p}.$$

Observe that $\alpha/(\alpha - \beta)$ is close to 1 when α/β, since $\alpha > |\beta|^{1+\eta}$. It then follows from Corollary 1 that

$$\log|\Lambda| \geq -c_7 \frac{(\log \alpha)(\log y)}{\eta \log \alpha} (10 + \log p)^2.$$

Since

$$\log|\Lambda| \leq -p \log y + (\nu p + 1) \log \beta$$

and $y \geq \alpha^\nu$, we get

$$\begin{aligned} p &\leq \frac{c_7}{\eta} (10 + \log p)^2 + \frac{(\nu p + 1) \log \beta}{\log y} \\ &\leq \frac{c_7}{\eta} (10 + \log p)^2 + (p + 1) \frac{\log \beta}{\log \alpha} \leq \frac{c_8 (\log p)^2}{\eta} + \frac{p + 1}{1 + \eta}, \end{aligned}$$

hence, an upper bound for p depending only on η, namely $p \leq c_9 \eta^{-2} (\log \eta^{-1})^2$. □

When α and β are both positive integers, it is possible to say something more, by using Lemma 1 above. The proof of the next result is left to the reader.

Theorem 14. *Let η be a positive real number. There exists an effectively computable positive integer $P_2(\eta)$, depending only upon η, with the following property. There are only finitely many Lucas pairs of integers (α, β) satisfying $|\alpha| \geq |\beta|^{1+\eta}$ and such that the equation*

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = y^p$$

with $n \equiv 1 \pmod{p}$ has a solution (n, y, p) with $|y| \geq 2$ and $p \geq P_2(\eta)$.

To conclude this section, we display another application of Corollary 1.

Theorem 15. *Let η be a positive real number. There exists an effectively computable positive integer $P_3(\eta)$, depending only upon η, with the following property. Let (α, β) be a Lucas pair of real algebraic integers with $\alpha \geq |\beta|^{1+\eta}$. Let p be a prime number. Assume that n_1 and n_2 are distinct positive integers such that $n_1 \equiv n_2 \pmod{p}$ and $u_{n_1}(\alpha, \beta)$, $u_{n_2}(\alpha, \beta)$ are perfect p-th powers. Then, we have $p \leq P_3(\eta)$.*

The proof of Theorem 15 follows the same lines as that of Theorem 1 of Shorey [33].

7. Equal values of Lucas sequences

In the present section, we briefly discuss a natural extension of the Goormaghtigh equation. Let $u_n = u_n(\alpha, \beta)$ and $v_n = v_n(\gamma, \delta)$ be two Lucas sequences. Consider the Diophantine equation

$$u_m(\alpha, \beta) = v_n(\gamma, \delta), \quad \text{in positive integers } m \text{ and } n. \tag{15}$$

Theorem 16. *Let η be a positive real number. Assume that $\alpha \geq |\beta|^{1+\eta}$ and $\gamma \geq |\delta|^{1+\eta}$. Let $\theta > 1$ and $d > 1$ be an integer. Suppose that $(\alpha, \beta, \gamma, \delta, m, n)$ with $\alpha > \gamma$ satisfies (15). Assume that*

$$\gcd(m-1, n-1) = d, \quad \frac{n-1}{m-1} \leq \theta.$$

Then, there exists an effectively constant $c_{10}(\theta, \eta)$, depending only on θ and on η, such that $d \leq c_{10}(\theta, \eta)$.

We outline the proof of Theorem 16, which follows the same lines as that of Lemma 1 from [15]. For sake of simplicity, we assume that α, β, γ and δ are rational integers. Furthermore, without any restriction, we assume that α, $|\beta|$, γ and $|\delta|$ are sufficiently large, and that $\eta \leq 1/2$. All the numerical constants implied by $\ll$ below are absolute and effectively computable.

Let $\theta > 1$ and $d > 1$ be an integer. We suppose that (15) with $\gcd(m-1, n-1) = d$ and $(m-1)/(n-1) \leq \theta$ is satisfied and we put

$$m - 1 = dr, \quad n - 1 = ds$$

where r and s are positive integers. We assume that $y \geq 7$. We re-write (15) as

$$\frac{\alpha}{\alpha - \beta}\, \alpha^{rd} - \frac{\gamma}{\gamma - \delta}\, \gamma^{sd} = \frac{\beta^{1+rd}}{\alpha - \beta} - \frac{\delta^{1+sd}}{\gamma - \delta}$$

which implies that

$$\Lambda := \left| \left(\frac{\alpha(\gamma-\delta)}{\gamma(\alpha-\beta)} \right) \left(\frac{\gamma^s}{\alpha^r} \right)^{-d} - 1 \right| \ll \alpha^{-\eta r d}. \tag{16}$$

We then infer from Corollary 1 and (16) that

$$\eta r d \log \alpha \ll \frac{(\log \alpha)(r \log \alpha)}{\eta \log \gamma} (\log d)^2 \ll \frac{(\log \alpha)(s \log \gamma)}{\eta \log \gamma} (\log d)^2 \ll \theta\, r (\log \alpha) \frac{(\log d)^2}{\eta},$$

since α^r is close to γ^s and $s \leq \theta r$. This implies the theorem. □

8. Conclusion

We have seen several applications of Shorey's very nice paper [30]. They give *uniform* bounds on the size of the solutions of Diophantine equations in several unknowns. In other words, one can treat as variables some parameters which were fixed in previous results, and be still able to conclude that a certain parameter is bounded.

We hope that many new, interesting applications remain to be found.

The p-adic case is not as nice as the rational case. A parameter E has been introduced by Bugeaud [9], but it must be acknowledged that there are only very few interesting applications of Bugeaud's estimate, and that these are far from being as spectacular as the results exposed in the present survey.

Ackonwledgements. I would like to warmly thank the referee for his very careful reading and his many comments.

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Yann Bugeaud
Université Louis Pasteur
U. F. R. de mathématiques
7, rue René Descartes
67084 STRASBOURG
FRANCE

e-mail : `bugeaud@math.u-strasbg.fr`