Weyl group action and semicanonical bases

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Abstract

Let $\mathbf{U}$ be the enveloping algebra of a symmetric Kac-Moody algebra. The Weyl group acts on $\mathbf{U}$, up to a sign. In addition, the positive subalgebra $\mathbf{U}^+$ contains a so-called semicanonical basis, with remarkable properties. The aim of this paper is to show that these two structures are as compatible as possible.

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1 INTRODUCTION

1.1 Let $A = (a_{ij})$ be a generalized Cartan matrix, with lines and columns indexed by a set I. From the datum of A, one builds a Kac-Moody algebra $\mathfrak{g}$; its derived algebra is generated by $\mathfrak{sl}_2$-triples (e_i, h_i, f_i), for $i \in I$. Let $\mathbf{U}$ be the enveloping algebra of $\mathfrak{g}$ and let $\mathbf{U}^+$ be the subalgebra of $\mathbf{U}$ generated by the elements e_i.

Let us fix $i \in I$. The element $s_i = \exp(-e_i)\exp(f_i)\exp(-e_i)$ belongs to the Kac-Moody group attached to $\mathfrak{g}$. Through the adjoint action, s_i defines an automorphism T_i of $\mathbf{U}$. We set $\mathbf{U}_i^+ = \mathbf{U}^+ \cap T_i^{-1}(\mathbf{U}^+)$ and ${}_i\mathbf{U}^+ = \mathbf{U}^+ \cap T_i(\mathbf{U}^+)$; thus T_i restricts to an isomorphism $\mathbf{U}_i^+ \to {}_i\mathbf{U}^+$. Moreover, the decompositions

$$\mathbf{U}^+ = \mathbf{U}^+ e_i \oplus \mathbf{U}_i^+ = e_i \mathbf{U}^+ \oplus {}_i\mathbf{U}^+$$

yield projections $\pi_i : \mathbf{U}^+ \to \mathbf{U}_i^+$ and ${}_i\pi : \mathbf{U}^+ \to {}_i\mathbf{U}^+$.

The algebra $\mathbf{U}^+$ contains a special basis $\mathbf{B}$, defined by Lusztig [6] and called the canonical basis. It induces a basis $\pi_i(\mathbf{B}) \setminus \{0\}$ in $\mathbf{U}_i^+$ and a basis ${}_i\pi(\mathbf{B}) \setminus \{0\}$ in ${}_i\mathbf{U}^+$. (The two subspaces $\mathbf{U}_i^+$ and ${}_i\mathbf{U}^+$ of $\mathbf{U}^+$ are in fact equal, but we distinguish them because the induced bases are different.) In [7], Lusztig shows that these two bases correspond to each other under $T_i : \mathbf{U}_i^+ \to {}_i\mathbf{U}^+$.

When A is symmetric, $\mathbf{U}^+$ can also be endowed with Lusztig's semicanonical basis [8]. Though not being algorithmically computable, this basis recently attracted some interest because of its relation with the theory of cluster algebra, see [4] for a recent survey. The main result of the present paper is a proof that the above statement about the canonical basis also holds true for the semicanonical basis.

1.2 Let $*$ be the antiautomorphism of $\mathbf{U}^+$ that fixes all the generators e_i. This involution exchanges $(\mathbf{U}^+e_i, \mathbf{U}_i^+, \pi_i)$ and $(e_i\mathbf{U}^+, {}_i\mathbf{U}^+, {}_i\pi)$. It leaves stable the canonical and the semicanonical bases, by 14.2.5 (c) in [6] and Theorem 3.8 in [8].

Let $B = B(-\infty)$ be the crystal associated to the crystal basis of $U_q^+(\mathfrak{g})$ (see Section 8 in [2]). Besides the maps wt, ε_i, φ_i and the operations $\tilde{e}_i$ and $\tilde{f}_i$, the set B is endowed with an involution $b \mapsto b^*$, which reflects the existence of $*$. This invites us to define the starred operators $\tilde{e}_i^* = \big(b \mapsto (\tilde{e}_i b^*)^*\big)$ and $\tilde{f}_i^* = \big(b \mapsto (\tilde{f}_i b^*)^*\big)$. Given $i \in I$, we set

$$B_i = \{b \in B \mid \varphi_i(b^*) = 0\}.$$

In Corollary 3.4.8 in [9], Saito defines a bijection $\sigma_i : B_i \to (B_i)^*$ by the rule

$$\sigma_i(b) = (\tilde{e}_i^*)^{\varepsilon_i(b)}(\tilde{f}_i)^{\max} b.$$

The canonical and semicanonical bases are naturally indexed by the crystal B: to each $b \in B$ correspond elements $G(b)$ and $S(b)$ in the canonical and in the semicanonical bases, respectively. By Theorem 14.3.2 in [6] and Theorem 3.1 in [8], both $\{G(b) \mid b \in B \setminus B_i\}$ and $\{S(b) \mid b \in B \setminus B_i\}$ are bases of $\mathbf{U}^+e_i$, therefore both $\{\pi_i(G(b)) \mid b \in B_i\}$ and $\{\pi_i(S(b)) \mid b \in B_i\}$ are bases of $\mathbf{U}_i^+$. Lusztig's result recalled above can then be given the more precise form:

$$\forall b \in B_i, \qquad T_i(\pi_i(G(b))) = {}_i\pi(G(\sigma_i b)).$$

Our aim is to prove the analog formula for the semicanonical basis:

$$\forall b \in B_i, \qquad T_i(\pi_i(S(b))) = {}_i\pi(S(\sigma_i b)). \tag{1}$$

We consider the dual framework. Let us denote by $\{S(b)^* \mid b \in B\}$ the dual semicanonical basis, namely the basis of $(\mathbf{U}^+)^*$ dual to the semicanonical basis. Then (1) is equivalent to

$$\forall (b', u) \in B_i \times \mathbf{U}_i^+, \qquad \langle S(b')^*, u\rangle = \langle S(\sigma_i b')^*, T_i(u)\rangle. \tag{2}$$

Taking $u = \pi_i(G(b))$ in (2), we get

$$\forall (b, b') \in (B_i)^2, \qquad \langle S(b')^*, G(b)\rangle = \langle S(\sigma_i b')^*, G(\sigma_i b)\rangle. \tag{3}$$

Relation (3) constrains the transition matrix between the canonical and the semicanonical bases.

1.3 We now briefly describe the plan of this paper. In Section 3, we recall Lusztig's construction of $\mathbf{U}^+$ in terms of representations of preprojective algebras; this is the natural framework for dealing with the semicanonical basis. Our main result is the theorem in Section 3.6, which describes the automorphism T_i in this setup. Equation (2) is derived from it in Section 3.8. The theorem is proved in three steps in Section 4. A combinatorial argument needed in the first step has been put aside in a proposition; this proposition is stated in Section 2.2 and is proved in Sections 2.3–2.8.

2 ADAPTED FILTRATIONS

2.1 Given an algebraic variety X over $\mathbb{C}$, we denote by $M(X)$ the $\mathbb{Q}$-vector space consisting of all constructible functions $f : X \to \mathbb{Q}$, that is, the $\mathbb{Q}$-vector space spanned by the indicator functions of the locally closed subsets of X. Let $\int_X : M(X) \to \mathbb{Q}$ be the linear form given by $f \mapsto \sum_{a\in\mathbb{Q}} a\, \chi(f^{-1}(a))$, where χ denotes the Euler characteristic with compact support.

2.2 We consider the following setup:

(a) $n \geq k \geq 0$ are integers;

(b) V is an n-dimensional $\mathbb{C}$-vector space;

(c) $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$ is a complete flag in V;

(d) x is an endomorphism of V which leaves the flag stable and is such that $x^2 = 0$;

(e) W is a k-dimensional subspace of V such that $\operatorname{im} x \subseteq W \subseteq \ker x$;

(f) J is a subset of $\{1, \ldots, n\}$ of cardinality k.

