The Hall algebra of the category of coherent sheaves on the projective line

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Abstract

To an abelian category $\mathcal{A}$ of homological dimension one satisfying certain finiteness conditions, one can associate an algebra, called the Hall algebra. Kapranov studied this algebra when $\mathcal{A}$ is the category of coherent sheaves over a smooth projective curve defined over a finite field, and observed analogies with quantum affine algebras. We recover here in an elementary way his results in the case when the curve is the projective line.

Résumé

A toute catégorie abélienne $\mathcal{A}$ de dimension homologique égale à un vérifiant certaines conditions de finitude, on peut associer une algèbre appelée l'algèbre de Hall. Kapranov a étudié cette algèbre lorsque $\mathcal{A}$ est la catégorie des faisceaux cohérents sur une courbe projective lisse définie sur un corps fini et a observé des analogies entre l'algèbre de Hall et les algèbres affines quantiques. Nous redémontrons de manière élémentaire ses résultats dans le cas où la courbe est la droite projective.

Introduction

[illegible]

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Keywords: Hall algebra, vector bundle on a curve, quantum affine algebra.

n a d a L n n o a y a
o o X n o a b a a a b a o $\mathcal{A}$ a d o
o b o n a b n a a a b a an n a a o n
o an a n a b a 7 n a n X o n $\mathbf{P}^1(\mathbf{F}_q)$ a an
d d d n a n n an b n a a n b a b a o
a a b a an a an o a o n an a n a b a $U_q(\widehat{\mathfrak{sl}}_2)$
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b o by B_1 an $H(\mathcal{A}_{\mathrm{tor}})$ an a o o a y n
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a o a n o nan a o o n n d
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1 Hall alge[redacted]

1.1 Initial da[redacted]

L k b a a an a b an a o y $\mathcal{A}$ a o b k n a o o
o n $\mathcal{A}$ a n o k o a o n o
b n a k b n a o a o n n o n o n $\mathcal{A}$ an b n n $\mathcal{A}$ o
n o a n o n o o §7 n° a n a o a a y k o

... k ... $\mathcal{A}$ ...

... $\mathrm{Iso}(\mathcal{A})$

... V W ... $\mathcal{A}$ ... k ... $\mathrm{Hom}_{\mathcal{A}}(V,W)$ and $\mathrm{Ext}^1_{\mathcal{A}}(V,W)$ ...

... $\mathrm{Ext}^2_{\mathcal{A}}(V,W)$ ... V W ... $\mathcal{A}$

... V ... $\mathcal{A}$ ... by $[V] \in \mathrm{Iso}(\mathcal{A})$ and ... by $[0]$ ... $M(\alpha)$ ... $\alpha \in \mathrm{Iso}(\mathcal{A})$ ... $\mathcal{A}$ ...

... V ... $\mathcal{A}$ ... $W_1 \oplus \cdots \oplus W_\ell$ ... W_i and ... $\mathcal{A}$ ... k ... $\mathrm{Hom}_{\mathcal{A}}(-,-)$ and $\mathrm{Ext}^1_{\mathcal{A}}(-,-)$

Grothendieck group ... $\mathcal{A}$ by ... $K(\mathcal{A})$ ... by ... $d(\alpha)$ ... $\alpha \in \mathrm{Iso}(\mathcal{A})$ ... $d(\beta) = d(\alpha) + d(\gamma)$ ...

$$0 \longrightarrow M(\gamma) \longrightarrow M(\beta) \longrightarrow M(\alpha) \longrightarrow 0.$$

V ... $\mathcal{A}$ ... $d(V)$ ... $d([V])$ ... $\langle \cdot,\cdot \rangle$ on $K(\mathcal{A})$ ... *Euler form* ... V W ... $\mathcal{A}$

$$\langle d(V), d(W) \rangle = \dim \mathrm{Hom}_{\mathcal{A}}(V,W) - \dim \mathrm{Ext}^1_{\mathcal{A}}(V,W).$$

... $\mathcal{A}$ ...

... $\mathcal{A}$ ...

$\mathcal{A}$ ... $K(\mathcal{A})$ ... $d(\alpha)$ ... $\alpha \in \mathrm{Iso}(\mathcal{A})$ ...