Let $\mathscr{F}$ be the set of all filtrations

$$\mathbf{X} : \quad 0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n = W$$

such that

(g) $\dim X_p/X_{p-1} = 1$ if $p \in J$ and $X_p = X_{p-1}$ if $p \notin J$.

We define a function $f : \mathscr{F} \to \mathbb{Q}$ as follows. Given $\mathbf{X} \in \mathscr{F}$, we set $f(\mathbf{X}) = 0$ except when

(h) $x(V_p) \subseteq X_p \subseteq V_p$ for any p.

When (h) holds true, we set $f(\mathbf{X}) = \prod_{p\in J} \eta_p$, where

$$\eta_p = \begin{cases} 1 & \text{if } X_p \not\subseteq V_{p-1}, \\ -1 & \text{if } x(V_p) \not\subseteq X_{p-1}, \\ 0 & \text{if both } X_p \subseteq V_{p-1} \text{ and } x(V_p) \subseteq X_{p-1}. \end{cases}$$

(This definition of η_p makes sense, because at least one of the inclusions $X_p \subseteq V_{p-1}$ or $x(V_p) \subseteq X_{p-1}$ holds true. In fact, if $x(V_p) \not\subseteq X_{p-1}$, then $X_p = X_{p-1} + x(V_p)$, because X_{p-1} is an hyperplane in X_p, and therefore $X_p \subseteq V_{p-1}$, because both X_{p-1} and $x(V_p)$ are contained in V_{p-1}.)

Likewise, let $\mathscr{G}$ be the set of all filtrations

$$\mathbf{Y} : \quad W = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_{n-1} \subseteq Y_n = V$$

such that

(j) $\dim Y_p/Y_{p-1} = 1$ if $p \notin J$ and $Y_p = Y_{p-1}$ if $p \in J$.

We define a function $g : \mathscr{G} \to \mathbb{Q}$ as follows. Given $\mathbf{Y} \in \mathscr{G}$, we set $g(\mathbf{Y}) = 0$ except when

(k) $V_p \subseteq Y_p \subseteq x^{-1}(V_p)$ for any p.

When (k) holds true, we set $g(\mathbf{Y}) = \prod_{p\notin J} \eta_p$, where

$$\eta_p = \begin{cases} 1 & \text{if } V_p \not\subseteq Y_{p-1}, \\ -1 & \text{if } Y_p \not\subseteq x^{-1}(V_{p-1}), \\ 0 & \text{if both } V_p \subseteq Y_{p-1} \text{ and } x(Y_p) \subseteq V_{p-1}. \end{cases}$$

PROPOSITION. With the above notation,

$$\int_{\mathscr{F}} f = \int_{\mathscr{G}} g. \tag{4}$$

The proof of this proposition occupies the rest of Section 2.

2.3 We consider the setup (a)–(e) of the previous paragraph. We show that one can construct a basis $\mathbf{e} = (e_1, \dots, e_n)$ of V such that

(l) $V_p = \operatorname{span}_{\mathbb{C}}\{e_1, \dots, e_p\}$ for each p;

(m) $x(e_p) \in \{0, e_1, \dots, e_{p-1}\}$ for each $p \geq 1$;

(n) $\operatorname{im} x$, $\ker x$ and W are coordinate subspaces w.r.t. the basis $\mathbf{e}$.

We will show in fact a slightly stronger statement, namely that we can replace the subspace W by an increasing sequence $\operatorname{im} x \subseteq W_1 \subseteq \cdots \subseteq W_\ell \subseteq \ker x$ and demand (n) for each W_p at once.

The existence of $\mathbf{e}$ is obvious if V is a line. To prove the general case, we use induction on $n = \dim V$.

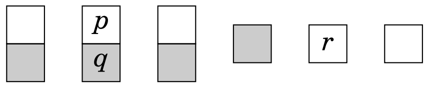
To the sequence W_p, we add $W_0 = \operatorname{im} x$ and $W_{\ell+1} = \ker x$. Since x leaves stable the hyperplane V_{n-1}, it induces an endomorphism of the line V_n/V_{n-1}, which is necessarily zero for x is nilpotent, and therefore $V_{n-1} \supseteq \operatorname{im} x$. Applying the induction hypothesis to $\widetilde{V} = V_{n-1}$, endowed with the flag $(V_p)_{0\leq p\leq n-1}$, the endomorphism $\tilde{x} = x\big|_{\widetilde{V}}$ and the subspaces $\widetilde{W}_p = W_p \cap \widetilde{V}$, we get a basis $\widetilde{\mathbf{e}} = (e_1, \dots, e_{n-1})$ of V_{n-1} satisfying (l) and (m) for $p < n$ and such that all the $\widetilde{W}_p$ are coordinate subspaces w.r.t. the basis $\widetilde{\mathbf{e}}$.

We now have to complete $\widetilde{\mathbf{e}}$ by adding a vector e_n. We distinguish two cases.

If $V_{n-1} \supseteq \ker x$, then $x(V_{n-1}) \subsetneq x(V_n) = \operatorname{im} x$. Now $\operatorname{im} x = \widetilde{W}_0$ is a coordinate subspace w.r.t. the basis $\widetilde{\mathbf{e}}$, whence an index p such that $e_p \in \operatorname{im} x \setminus x(V_{n-1})$. We then choose $e_n \in x^{-1}(e_p)$ and check that conditions (l)–(n) are satisfied.

Otherwise, there is an index $p \geq 1$ such that $V_{n-1} \not\supseteq W_p$ but $V_{n-1} \supseteq W_{p-1}$. Let us choose $e_n \in W_p \setminus V_{n-1}$ and set $\mathbf{e} = (e_1, \dots, e_n)$. For each $q \geq p$, we have then $e_n \in W_q \setminus \widetilde{W}_q$, so $W_q = \widetilde{W}_q + \mathbb{C}e_n$, and thus W_q is a coordinate subspace w.r.t. the basis $\mathbf{e}$. Conditions (l) and (n) are therefore satisfied, and condition (m) follows from $e_n \in \ker x$.

2.4 We consider again the setup of paragraph 2.2. Using 2.3, we find a basis $(e_1,\dots,e_n)$ that satisfies (l)–(n). The situation can be depicted on a diagram of the following form.



Each column contains two indices p,q such that $x(e_p)=e_q$, an isolated box contains an index r such that $e_r \in (\ker x \setminus \operatorname{im} x)$, and the gray boxes are those that contain an index s such that $e_s \in W$. The above picture represents a situation with $n=9$, $k=4$, $\dim \operatorname{im} x = 3$, $\dim \ker x = 6$.

2.5 To compute the left-hand side of (4), we may eliminate the filtrations $\mathbf{X}$ such that $f(\mathbf{X}) = 0$, hence we may replace $\mathscr{F}$ by the set $\mathscr{F}_0$ of all filtrations $\mathbf{X}$ that satisfy (g), (h) and

(i) if $p \in J$, either $X_p \not\subseteq V_{p-1}$ or $x(V_p) \not\subseteq X_{p-1}$.

We want to describe $\mathscr{F}_0$ in coordinates. To this aim, we denote by K the set of labels in the gray boxes; in other words, $K = \{p \in [1,n] \mid e_p \in W\}$. In addition, let $(e_1^*,\dots,e_n^*)$ be the dual basis to $\mathbf{e}$.

We first note that if $\mathscr{F}_0 \neq \varnothing$, then

(A) Each $p \in J$ either lies in the top box of a column or belongs to K.

In fact, if $p \in J$ does not lie in the top box of a column, then $e_p \in \ker x$. Picking $\mathbf{X} \in \mathscr{F}_0$, we thus have $x(V_p) = x(V_{p-1}) \subseteq X_{p-1}$. Condition (i) then says that $X_p \not\subseteq V_{p-1}$. A fortiori $W \cap V_p \not\subseteq V_{p-1}$, and therefore $e_p \in W$.

We now claim that when condition (A) is satisfied, a filtration $\mathbf{X} \in \mathscr{F}_0$ is uniquely described by a matrix $\Xi = (\xi_{pq})$ of complex numbers with the following properties:

(B) The lines of Ξ are indexed by J; the columns of Ξ are indexed by K.

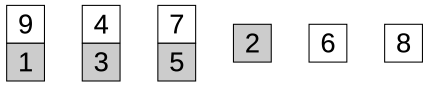
(C) If $p \in J$ does not lie in the top box of a column, then $\xi_{pp} = 1$ and all the other entries in p-th line of Ξ are zero.

(D) If $p \in J$ lies in the top box of a column, if q is the index in the box below $\boxed{p}$, then $\xi_{pq} = 1$. In addition, if $p' \neq p$ also lies in the top box of a column, if q' is the index in the box below $\boxed{p'}$, and if either $p' < p$ or $p' \in J$, then $\xi_{pq'} = 0$. Also, if $q'' > p$ does not lie in the top box of a column and if $q'' \in J$, then $\xi_{pq''} = 0$.

(E) Let $p \in K \setminus J$ and let $Y = (y_q)$ be the line-vector given by $y_p = 1$ and $y_q = 0$ for $q \neq p$. Then Y belongs to the linear span of the rows of Ξ with index $> p$.

(F) Ξ is invertible.

As an example, consider the following diagram

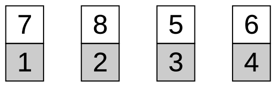


and take $J = \{2,3,7,9\}$ and $K = \{1,2,3,5\}$. Condition (A) is satisfied. Conditions (B)–(D) impose Ξ has the form

$$\begin{array}{c} \\ 2 \\ 3 \\ 7 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \xi_{72} & 0 & 1 \\ 1 & \xi_{92} & 0 & 0 \end{pmatrix} \end{array}.$$

Condition (F) is then automatically fulfilled, while condition (E) amounts to $\xi_{72} = 0$.

As another example, we consider the diagram



and take $J = \{3,4,5,6\}$ and $K = \{1,2,3,4\}$. Condition (A) is satisfied. Conditions (B)–(D) impose Ξ has the form

$$\begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \xi_{51} & \xi_{52} & 1 & 0 \\ \xi_{61} & \xi_{62} & 0 & 1 \end{pmatrix} \end{array}.$$

Conditions (E) and (F) demand that $\begin{vmatrix} \xi_{51} & \xi_{52} \\ \xi_{61} & \xi_{62} \end{vmatrix} \neq 0$.

2.6 We prove here the claim of 2.5.

Given the matrix Ξ, we set $\varphi_p = \sum_{q \in K} \xi_{pq} e_q^*$. The correspondence $\Xi \mapsto \mathbf{X}$ is given by the rule

$$X_p = \{v \in W \mid \varphi_{p'}(v) = 0 \text{ for each } p' \in J \cap [p+1, n]\}.$$

We first check that $\mathbf{X}$ satisfies conditions (g)–(i) when Ξ satisfies conditions (B)–(F).

By construction, $X_n = W$, $X_p = X_{p-1}$ if $p \notin J$, and X_{p-1} has codimension at most one in X_p if $p \in J$. In addition, $X_0 = 0$ thanks to (F). All this gives (g).

To establish the inclusion $X_p \subseteq V_p$, we proceed by decreasing induction on p. So let us assume that $X_p \subseteq V_p$ and let us show that $X_{p-1} \subseteq V_{p-1}$. If $p \notin K$, then $X_p \subseteq V_p \cap W \subseteq V_{p-1}$, and a fortiori $X_{p-1} \subseteq V_{p-1}$. If $p \in J \cap K$, then p does not lie in the top box of a column, so $\varphi_p = e_p^*$ by condition (C), and therefore $X_{p-1} = \{v \in X_p \mid \varphi_p(v) = 0\}$ is contained in $\{v \in V_p \mid \varphi_p(v) = 0\} = V_{p-1}$. Lastly, if $p \in K \setminus J$, then $e_p^*\big|_W$ is a linear combination of the forms $\varphi_{p'}\big|_W$, for $p' \in J \cap [p+1, n]$, thanks to condition (E); it follows that e_p^* vanishes on X_p, hence that X_p is already contained in V_{p-1}.

Let now p be in the top box of a column and let q be the index in the box below $\boxed{p}$. Take $p' \in J \cap [p+1,n]$. If p' does not lie in the top box of a column, then $\varphi_{p'} = e^*_{p'}$ and $p' > p > q$. If p' lies in the top box of a column, then $\xi_{p'q} = 0$ by condition (D). In either case, $\varphi_{p'}(e_q) = 0$. Since this holds for any $p' \in J \cap [p+1,n]$, we conclude that $e_q \in X_p$. This can be rewritten $x(e_p) \in X_p$. In fact, $x(e_p) \in X_p$ holds for any p, because $x(e_p) = 0$ when p is not in a top box. It follows that $x(V_p) \subseteq X_p$. Combined with the inclusion $X_p \subseteq V_p$ proved in the previous paragraph, we get (h).

Fix $p \in J$. If p does not lie in the top box of a column, then $X_{p-1} = \{v \in X_p \mid e^*_p(v) = 0\}$ by condition (C); in addition, we have already proved (g), so we know that $X_{p-1} \neq X_p$; these two facts imply that X_p cannot be contained in $\ker e^*_p$, that is, $X_p \not\subseteq V_{p-1}$. If p lies in the top box of a column, then $\varphi_p(x(e_p)) = 1$ by condition (D), so $x(e_p) \notin X_{p-1}$, and therefore $x(V_p) \not\subseteq X_{p-1}$. This shows (i).

Therefore the map $\Xi \mapsto \mathbf{X}$ is well defined. To show its bijectivity, we construct its inverse. We thus start with a filtration $\mathbf{X} \in \mathscr{F}_0$ and look for the matrix Ξ.

Take $p \in J$. The p-th line of Ξ displays the coordinates of a linear form $\varphi_p \in \operatorname{span}_{\mathbb{C}}\{e^*_q \mid q \in K\}$ such that $X_{p-1} = \{v \in X_p \mid \varphi_p(v) = 0\}$. We need to show the existence of φ_p and to normalize it in a unique way. By induction, we may assume that the p'-th line of Ξ has been determined for each $p' \in J$ greater than p.

Assume first that p does not lie in the top box of a column. Then by (h), we have $x(V_p) = x(V_{p-1}) \subseteq X_{p-1}$, and thus $X_p \not\subseteq V_{p-1}$, thanks to condition (i). It follows that $V_{p-1} \cap X_p$ is strictly contained in X_p. However X_{p-1} is contained in $V_{p-1} \cap X_p$ and is an hyperplane of X_p, by (g), so we necessarily have $X_{p-1} = V_{p-1} \cap X_p$. Therefore X_{p-1} is obtained by cutting X_p by the hyperplane of equation $\varphi_p = e^*_p$. This gives condition (C) on the matrix Ξ.