1.2 Hall numbers

... k ... $\mathbf{F}_q$ ... q ... $\alpha \in \mathrm{Iso}(\mathcal{A})$ ... $\mathrm{Aut}_{\mathcal{A}}(M(\alpha))$ by g_α

... $\alpha, \beta, \gamma \in \mathrm{Iso}(\mathcal{A})$ ... by $\phi^\beta_{\alpha\gamma}$ ... $X \subseteq M(\beta)$ ... $X \in \gamma$ and $M(\beta)/X \in \alpha$ ... $S(\alpha,\beta,\gamma)$ ...

$$(f,g) \in \mathrm{Hom}_{\mathcal{A}}(M(\gamma), M(\beta)) \times \mathrm{Hom}_{\mathcal{A}}(M(\beta), M(\alpha))$$

$$0 \longrightarrow M(\gamma) \xrightarrow{f} M(\beta) \xrightarrow{g} M(\alpha) \longrightarrow 0$$

… $\mathrm{Aut}_{\mathcal{A}}(M(\alpha)) \times \mathrm{Aut}_{\mathcal{A}}(M(\gamma))$ … $S(\alpha,\beta,\gamma)$ … $\phi^{\beta}_{\alpha\gamma}$ by … $S(\alpha,\beta,\gamma)/(\mathrm{Aut}_{\mathcal{A}}(M(\alpha)) \times \mathrm{Aut}_{\mathcal{A}}(M(\gamma)))$ … $\phi^{\beta}_{\alpha\gamma}$ … *Hall number* … $d(\beta) \neq d(\alpha) + d(\gamma)$ …

Proposition 1 *If $\alpha, \beta, \gamma, \delta \in \mathrm{Iso}(\mathcal{A})$ are isomorphism classes, then*

(i) there are only finitely many isomorphism classes λ such that $\phi^{\lambda}_{\alpha\gamma} \neq 0$;

(ii) if (H4) holds, there are only finitely many pairs $(\rho, \sigma) \in \mathrm{Iso}(\mathcal{A})^2$ such that $\phi^{\beta}_{\rho\sigma} \neq 0$;

(iii) $\phi^{\beta}_{\alpha[0]} = \delta_{\alpha\beta}$ and $\phi^{\beta}_{[0]\gamma} = \delta_{\beta\gamma}$ (Kronecker symbols);

(iv) $\displaystyle\sum_{\lambda \in \mathrm{Iso}(\mathcal{A})} \phi^{\lambda}_{\alpha\beta}\phi^{\delta}_{\lambda\gamma} = \sum_{\lambda \in \mathrm{Iso}(\mathcal{A})} \phi^{\delta}_{\alpha\lambda}\phi^{\lambda}_{\beta\gamma}$;

(v) the number $q^{\dim \mathrm{Hom}_{\mathcal{A}}(M(\alpha),M(\gamma))}\, \phi^{\beta}_{\alpha\gamma}\, g_\alpha g_\gamma / g_\beta$ is an integer;

(vi) $\displaystyle\sum_{\lambda \in \mathrm{Iso}(\mathcal{A})} \phi^{\lambda}_{\alpha\gamma}\, g_\alpha g_\gamma / g_\lambda = q^{-\langle d(\alpha), d(\gamma)\rangle}$;

(vii) if $M(\alpha)$ and $M(\gamma)$ are indecomposable objects and $M(\beta)$ is a decomposable object, then $q-1$ divides $\phi^{\beta}_{\alpha\gamma} - \phi^{\beta}_{\gamma\alpha}$;

(viii) the following formula holds:

$$g_\alpha g_\beta g_\gamma g_\delta \sum_{\lambda \in \mathrm{Iso}(\mathcal{A})} \phi^{\lambda}_{\alpha\beta}\, \phi^{\lambda}_{\gamma\delta} / g_\lambda = \sum_{\rho,\sigma,\tau,\upsilon \in \mathrm{Iso}(\mathcal{A})} q^{-\langle d(\rho), d(\upsilon)\rangle}\, \phi^{\alpha}_{\rho\sigma}\, \phi^{\beta}_{\tau\upsilon}\, \phi^{\gamma}_{\rho\tau}\, \phi^{\delta}_{\sigma\upsilon}\, g_\rho g_\sigma g_\tau g_\upsilon .$$

[illegible]

Proof. … $\mathrm{Ext}^1_{\mathcal{A}}(M(\alpha), M(\gamma))$ … $d : \mathrm{Iso}(\mathcal{A}) \to K(\mathcal{A})$ … [illegible] … 8 … □