Now assume that p lies in the top box of a column. Then $V_{p-1} \cap W = V_p \cap W$, so $V_{p-1} \cap X_p = V_p \cap X_p = X_p$, whence $X_p \subseteq V_{p-1}$. Condition (i) then says that $x(V_p) \not\subseteq X_{p-1}$. Looking at conditions (g) and (h), we deduce that X_{p-1} is produced by cutting X_p by an hyperplane that contains $x(V_{p-1})$ but not $x(V_p)$. If q is the index in the box below $\boxed{p}$, then the equation of this hyperplane can be uniquely written as $\varphi_p = e^*_q + \sum_{r \neq q} \xi_{pr} e^*_r$. If $p' < p$ lies in the top box of a column and if q' is the index of the box below $\boxed{p'}$, then $\xi_{pq'} = 0$, because $x(e_{p'}) \in x(V_{p-1})$ and $x(V_{p-1}) \subseteq \ker \varphi_p$. We now modify φ_p so as to obtain the form specified in condition (D), without altering the property $X_{p-1} = \{v \in X_p \mid \varphi_p(v) = 0\}$:

- Since $X_p \subseteq W$, we may subtract from φ_p all the terms $\xi_{pr} e^*_r$ such that $r \notin K$.
- If $p' > p$ lies in the top box of a column and if $p' \in J$, then $X_p \subseteq X_{p'-1} \subseteq \ker \varphi_{p'}$. Let q' be the index in the box below $\boxed{p'}$; we may then subtract $\xi_{pq'} \varphi_{p'}$ from φ_p, which yields a new φ_p with $\xi_{pq'} = 0$.
- We repeat the previous step for each relevant p' in order to annihilate all $\xi_{pq'}$.
- Lastly, if $q'' > p$ does not lie in the top box of a column and if $q'' \in J$, then $X_p \subseteq V_p \subseteq \ker e^*_{q''}$, so we may subtract the term $\xi_{pq''} e^*_{q''}$ from φ_p.

It remains to check that the matrix Ξ satisfies (E) and (F). We note that by construction,

$$X_p = \{v \in W \mid \varphi_{p'}(v) = 0 \text{ for each } p' \in J \cap [p+1,n]\}.$$

Condition (F) then comes from $X_0 = 0$. Now take $p \in K \setminus J$. By (g) and (h), we have $X_p = X_{p-1} \subseteq V_{p-1}$, so e_p^* vanishes on X_p. From the above description of X_p, it then follows that $e_p^*\big|_W$ belongs to the linear span of the elements $\varphi_{p'}\big|_W$ with $p' \in J \cap [p+1, n]$. This is condition (E).

2.7 Sections 2.5 and 2.6 describe $\mathscr{F}_0$ as a set of matrices. This allows to compute the Euler characteristic of $\mathscr{F}_0$: it is equal to zero or one, the latter case happening precisely when each column of the diagram contains exactly one element of J and when the remaining elements of J occupy the isolated gray boxes. The aim of this section is to show this result.

We begin with a closer look at conditions (B)–(F). Conditions (C) and (D) prescribe that certain matrix elements ξ_{pq} have value zero or one. The other matrix entries of Ξ can be freely chosen in $\mathbb{C}$; we call them the free entries. The set $\mathscr{V}$ of all matrices defined by conditions (B)–(D) is thus an affine space; setting the free entries to zero provides an origin O. Conditions (E) and (F) define a locally closed subset $\mathscr{W} \subseteq \mathscr{V}$.

The position of the zeros and the ones in a matrix $\Xi \in \mathscr{V}$ obey a special pattern, which can be described as follows. Up to a reordering of the rows, Ξ can be viewed as the pile of two matrices Ξ_1 and Ξ_2. The matrix Ξ_1 gathers the rows with index in $J \cap K$; by condition (C), each row of Ξ_1 is a basis row-vector, that is, its entries are all zero except one entry equal to one. The matrix Ξ_2 gather the rows with index in $J \setminus K$; all free entries of Ξ are in Ξ_2. Condition (D) implies that a column of Ξ_2 either is a basis column-vector, or its entries are zero or free entries. In other words, if a column of Ξ_2 contains a one, then it is a basis column-vector.

With the help of this description, one easily sees that a minor of Ξ is a homogeneous polynomial in the free entries. In fact, let H be a matrix obtained from Ξ by removing some lines and some columns. Like Ξ, the matrix H has a description as a pile of two matrices H_1 and H_2; the only difference with Ξ is that a row of H_1 is either a basis row-vector or zero. To compute the determinant of H, we pick a row in H_1. If the row is zero, then the determinant vanishes; otherwise, this row contains a one, and the determinant is changed at most by a sign if we remove the line and the column that contains this one. Repeating this operation, we get rid of all the lines of H_1 and end up with a matrix obtained by removing columns from H_2. Again, a column of this matrix either is a basis column-vector, or its entries are zero or free entries. Pursuing the expansion of the determinant, we remove the basis vector-columns together with some lines. We wind up with a matrix H_3 whose entries are zeros or free entries, and we have $\det H = \pm \det H_3$. This establishes our claim.

In view of this homogeneity property, conditions (E) and (F) define $\mathscr{W}$ as a cone of vertex O. Thus $\mathscr{W}_0 = \mathscr{W} \setminus \{O\}$ is endowed with a free action of $\mathbb{C}^*$. The principal $\mathbb{C}^*$-bundle $\mathscr{W}_0 \to \mathscr{W}_0/\mathbb{C}^*$ then gives for the Euler characteristic with compact support

$$\chi(\mathscr{W}_0) = \chi(\mathscr{W}_0/\mathbb{C}^*)\,\chi(\mathbb{C}^*) = 0.$$

We conclude that $\chi(\mathscr{F}_0) = \chi(\mathscr{W})$ is zero if $\mathscr{W} = \mathscr{W}_0$ and is one if $\mathscr{W} = \mathscr{W}_0 \sqcup \{O\}$. To finish the proof, it remains to determine when O belongs to $\mathscr{W}$.

According to the computation explained above, $\det \Xi = \pm \det \Xi_3$, where the entries of Ξ_3 are either zero or free. In view of condition (F), the origin O belongs to $\mathscr{W}$ if and only if Ξ_3 is the empty matrix. A scrutiny of the construction of Ξ_3 reveals that the columns of Ξ_3 are indexed by columns $\begin{array}{|c|}\hline p \\ \hline q \\ \hline \end{array}$ with $p,q \notin J$ and boxes $\boxed{r}$ with $r \notin J$. Therefore Ξ_3 is the empty matrix if and only if each column of the diagram contains at least one element of J and each isolated gray box contains an element of J. By cardinality, the latter condition is equivalent to the criterion given in our claim.

2.8 We are now in a position to prove the proposition in Section 2.2.

We first consider the left-hand side of the equality. By definition of f, given $\mathbf{X} \in \mathscr{F}$, we have $f(\mathbf{X}) \neq 0$ if and only if $\mathbf{X} \in \mathscr{F}_0$. In addition, in the case $\mathbf{X} \in \mathscr{F}_0$, we have $x(V_p) \not\subseteq X_{p-1}$ if and only if p lies in the top box of the diagram, by Section 2.6. It follows that f assumes only two values on $\mathscr{F}$: it vanishes on $\mathscr{F} \setminus \mathscr{F}_0$ and it is equal to $(-1)^N$ on $\mathscr{F}_0$, where N is the number of indices $p \in J$ in the top boxes of columns. Therefore $\int_{\mathscr{F}} f = (-1)^N \chi(\mathscr{F}_0)$. The computation in Section 2.7 then gives us the following combinatorial recipe for the value of $\int_{\mathscr{F}} f$:

- It is zero if there is an isolated white box whose index belongs to J or if there is a column whose two indices belong to J;
- Otherwise, it is $(-1)^N$, where $N = \left|J \setminus K\right|$.

The right-hand side $\int_{\mathscr{G}} g$ can be computed by duality. Indeed, we define a twisted setup as follows:

(ã) $\tilde{n} = n$, $\tilde{k} = n - k$;

(b̃) $\widetilde{V}$ is the dual of V;

(c̃) The p-dimensional subspace $\widetilde{V}_p$ of the flag of $\widetilde{V}$ is the orthogonal of V_{n-p};

(d̃) $\tilde{x}$ is the transpose of x;

(ẽ) $\widetilde{W}$ is the orthogonal of W;

(f̃) $\widetilde{J} = \{n+1-p \mid p \in [1,n] \setminus J\}$.

In addition, duality also gives us a bijective correspondence between $\mathbf{Y}$ and the set $\widetilde{\mathbf{X}}$ of filtrations for the twisted setup; the map g is thereby transported to a map $\tilde{f}$. We then have $\int_{\mathscr{G}} g = \int_{\widetilde{\mathscr{F}}} \tilde{f}$.

We can then use again our combinatorial recipe. We observe that the diagram of the twisted situation is obtained by turning upside-down the diagram in Section 2.4, by changing each entry p to $n+1-p$, and by inverting the colors in all the boxes. Comparing this with the change described in (f̃), we retrieve for $\int_{\widetilde{\mathscr{F}}} \tilde{f}$ the same value as for $\int_{\mathscr{F}} f$. Therefore both sides of (4) are equal.

3 THE PREPROJECTIVE MODEL FOR $\mathbf{U}^+$

3.1 We consider a finite non-empty graph without loops. This is the same as giving two finite sets I and H with $I \neq \varnothing$, a fixed point free involution $h \mapsto h^*$ of H, and two maps $s,t: H \to I$ such that $s(h^*) = t(h) \neq s(h)$ for all $h \in H$.

For $i,j \in I$, we set $a_{ij} = -\left|\{h \in H \mid s(h) = i,\ t(h) = j\}\right|$ if $i \neq j$ and $a_{ij} = 2$ if $i = j$. Then $A = (a_{ij})$ is a symmetric generalized Cartan matrix.

Let $\mathbf{U}^+$ be the $\mathbb{Q}$-algebra defined by generators e_i for $i \in I$, submitted to the Serre relations

$$\sum_{p+q=1-a_{ij}} (-1)^p \frac{e_i^p}{p!}\, e_j\, \frac{e_i^q}{q!} = 0.$$

The weight lattice is the free $\mathbb{Z}$-module with basis $\{\alpha_i \mid i \in I\}$; we denote it by $\mathbb{Z}I$. We write a typical element in $\mathbb{Z}I$ as $\nu = \sum_{i \in I} \nu_i \alpha_i$; when all the ν_i are non-negative, we write $\nu \in \mathbb{N}I$. Given $\nu \in \mathbb{N}I$, we denote by $\mathbf{U}^+_\nu$ the subspace of $\mathbf{U}^+$ spanned by the monomials $e_{i_1} \cdots e_{i_n}$ for sequences $i_1, \ldots, i_n$ in which i appears ν_i times.

3.2 For $i \neq j$ and $0 \leq m \leq -a_{ij}$, let

$$f_{i,j,m} = \sum_{p+q=m} (-1)^p \frac{e_i^p}{p!}\, e_j\, \frac{e_i^q}{q!}.$$

For a fixed $i \in I$, we denote by $\mathbf{U}^+_i$ the subalgebra of $\mathbf{U}^+$ generated by the elements $f_{i,j,m}$, for all possible (j,m). We set $\mathbf{U}^+_{i,\nu} = \mathbf{U}^+_i \cap \mathbf{U}^+_\nu$.

We define an automorphism T_i of $\mathbf{U}^+_i$ by the requirement

$$T_i(f_{i,j,m}) = (-1)^m f_{i,j,-a_{ij}-m}.$$

Lemma 38.1.3 in [6] shows that this definition makes sense; in fact, T_i coincides with the restriction to $\mathbf{U}^+_i$ of (the specialization at $v = 1$ of) the automorphism $T'_{i,-1}$. Lemma 38.1.2 in [6] then implies that the subalgebra $\mathbf{U}^+_i$ just defined coincides with the subalgebra $\mathbf{U}^+_i$ of the introduction.

One may here note that $\mathbf{U}^+_i$ is the smallest subalgebra of $\mathbf{U}^+$ that contains all the elements e_j for $j \neq i$ and that is stable by the derivation $D = \mathrm{ad}(e_i)$. This assertion follows from the equality

$$f_{i,j,m} = \frac{(-1)^m}{m!} D^m(e_j).$$

3.3 We fix a function $\varepsilon : H \to \{\pm 1\}$ such that $\varepsilon(h) + \varepsilon(h^*) = 0$ for all $h \in H$. We view the tuple (I,H,s,t) as directed graph, s and t being the source and target maps. We work over the field of complex numbers.

The preprojective algebra is defined as the quotient of the path algebra of this graph by the ideal generated by

$$\sum_{h \in H} \varepsilon(h) h^* h.$$

Thus, a finite dimensional representation M of the preprojective algebra is the datum of finite dimensional $\mathbb{C}$-vector spaces M_i together with a tuple

$$(M_h) \in \prod_{h\in H} \mathrm{Hom}_{\mathbb{C}}(M_{s(h)}, M_{t(h)})$$

such that for any $i \in I$,

$$\sum_{\substack{h\in H \\ s(h)=i}} \varepsilon(h) M_{h^*} M_h = 0$$

as a linear map $M_i \to M_i$. The dimension-vector of M is defined as

$$\underline{\dim} M = \sum_{i\in I} (\dim M_i)\alpha_i.$$

Each vertex $i \in I$ affords a one-dimensional representation S_i of the preprojective algebra; explicitly, $\underline{\dim} S_i = \alpha_i$ and the arrows act by zero on S_i. A finite dimensional representation M of the preprojective algebra is said to be nilpotent if all its Jordan-Hölder components are one-dimensional, that is, are isomorphic to a S_i. Nilpotent representations of the preprojective algebra are the same as finite dimensional Π-modules, where Π is the completion of the preprojective algebra with respect to the ideal generated by the arrows.

We can here fix the vector space M_i and let the linear maps M_h vary: we denote by $\mathscr{C}$ the category of finite dimensional I-graded $\mathbb{C}$-vector spaces $\mathbf{V} = \bigoplus_{i\in I} V_i$, and for $\mathbf{V} \in \mathscr{C}$, we denote by $\Lambda_{\mathbf{V}}$ the set of all elements $x = (x_h)$ in

$$E_{\mathbf{V}} = \prod_{h\in H} \mathrm{Hom}_{\mathbb{C}}(V_{s(h)}, V_{t(h)})$$

such that $(\mathbf{V}, x)$ is a nilpotent representation of the preprojective algebra. An isomorphism $\mathbf{V} \cong \mathbf{W}$ in $\mathscr{C}$ induces an isomorphism $\Lambda_{\mathbf{V}} \cong \Lambda_{\mathbf{W}}$; in particular, the group $G_{\mathbf{V}} = \prod_{i\in I} \mathbf{GL}(V_i)$ acts on $\Lambda_{\mathbf{V}}$. Isomorphism classes of Π-modules of dimension-vector $\underline{\dim}\mathbf{V}$ are in one-to-one correspondence with $G_{\mathbf{V}}$-orbits in $\Lambda_{\mathbf{V}}$.

3.4 Recall the notation defined in Section 2.1.

For $\mathbf{V} \in \mathscr{C}$, let $\widetilde{M}(\Lambda_{\mathbf{V}})$ be the space of all functions $f \in M(\Lambda_{\mathbf{V}})$ that are constant on any $G_{\mathbf{V}}$-orbit in $\Lambda_{\mathbf{V}}$. If $\mathbf{V}, \mathbf{W} \in \mathscr{C}$ have the same dimension-vector, say ν, then $\widetilde{M}(\Lambda_{\mathbf{V}})$ and $\widetilde{M}(\Lambda_{\mathbf{W}})$ are canonically isomorphic. One can therefore identify these spaces and safely denote them by $\widetilde{M}_\nu$.