1.3 The Hall algebra and the Ringel-Green bialgebra

… $\widetilde{\mathbf{Z}} = \mathbf{Z}[v, v^{-1}]/(v^2 - q)$ a … $H(\mathcal{A})$ … $\widetilde{\mathbf{Z}}$ … α … α … $\mathrm{Iso}(\mathcal{A})$ …

$$\alpha \cdot \gamma = \sum_{\beta \in \mathrm{Iso}(\mathcal{A})} \phi^{\beta}_{\alpha\gamma}\, \beta$$

… $H(\mathcal{A})$ … $\widetilde{\mathbf{Z}}$ … [0] … *Hall algebra* … $\mathcal{A}$ … $K(\mathcal{A})$ … α … $d(\alpha)$ … $K(\mathcal{A})$ … $*$ … $H(\mathcal{A})$ …

$$\alpha * \gamma = v^{\langle d(\alpha), d(\gamma)\rangle}\, \alpha \cdot \gamma.$$

… $\widetilde{\mathbf{Z}}$ … *Ringel algebra.* … $\Delta : H(\mathcal{A}) \to H(\mathcal{A}) \otimes_{\widetilde{\mathbf{Z}}} H(\mathcal{A})$ … $\varepsilon : H(\mathcal{A}) \to \widetilde{\mathbf{Z}}$ …

$$\Delta(\beta) = \sum_{\alpha,\gamma \in \mathrm{Iso}(\mathcal{A})} v^{\langle d(\alpha), d(\gamma)\rangle}\, \frac{g_\alpha g_\gamma}{g_\beta}\, \phi^{\beta}_{\alpha\gamma}\, (\alpha \otimes \gamma) \qquad \text{and} \qquad \varepsilon(\beta) = \delta_{\beta[0]},$$

… $\beta \in \mathrm{Iso}(\mathcal{A})$ … $H(\mathcal{A})$ … $\widetilde{\mathbf{Z}}$ … Δ … $H(\mathcal{A}) \otimes_{\widetilde{\mathbf{Z}}} H(\mathcal{A})$ …

$$(\alpha \otimes \beta) * (\gamma \otimes \delta) = v^{\langle d(\beta), d(\gamma)\rangle + \langle d(\gamma), d(\beta)\rangle}\, (\alpha * \gamma) \otimes (\beta * \delta),$$

… $\alpha, \beta, \gamma, \delta \in \mathrm{Iso}(\mathcal{A})$ … $*$ … Δ … ε … $\widetilde{\mathbf{Z}}$ … $H(\mathcal{A})$ … *twisted Ringel-Green bialgebra*

2 Coherent sheaves over the projective line

… k … $\mathcal{A}$ … $\mathbf{P}^1(k)$ … $\mathcal{A}$ … $H(\mathcal{A})$ … k …

2.1 Generalities on coherent sheaves on $\mathbf{P}^1(k)$

… $(t : u)$ … $\mathbf{P}^1(k)$ …

$$U' = \{(t : u) \mid t \neq 0\} \quad \text{and} \quad U'' = \{(t : u) \mid u \neq 0\}$$

$\mathbf{P}^1(k)$ an a $z = u/t$ an $z^{-1} = t/u$ n na n U' an U'' y n $k[z]$ an $k[z^{-1}]$ a n a n n U' an U''

n n n A a a n M an A an z an n A n M_z n a A a n M by n n z An ana n a n

An y $\mathcal{A}$ *coherent sheaves* n $\mathbf{P}^1(k)$ a (M', M'', φ) M' a n y n a $k[z]$ M'' a n y n a $k[z^{-1}]$ an $\varphi : M'_z \to M''_{z^{-1}}$ an $k[z, z^{-1}]$ A n $\mathcal{A}$ n a (M', M'', φ) n a (N', N'', ψ) a a a (f', f'') $f' : M' \to N'$ a $k[z]$ n a a an $f'' : M'' \to N''$ a $k[z^{-1}]$ n a a a $\psi \circ f'_z = f''_{z^{-1}} \circ \varphi$ On a n n an ay n n an a n n $\mathcal{A}$ a $\mathcal{A}$ an a an k n a a y n n $\mathcal{A}$ a n an a n n a an n n an n S n [redacted]

y n a a a n y a y a an y n n a n a a n a $\mathcal{F} = (M', M'', \varphi)$ a *locally free* M' an M'' a $k[z]$ an $k[z^{-1}]$ y a $\mathcal{F}$ a a *torsion sheaf* M' a n $k[z]$ a n a a n M'_z $M''_{z^{-1}}$ M'' a y $\mathcal{A}$ n n a a y a n by $\mathcal{A}_{\mathrm{lf}}$ a y $\mathcal{A}$ n n a n a n by $\mathcal{A}_{\mathrm{tor}}$

y n n a n y n a a n L a n n n

A *closed point* x $\mathbf{P}^1(k)$ an n yn a $P \in k[T, U]$ P na yn a T y U n n n a n y $\infty \in U''$ y n $0 \in U'$ O x an a n U' yn a $P(1, z) \in k[z]$ an a n U'' yn a $P(z^{-1}, 1) \in k[z^{-1}]$ n any a x n P a n n a a an $\deg x$ x n a P

x n a n n U' y U'' n n a n $\mathcal{O}_{\mathbf{P}^1(k),x}$ a na n n a n a x a a a n $k[z]$ a a n a by $P(1, z)$ y a a a n $k[z^{-1}]$ a a n a by $P(z^{-1}, 1)$ x n U' an U'' n n n $\mathcal{O}_{\mathbf{P}^1(k),x}$ y n a n a π_x a a a $\mathcal{O}_{\mathbf{P}^1(k),x}$ n an n by $\mathcal{O}_{\mathbf{P}^1(k),x}$ modf a y $\mathcal{O}_{\mathbf{P}^1(k),x}$ n n a a a n y n a an ann a by π_x

L $\mathcal{F} = (M', M'', \varphi)$ a n a an x a n n by yn a P an n n U' y U'' *stalk* $\mathcal{F}_x$ a $\mathcal{F}$ a x $\mathcal{O}_{\mathbf{P}^1(k),x}$ a n by a n $k[z]$ M' a a n a by $P(1, z)$ y $k[z^{-1}]$ M'' a a n a by $P(z^{-1}, 1)$ x n U' an U'' n n n a a n On ay a n x n *support* $\mathcal{F}$ a $\mathcal{F}_x$ n an a a n n a a ay a n n n a x n by $\mathcal{A}_{\{x\}}$ a y $\mathcal{A}$ n n a n a n d d n $\{x\}$

Proposition 2 *(i) The category $\mathcal{A}$ is k-linear, abelian, and satisfies Conditions (H1)–(H3) of Section 1.1. The subcategories $\mathcal{A}_{\mathrm{lf}}$, $\mathcal{A}_{\mathrm{tor}}$, and $\mathcal{A}_{\{x\}}$ of $\mathcal{A}$ are closed under extensions. The categories $\mathcal{A}_{\mathrm{tor}}$ and $\mathcal{A}_{\{x\}}$ are k-linear, abelian, and satisfy Conditions (H1)–(H4) of Section 1.1.*

(ii) If $\mathcal{F}$ is a locally free sheaf and $\mathcal{G}$ is a torsion sheaf, then $\mathrm{Hom}_{\mathcal{A}}(\mathcal{G},\mathcal{F}) = \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F},\mathcal{G}) = 0$.

(iii) If $\mathcal{F}$ and $\mathcal{G}$ are torsion sheaves with disjoint supports, then $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G}) = \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F},\mathcal{G}) = 0$.

(iv) Every coherent sheaf $\mathcal{F}$ can be written as a direct sum of a torsion sheaf and a locally free sheaf. Every torsion sheaf can be written as a finite direct sum of sheaves whose supports are closed points.

(v) The category $\mathcal{A}_{\mathrm{tor}}$ is the direct sum of the subcategories $\mathcal{A}_{\{x\}}$, where x runs over the set of all closed points of $\mathbf{P}^1(k)$.

(vi) Going from a sheaf to its stalk at a closed point x gives an isomorphism of categories from $\mathcal{A}_{\{x\}}$ to $\mathcal{O}_{\mathbf{P}^1(k),x}$-modf.