Let $\nu', \nu'' \in \mathbb{N}I$ and set $\nu = \nu' + \nu''$. We define a bilinear map $\star : \widetilde{M}_{\nu'} \times \widetilde{M}_{\nu''} \to \widetilde{M}_\nu$ by the following recipe. We choose $\mathbf{V}, \mathbf{V}', \mathbf{V}'' \in \mathscr{C}$ such that $\nu = \underline{\dim}\mathbf{V}$, $\nu' = \underline{\dim}\mathbf{V}'$ and $\nu'' = \underline{\dim}\mathbf{V}''$. For $(f', f'') \in \widetilde{M}(\Lambda_{\mathbf{V}'}) \times \widetilde{M}(\Lambda_{\mathbf{V}''})$ and $x \in \Lambda_{\mathbf{V}}$, we set $(f' \star f'')(x) = \int_{\mathscr{H}} \phi$, where the following notation is used:

- $\mathscr{H}$ is the variety consisting of all I-graded subspaces $\mathbf{W} \subseteq \mathbf{V}$ such that $\underline{\dim}\mathbf{W} = \underline{\dim}\mathbf{V}''$ (this is a product of Grassmannian varieties).
- Given $\mathbf{W} \in \mathscr{H}$, the number $\phi(\mathbf{W})$ is zero except if $x_h(W_{s(h)}) \subseteq W_{t(h)}$ for all $h \in H$. In the latter case, let $\tilde{x}' \in \Lambda_{\mathbf{V}/\mathbf{W}}$ and $\tilde{x}'' \in \Lambda_{\mathbf{W}}$ be the elements induced by x, and let $x' \in \Lambda_{\mathbf{V}'}$ and $x'' \in \Lambda_{\mathbf{V}''}$ be the elements obtained by transporting $\tilde{x}'$ and $\tilde{x}''$ through isomorphisms $\mathbf{V}' \cong \mathbf{V}/\mathbf{W}$ and $\mathbf{V}'' \cong \mathbf{W}$; then $\phi(\mathbf{W}) = f'(x')f''(x'')$.

The maps $\star$ combine to endow the $\mathbb{Q}$-vector space $\widetilde{M} = \bigoplus_{\nu \in \mathbb{N}I} \widetilde{M}_\nu$ with the structure of a $\mathbb{N}I$-graded algebra.

With this notation, there is a unique morphism of algebras $\kappa : \mathbf{U}^+ \to \widetilde{M}$ such that for any $i \in I$ and any $p \geq 1$, for any $\mathbf{V} \in \mathscr{C}$ of dimension-vector $p\alpha_i$, the element $\kappa(e_i^p/p!)$ is the function $f \in \widetilde{M}(\Lambda_\mathbf{V})$ with value 1 on the point $\Lambda_\mathbf{V}$. The morphism κ is injective (Theorem 2.7 (c) in [8]).

3.5 In Section 2.2 of [1], endofunctors Σ_i and Σ_i^* of the category Π-mod, called reflection functors, are defined for each vertex $i \in I$. In this paper, we will use only Σ_i^*; it can be quickly defined as $\Sigma_i^* = I_i \otimes_\Pi ?$, where I_i is the annihilator of the simple Π-module S_i.

To concretely describe Σ_i^*, we introduce a special notation, that analyzes a Λ-module M locally around the vertex i. We break the datum of M in two parts: the first part consists of the vector spaces M_j for $j \neq i$ and of the linear maps between them; the second part consists of the vector spaces and of the linear maps that appear in the diagram

$$\bigoplus_{\substack{h \in H \\ s(h)=i}} M_{t(h)} \xrightarrow{(M_{h^*})} M_i \xrightarrow{(\varepsilon(h)M_h)} \bigoplus_{\substack{h \in H \\ s(h)=i}} M_{t(h)}.$$

For brevity, we will write the latter as

$$\widetilde{M}_i \xrightarrow{M_{\mathrm{in}(i)}} M_i \xrightarrow{M_{\mathrm{out}(i)}} \widetilde{M}_i. \tag{5}$$

The relations of the preprojective algebra imply

$$M_{\mathrm{in}(i)} M_{\mathrm{out}(i)} = 0. \tag{6}$$

With this notation, the module $\Sigma_i^* M$ is obtained by replacing (5) with

$$\widetilde{M}_i \twoheadrightarrow \operatorname{coker} M_{\mathrm{out}(i)} \xrightarrow{M_{\mathrm{out}(i)} \overline{M}_{\mathrm{in}(i)}} \widetilde{M}_i,$$

where the map $\overline{M}_{\mathrm{in}(i)} : \operatorname{coker} M_{\mathrm{out}(i)} \to M_i$ is induced by $M_{\mathrm{in}(i)}$. The vector spaces M_j for $j \neq i$ and the linear maps between them are not affected by the functor Σ_i^*.

3.6 Let $\nu \in \mathbb{N}I$ and let $\mathbf{V} \in \mathscr{C}$ be of dimension-vector ν. The evaluation at a point $x \in \Lambda_\mathbf{V}$ is a linear form on $\widetilde{M}(\Lambda_\mathbf{V}) = \widetilde{M}_\nu$, hence gives via κ a linear form δ_x on $\mathbf{U}_\nu^+$. If M is a Π-module of dimension-vector ν, we write δ_M instead of δ_x, where $x \in \Lambda_\mathbf{V}$ is chosen so that $(\mathbf{V}, x)$ is isomorphic to M.

THEOREM. Let M be a Π-module of dimension-vector ν. If $\ker M_{\mathrm{out}(i)} = 0$, then

$$\forall u \in \mathbf{U}_{i,\nu}^+, \qquad \langle \delta_M, u \rangle = \langle \delta_{\Sigma_i^* M}, T_i(u) \rangle. \tag{7}$$

In the remainder of Section 3, we explain how to deduce (2) from this theorem.

3.7 Given an algebraic variety X, we denote by $\operatorname{Irr} X$ the set of irreducible components of V.

Let $\nu \in \mathbb{N}I$. Between two objects $\mathbf{V}, \mathbf{W} \in \mathscr{C}$ of dimension-vector ν, there is always an isomorphism $\mathbf{V} \cong \mathbf{W}$, unique up to right multiplication by an element of $G_{\mathbf{V}}$. The resulting isomorphism $\Lambda_{\mathbf{V}} \cong \Lambda_{\mathbf{W}}$ induces a bijection $\operatorname{Irr} \Lambda_{\mathbf{V}} \cong \operatorname{Irr} \Lambda_{\mathbf{W}}$, which is canonical since $G_{\mathbf{V}}$ is connected. One can therefore identify these sets and safely denote them by B_ν.

We refer the reader to Section 7 in [2] for the notion of crystal in the sense of Kashiwara. In Section 8 of [5], Lusztig endows the set $B = \bigsqcup_{\nu \in \mathbb{N}I} B_\nu$ with the structure of a crystal. Through the study of the properties of an involution $b \mapsto b^*$, Kashiwara and Saito prove (Theorem 5.3.2 in [3]) that B is isomorphic to the crystal $B(-\infty)$ associated to the crystal basis of $U_q^+(\mathfrak{g})$. In this model, the values $\varphi_i(b)$ and $\varphi_i(b^*)$ are given as follows: if $b \in B(-\infty)$ corresponds to the irreducible component $Z \in \operatorname{Irr} \Lambda_{\mathbf{V}}$ under Kashiwara and Saito's isomorphism and if x is general enough in Z, then

$$\varphi_i(b) = \dim \operatorname{coker} M_{\mathrm{in}(i)} \quad \text{and} \quad \varphi_i(b^*) = \dim \ker M_{\mathrm{out}(i)},$$

where $M = (\mathbf{V}, x)$.

In addition, Saito's reflection σ_i (which we defined in Section 1.2) also has a nice interpretation: if $b \in B_i$ corresponds to $Z' \in \operatorname{Irr} \Lambda_{\mathbf{V}'}$, if $\sigma_i b$ corresponds to $Z'' \in \operatorname{Irr} \Lambda_{\mathbf{V}''}$, then any non-empty open subset of $Z' \times Z''$ contains an element (x', x'') such that

$$\Sigma_i^*(\mathbf{V}', x') \cong (\mathbf{V}'', x'').$$

For a proof, see Proposition 18 in [1].