Proof. a n ab $\mathcal{A}$ a an a a b a y a n n a $\mathcal{A}_{\mathrm{lf}}$ $\mathcal{A}_{\mathrm{tor}}$ an $\mathcal{A}_{\{x\}}$ a n n n n $\mathcal{A}$ S n b b an n n a a n a $\mathcal{A}_{\mathrm{tor}}$ an ab an b a y $\mathcal{A}$ an Ext a b n b n $\mathcal{A}_{\mathrm{tor}}$ a a n n $\mathcal{A}$ n $\mathcal{A}_{\mathrm{tor}}$ $\mathcal{A}_{\mathrm{tor}}$ a Con n na y a a n y n a n n a a a n a an a $\mathcal{A}_{\mathrm{tor}}$ a Con n A a a n a $\mathcal{A}_{\{x\}}$ an ab an b a y $\mathcal{A}$ a Con n

n a a y a $\mathcal{F}$ an a n a $\mathcal{G}$ an n $\mathrm{Hom}_{\mathcal{A}}(\mathcal{G},\mathcal{F})$ a n n n a $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F},\mathcal{G})$ S an n b n

an a n an a n y n a n n a a n $k[z]$ an $k[z^{-1}]$

L $\mathcal{F}$ b a n a On an n n an ay n b a $\mathrm{tor}(\mathcal{F})$ $\mathcal{F}$ an n a $\mathcal{F}/\mathrm{tor}(\mathcal{F})$ a y y A n $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F}/\mathrm{tor}(\mathcal{F}), \mathrm{tor}(\mathcal{F}))$ an a n

$$0 \longrightarrow \mathrm{tor}(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathrm{tor}(\mathcal{F}) \longrightarrow 0$$

an n n $\mathcal{F} \simeq \mathrm{tor}(\mathcal{F}) \oplus (\mathcal{F}/\mathrm{tor}(\mathcal{F}))$ a n a n n a n a y n n y n a n n a a n $k[z]$ an $k[z^{-1}]$

A n an

□

2.2 Indecomposable coherent sheaves over $\mathbf{P}^1(k)$

y A on n d ab b $\mathcal{A}$ a a y a n d n $\{x\}$

any $n \in \mathbf{Z}$ n a a y n a (M', M'', φ) by n $M' = k[z]$ $M'' = k[z^{-1}]$ and $\varphi : k[z, z^{-1}] \to k[z, z^{-1}]$ b a n by z^{-n} A a a b d n by $\mathcal{O}(n)$ A n d a a any a y n a a d $\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_r)$ n $n_1 \leq \cdots \leq n_r$ n y d n d n a $\mathcal{O}(n)$ a n d ab a y n a

any $m, n \in \mathbf{Z}$ a n yn a $F \in k[T, U]$ d $n - m$ na a y a $\mathrm{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(n))$ n a a F a a (f', f'') $f' : k[z] \to k[z]$ a n by $F(1, z)$ and $f'' : k[z^{-1}] \to k[z^{-1}]$ a n by $F(z^{-1}, 1)$

n a a n an d b n yn a $P \in k[T, U]$ d d and an n $r \geq 1$ r yn a P^r d n a $\mathcal{O}(-rd)$ $\mathcal{O}(0)$ n n a (M', M'', φ) $M' = k[z]/(P(1, z)^r)$ $M'' = k[z^{-1}]/(P(z^{-1}, 1)^r)$ and φ n d d by d n y $k[z, z^{-1}]$ Ca n x d n d n d by P d n n a by $\mathcal{O}_{r[x]}$ $\{x\}$ a a n a n d $\mathcal{O}_{\mathbf{P}^1(k),x}$ d $\mathcal{O}_{\mathbf{P}^1(k),x}/(\pi_x^r)$ Con n y a $\mathcal{O}_{r[x]}$ a n d ab b a y $\mathcal{A}_{\{x\}}$

d a n a n n y n n d n a n A d n and a d n y a a n a n n a n n n n a n n y n y any n n $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ $\lambda_i = 0$ i b n n λ a n $\ell \geq 0$ a $\lambda_{\ell+1} = 0$ and $|\lambda|$ λ n n λ_i W a $n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$ a n y a n a n n n a a n r n a a a 1 d n d by (1^r) a n n n a a r d n d by (r) na y n a d n x and a a n $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ n ℓ d n n a

$$\mathcal{O}_{\lambda[x]} = \mathcal{O}_{\lambda_1[x]} \oplus \cdots \oplus \mathcal{O}_{\lambda_\ell[x]}.$$

n an $\mathcal{O}_{(1^r)[x]} = \left(\mathcal{O}_{[x]}\right)^{\oplus r}$ and $\mathcal{O}_{(r)[x]} = \mathcal{O}_{r[x]}$

2.3 The Grothendieck group and the Euler form

W d n an and d an n d ab a by

$$\mathrm{rk}\, \mathcal{O}(n) = 1, \quad \deg \mathcal{O}(n) = n, \quad \mathrm{rk}\, \mathcal{O}_{r[x]} = 0, \quad \text{and} \quad \deg \mathcal{O}_{r[x]} = r \deg x.$$

S n y n a an b n n an n a y n ay a a n d ab a ay n a d dd y n n an and d a b a y n a n n a b a y a an and d a

a n d n $K(\mathcal{A})$ d nn a ab an
$K(\mathcal{A}) \to \mathbf{Z}^2$ by

$$d(\mathcal{F}) \mapsto (\operatorname{rk}\mathcal{F}, \deg\mathcal{F}).$$

Proposition 3 *The Euler form on $K(\mathcal{A})$ is given for all coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ by*

$$\langle d(\mathcal{F}), d(\mathcal{G})\rangle = \operatorname{rk}\mathcal{F}\ \operatorname{rk}\mathcal{G} + \operatorname{rk}\mathcal{F}\ \deg\mathcal{G} - \deg\mathcal{F}\ \operatorname{rk}\mathcal{G}.$$

Proof. n ann a n an an a d d d
a y n an 7 n b a n

$$\begin{aligned}
\langle d(\mathcal{O}(m)), d(\mathcal{O}(n))\rangle &= \dim \operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(n)) - \dim \operatorname{Ext}^1_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(n)) \\
&= \dim \operatorname{Hom}_{\mathcal{A}}(\mathcal{O}_{\mathbf{P}^1(k)}, \mathcal{O}(n-m)) - \dim \operatorname{Ext}^1_{\mathcal{A}}(\mathcal{O}_{\mathbf{P}^1(k)}, \mathcal{O}(n-m)) \\
&= \dim H^0(\mathbf{P}^1(k), \mathcal{O}(n-m)) - \dim H^1(\mathbf{P}^1(k), \mathcal{O}(n-m)) \\
&= 1 + \deg(\mathcal{O}(n-m)) \\
&= 1 + n - m \\
&= \operatorname{rk}\mathcal{O}(m)\ \operatorname{rk}\mathcal{O}(n)\ +\ \operatorname{rk}\mathcal{O}(m)\ \deg\mathcal{O}(n)\ -\ \deg\mathcal{O}(m)\ \operatorname{rk}\mathcal{O}(n)
\end{aligned}$$

a $m, n \in \mathbf{Z}$ n $\mathcal{F}$ an $\mathcal{G}$ a y a an S n
a a n a $K(\mathcal{A})$ n a by b a dd y b
d □

2.4 Some extensions of sheaves

n n y n a an n n n a y $\mathcal{A}$ an
d d n a a d d n n n S n

Lemma 4 *For all $m, n \in \mathbf{Z}$,*

(i) any non-zero element in $\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(n))$ is a monomorphism;

(ii) as a k-algebra, $\operatorname{End}_{\mathcal{A}}(\mathcal{O}(n)) \simeq k$;

(iii) the k-vector space $\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(n))$ has dimension $\max(0, n-m+1)$;

(iv) for any closed point x and any partition λ, the k-vector space $\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(n), \mathcal{O}_{\lambda[x]})$ has dimension $|\lambda| \deg x$.

O d b n n b n n ab a y a
n a a d n

$$0 \longrightarrow \mathcal{O}(m) \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{O}(n) \longrightarrow 0,$$

n a $\mathcal{F}$ n a y a y an an a n p an
$q \in \mathbf{Z}$ a $\mathcal{F}$ a $\mathcal{O}(p) \oplus \mathcal{O}(q)$

Proposition 5 *Let m, n, p, q be integers, and consider a sequence of the form*

$$0 \longrightarrow \mathcal{O}(m) \xrightarrow{f} \mathcal{O}(p) \oplus \mathcal{O}(q) \xrightarrow{g} \mathcal{O}(n) \longrightarrow 0.$$

Let

$$h \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(p)), \qquad j \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}(p), \mathcal{O}(n)),$$
$$i \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(q)), \qquad \ell \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}(q), \mathcal{O}(n)),$$

be defined by $f = h \oplus i$ and $g = j \oplus \ell$, and call H, I, J, L the homogeneous polynomials in $k[T,U]$ representing h, i, j, ℓ, respectively. Then the sequence is a non-split short exact sequence if and only if the following three conditions are satisfied:

(a) $m < \min(p,q)$, $\max(p,q) < n$, *and* $p + q = m + n$.