3.8 Let $\nu \in \mathbb{N}I$, let $\mathbf{V} \in \mathscr{C}$ be of dimension-vector ν, and let $Z \in \operatorname{Irr} \Lambda_{\mathbf{V}}$. Given $u \in \mathbf{U}_\nu^+$, the function $\kappa(u) = (x \mapsto \delta_x(u))$ is constructible on $\Lambda_{\mathbf{V}}$, so takes a constant value on a non-empty open subset of Z. We denote this value by $\delta_Z(u)$. Since $\mathbf{U}_\nu^+$ is finite dimensional, the open subset can be chosen independently of u: there is a non-empty open subset $\Omega \subseteq Z$ such that for any $x \in \Omega$, we have $\delta_x = \delta_Z$ as linear forms $\mathbf{U}_\nu^+ \to \mathbb{Q}$.

By Theorem 2.7 in [8], $\{\delta_Z \mid Z \in \operatorname{Irr} \Lambda_{\mathbf{V}}\}$ is a basis of $(\mathbf{U}_\nu^+)^*$. This basis does not depend on the choice of $\mathbf{V}$ and is called the dual semicanonical basis. If Z corresponds to $b \in B_\nu$ under Kashiwara and Saito's bijection, then the element $S(b)^*$ used in Section 1.2 is $S(b)^* = \delta_Z$.

Now take $b \in B_i$. Identify b to an irreducible component $Z' \in \operatorname{Irr} \Lambda_{\mathbf{V}'}$ and identify $\sigma_i b$ to an irreducible component $Z'' \in \operatorname{Irr} \Lambda_{\mathbf{V}''}$, as we did in Section 3.7. Let $\Omega' \subseteq Z'$ be a non-empty open subset such that $\delta_{x'} = \delta_{Z'}$ for any $x' \in \Omega'$. Likewise, let $\Omega'' \subseteq Z''$ be a non-empty open subset such that $\delta_{x''} = \delta_{Z''}$ for any $x'' \in \Omega''$. By shrinking Ω' if necessary, we can assume that

$$0 = \varphi_i(b^*) = \dim \ker M_{\mathrm{out}(i)}$$

for any $x' \in \Omega'$, where $M = (\mathbf{V}', x')$.

Take $(x', x'') \in \Omega' \times \Omega''$ such that

$$\Sigma_i^*(\mathbf{V}', x') \cong (\mathbf{V}'', x'').$$

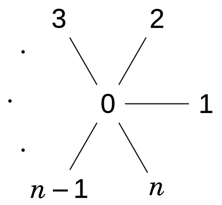
Applying Theorem 3.6 to $M = (\mathbf{V}', x')$, we get

$$\forall u \in \mathbf{U}_{i,\nu}^+, \qquad \langle \delta_{Z'}, u \rangle = \langle \delta_{Z''}, T_i(u) \rangle,$$

where ν is the weight of b. This equation is (2), with b instead of b'.

4 Proof of Theorem 3.6

4.1 We first look at a particular case of Theorem 3.6. Namely, we consider the following star-shaped graph with set of vertices $I = \{0, \dots, n\}$.



(8)

Let $0 \le k \le n$ and let M be a Π-module of dimension-vector $\nu = k\alpha_0 + (\alpha_1 + \cdots + \alpha_n)$. We claim that if $\ker M_{\mathrm{out}(0)} = 0$, then Theorem 3.6 holds for $i = 0$.

By linearity, it suffices to check (7) for u of the form $u = f_{0,\pi(n),m_n} f_{0,\pi(n-1),m_{n-1}} \cdots f_{0,\pi(1),m_1}$, where π is a permutation of $[1,n]$ and where each $m_j \in \{0,1\}$. The condition that u has weight ν imposes that $J = \{j \in [1,n] \mid m_j = 1\}$ has cardinality k.

Let us set

$$V = \widetilde{M}_0 = M_1 \oplus \cdots \oplus M_n, \quad W = \operatorname{im} M_{\mathrm{out}(0)} \quad \text{and} \quad x = M_{\mathrm{out}(0)} M_{\mathrm{in}(0)}.$$

The preprojective relation $M_{\mathrm{in}(0)} M_{\mathrm{out}(0)} = 0$ ensures that $x^2 = 0$ and that $W \subseteq \ker x$. We further set $V_0 = 0$ and $V_p = M_{\pi(1)} \oplus \cdots \oplus M_{\pi(p)}$ for each $p \in [1,n]$; this gives a complete flag in V.

If x does not leaves this flag stable, then both sides of (7) vanish, so (7) holds true.

If x leaves this flag stable, then we are in the setup of Section 2.2 and the two sides of (7) are given by the integrals on the two sides of (4). Our proposition in Section 2.2 therefore ensures that our particular case of Theorem 3.6 holds true.

4.2 We now tackle the general case of Theorem 3.6.

Since $\mathbf{U}_i^+$ is generated by the elements $f_{i,j,m}$, it is enough to consider the case of a monomial $u = f_{i,j_r,m_r} \cdots f_{i,j_1,m_1}$, where $j_s \in I \setminus \{i\}$ and $0 \le m_s \le -a_{ij_s}$ for each $s \in [1,r]$. The weight of u is of course $\nu = m\alpha_i + \alpha_{j_1} + \cdots + \alpha_{j_r}$, where $m = m_1 + \cdots + m_r$. We consider a Π-module M of dimension-vector ν such that $\ker M_{\mathrm{out}(i)} = 0$. We write $M = (\mathbf{V}', x')$ and $\Sigma_i^* M = (\mathbf{V}'', x'')$, where $\mathbf{V}' \in \mathscr{C}$, $\mathbf{V}'' \in \mathscr{C}$, $x' \in \Lambda_{\mathbf{V}'}$ and $x'' \in \Lambda_{\mathbf{V}''}$. In fact, the reflection functor Σ_i^* only touches the space attached to vertex i, so for $k \ne i$, we may identify V_k'' to V_k' and write simply V_k.

The left-hand side of (7) is the evaluation at M of the function $\kappa(f_{i,j_r,m_r}) \star \cdots \star \kappa(f_{i,j_1,m_1})$. By the definition in Section 3.4, this number is an integral $\int_{\mathscr{H}'} \phi'$, where $\mathscr{H}'$ is the set of all filtrations

$$0 = \mathbf{V}_0' \subseteq \mathbf{V}_1' \subseteq \cdots \subseteq \mathbf{V}_{r-1}' \subseteq \mathbf{V}_r' = \mathbf{V}'$$

such that $\underline{\dim}\,(\mathbf{V}_s' / \mathbf{V}_{s-1}') = \alpha_{j_s} + m_s \alpha_i$ for each $s \in [1,r]$ and where $\phi' : \mathscr{H} \to \mathbb{Q}$ is a function given by the product of the $\kappa(f_{i,j_s,m_s})$.

Let $\mathscr{H}_*$ be the product for $k \ne i$ of the complete flag varieties of the vector spaces V_k. Let $\mathscr{H}_i'$ be the set of all filtrations

$$0 = V_{i,0}' \subseteq V_{i,1}' \subseteq \cdots \subseteq V_{i,r-1}' \subseteq V_{i,r}' = V_i'$$

such that $\dim V'_{i,s}/V'_{i,s-1} = m_s$ for each $s \in [1,r]$. Certainly, $\mathscr{H}'$ is isomorphic to the product $\mathscr{H}'_i \times \mathscr{H}_*$. Our integral can then be computed with the help of the Fubini theorem:

$$\langle \delta_M, u \rangle = \int_{\mathscr{H}'} \phi' = \int_{\mathscr{H}_*} \left(\int_{\mathscr{H}'_i} \phi' \right).$$

The right-hand side of (7) can be computed by a similar convolution product. Let $\mathscr{H}''_i$ be the set of all filtrations

$$0 = V''_{i,0} \subseteq V''_{i,1} \subseteq \cdots \subseteq V''_{i,r-1} \subseteq V''_{i,r} = V''_i$$

such that $\dim V''_{i,s}/V''_{i,s-1} = -a_{ij_s} - m_s$ for each $s \in [1,r]$. Then

$$\langle \delta_M, T_i(u) \rangle = \int_{\mathscr{H}_*} \left(\int_{\mathscr{H}''_i} \phi'' \right),$$

where ϕ'' is a function given by the product of the $\kappa\big((-1)^{m_s} f_{i,j_s,-a_{ij_s}-m_s}\big)$.

Equation (7) can thus be written

$$\int_{\mathscr{H}_*} \left(\int_{\mathscr{H}'_i} \phi' \right) = \int_{\mathscr{H}_*} \left(\int_{\mathscr{H}''_i} \phi'' \right).$$

Therefore to prove Theorem 3.6, one only has to show

$$\int_{\mathscr{H}'_i} \phi' = \int_{\mathscr{H}''_i} \phi'', \tag{9}$$

where each side depends on the choice of a point in $\mathscr{H}_*$.

4.3 We keep the notations of Section 4.2.

For each $k \neq i$, let A_k be the vector space with basis $\{h \in H \mid (s(h), t(h)) = (i,k)\}$. Let $\widetilde{V} = \widetilde{M}_i$; in other words,

$$\widetilde{V} = \bigoplus_{\substack{h \in H \\ s(h)=i}} V_{t(h)} = \bigoplus_{k \neq i} V_k \otimes_{\mathbb{C}} A_k.$$

We also set $x = M_{\mathrm{out}(i)} M_{\mathrm{in}(i)}$.

We have chosen a point in $\mathscr{H}_*$, that is, a filtration in V_k for each $k \neq i$:

$$0 = V_{k,0} \subseteq V_{k,1} \subseteq \cdots \subseteq V_{k,r-1} \subseteq V_{k,r} = V_k,$$

such that $\dim V_{k,s}/V_{k,s-1} = 1$ if $k = j_s$ and $V_{k,s} = V_{k,s-1}$ if $k \neq j_s$. This induces a filtration

$$0 = \widetilde{V}_0 \subseteq \widetilde{V}_1 \subseteq \cdots \subseteq \widetilde{V}_{r-1} \subseteq \widetilde{V}_r = \widetilde{V}, \tag{10}$$

namely

$$\widetilde{V}_s = \bigoplus_{k \neq i} V_{k,s} \otimes_{\mathbb{C}} A_k.$$

We set $n = \dim \widetilde{V}$. For each $s \in [1,r]$, we set $p_s = 1 + \dim \widetilde{V}_{s-1}$ and $q_s = \dim \widetilde{V}_s$. We thus have a partition $[1,n] = [p_1, q_1] \sqcup \cdots \sqcup [p_r, q_r]$ as a union of disjoint intervals.

Let us first assume that x does not leave stable the filtration (10). In this case, both sides of (9) vanish, so (9) holds true.

Let us now assume that x leaves stable each $\widetilde{V}_s$. Noting that x induces a nilpotent endomorphism on each quotient $\widetilde{V}_s/\widetilde{V}_{s-1}$, we can find a basis $\mathbf{e} = (e_1, \dots, e_n)$ of $\widetilde{V}$ in which the matrix of x is strictly upper triangular and such that $(e_1, \dots, e_{q_s})$ is a basis of $\widetilde{V}_s$, for each $s \in [1, r]$.

We then consider the graph (8). The objects attached to this graph will be underlined: the enveloping algebra is $\underline{\mathbf{U}}^+$ and is generated by elements $\underline{e}_i$ with $i \in \{0, 1, \dots, n\}$; the completed preprojective algebra is $\underline{\Pi}$.

We set $\underline{M}_0 = V_i'$ and $\underline{M}_j = \mathbb{C}e_j$ for all $j \in [1, n]$; thus $M_i = \underline{M}_0$ and $\widetilde{M}_i = \underline{M}_1 \oplus \cdots \oplus \underline{M}_n$. Writing the linear maps $M_{\mathrm{in}(i)}$ and $M_{\mathrm{out}(i)}$ as block matrices with respect to this decomposition of $\widetilde{M}_i$, we get maps $\underline{M}_j \to \underline{M}_0$ and $\underline{M}_0 \to \underline{M}_j$. These maps satisfy the preprojective relations of $\underline{\Pi}$, thanks to Equation (6) and to the fact that the matrix of x in $\mathbf{e}$ is strictly upper triangular. We therefore get a $\underline{\Pi}$-module $\underline{M}$ such that $\underline{M}_{\mathrm{out}(0)} = M_{\mathrm{out}(i)}$ and $\underline{M}_{\mathrm{in}(0)} = M_{\mathrm{in}(i)}$.

With these notations, the integral on the left-hand side of (9) computes

$$\langle \delta_{\underline{M}}, \underline{g}_r' \cdots \underline{g}_1' \rangle, \qquad \text{where} \quad \underline{g}_s' = \sum_{p+q=m_s} (-1)^p \, \frac{\underline{e}_0^p}{p!} \, \underline{e}_{q_s} \cdots \underline{e}_{p_s} \, \frac{\underline{e}_0^q}{q!}.$$

Replacing M by $\Sigma_i^* M$ amounts to replace M_i by $\operatorname{coker} M_{\mathrm{out}(i)}$ without changing $\widetilde{M}_i$ nor x. The analogous of $\underline{M}$ for $\Sigma_i^* M$ is therefore $\Sigma_0^* \underline{M}$. Thus the right-hand side of (9) is equal to

$$\langle \delta_{\Sigma_0^* \underline{M}}, \underline{g}_r'' \cdots \underline{g}_1'' \rangle, \qquad \text{where} \quad \underline{g}_s'' = (-1)^{m_s} \sum_{p+q=-a_{ij_s}-m_s} (-1)^p \, \frac{\underline{e}_0^p}{p!} \, \underline{e}_{q_s} \cdots \underline{e}_{p_s} \, \frac{\underline{e}_0^q}{q!}.$$

Let $D = \mathrm{ad}(\underline{e}_0)$, a derivation of the algebra $\underline{\mathbf{U}}^+$. The Serre relations say that $D^2(\underline{e}_j) = 0$ for any $j \in [1, n]$. Using Leibniz's formula for the m_s-th derivative of a product, we obtain

$$\underline{g}_s' = \frac{(-1)^{m_s}}{m_s!} D^{m_s}(\underline{e}_{q_s} \cdots \underline{e}_{p_s}) = \sum_{\substack{J \subseteq [p_s, q_s] \\ |J| = m_s}} \underline{v}_{q_s}^J \cdots \underline{v}_{p_s}^J,$$

where

$$\underline{v}_j^J = \begin{cases} [\underline{e}_j, \underline{e}_0] = \underline{f}_{0,j,1} & \text{if } j \in J, \\ \underline{e}_j = \underline{f}_{0,j,0} & \text{if } j \notin J. \end{cases}$$

With the same notation,

$$\underline{g}_s'' = \frac{(-1)^{a_{ij_s}}}{(-a_{ij_s} - m_s)!} D^{-a_{ij_s} - m_s}(\underline{e}_{q_s} \cdots \underline{e}_{p_s}) = (-1)^{m_s} \sum_{\substack{K \subseteq [p_s, q_s] \\ |K| = -a_{ij_s} - m_s}} \underline{v}_{q_s}^K \cdots \underline{v}_{p_s}^K.$$

Matching the summand indexed by J in the expansion of $\underline{g}_s'$ with the summand indexed by $K = [p_s, q_s] \setminus J$ in the expansion of $\underline{g}_s''$, we get $\underline{g}_s'' = T_0(\underline{g}_s')$, so

$$\langle \delta_{\Sigma_0^* \underline{M}}, \underline{g}_r'' \cdots \underline{g}_1'' \rangle = \langle \delta_{\Sigma_0^* \underline{M}}, T_0(\underline{g}_r' \cdots \underline{g}_1') \rangle.$$

We then see that (9) is simply the result of Section 4.1 applied to $\underline{M}$ and to $\underline{u} = \underline{g}_r' \cdots \underline{g}_1'$.

This last argument finishes the proof of (9), hence of Theorem 3.6.

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