(b) J *and* L *are coprime polynomials.*

(c) *There is a non-zero scalar E such that $H = EL$ and $I = -EJ$.*

Proof. [illegible] Condition [illegible] h [illegible] i [illegible] f [illegible] h, i, j and ℓ [illegible] $m < \min(p,q)$ and $\max(p,q) < n$ [illegible] $m + n = p + q$ [illegible] Condition [illegible]

[illegible] Condition [illegible] $k[T,U]$ [illegible] D [illegible] J and L [illegible] D [illegible] $d \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}(n - \deg D), \mathcal{O}(n))$ [illegible]

$$\begin{array}{ccc} \mathcal{O}(p) \oplus \mathcal{O}(q) & \xrightarrow{\quad g \quad} & \mathcal{O}(n) \\ & \searrow \qquad \nearrow_{d} & \\ & \mathcal{O}(n - \deg D) & \end{array}.$$

Since g [illegible] d [illegible] d [illegible] $\deg D = 0$ [illegible] J and L [illegible] Condition [illegible] $g \circ f = 0$ [illegible] $HJ + IL = 0$. Condition [illegible] $E \in k[T,U]$ [illegible] $H = EL$ and $I = -EJ$. Since

$$2 \deg E = \deg H + \deg I - \deg L - \deg J = (p - m) + (q - m) - (n - q) - (n - p) = 0,$$

E [illegible] Condition [illegible]

[illegible] U' [illegible]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(m)(U') & \xrightarrow{f_{U'}=h_{U'}\oplus i_{U'}} & \mathcal{O}(p)(U')\oplus\mathcal{O}(q)(U') & \xrightarrow{g_{U'}=j_{U'}\oplus \ell_{U'}} & \mathcal{O}(n)(U') & \longrightarrow & 0, \\ & & \| & & \| & & \| & & \\ & & k[z] & & k[z]\oplus k[z] & & k[z] & & \end{array}$$

[illegible] $h_{U'}$, $i_{U'}, \ldots$ [illegible] $H(1,z)$, $I(1,z), \ldots$ [illegible] $J(1,z)$ [illegible] $L(1,z)$ [illegible] $k[z]$ [illegible] $g_{U'}$ [illegible] $\ker g_{U'} = \operatorname{im} f_{U'}$ [illegible] U' [illegible] U'' [illegible] □

Corollary 6 *If $m, n \in \mathbf{Z}$ are integers satisfying $n \leq m+1$, then the extension group $\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{O}(n), \mathcal{O}(m))$ vanishes.*

[illegible]

Lemma 7 *Let x be a closed point of $\mathbf{P}^1(k)$, corresponding to an irreducible homogeneous polynomial $P \in k[T,U]$. Let $m, n \in \mathbf{Z}$, let $F \in k[T,U]$ be a non-zero homogeneous polynomial of degree $n-m$, and let $f \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(n))$ be the morphism defined by F. If the support of the cokernel of f is included in $\{x\}$, then there exists an integer $r \geq 1$ such that $F = P^r$, up to a non-zero scalar, and one has $\operatorname{coker} f \simeq \mathcal{O}_{r[x]}$.*

Proof. [illegible] F [illegible] $\widetilde{f} : \mathcal{O}(m-n) \to \mathcal{O}(0)$ [illegible] $F = P_1^{r_1} \cdots P_t^{r_t}$ [illegible] $r_1, \ldots, r_t$ [illegible] $P_1, \ldots, P_t$ [illegible] $1 \leq i \leq t$ [illegible] x_i [illegible] $\mathbf{P}^1(k)$ [illegible] P_i [illegible] $P_i^{r_i}$ [illegible] $\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(-r_i \deg P_i), \mathcal{O}(0))$ [illegible] $\mathcal{O}_{r_i[x_i]}$ [illegible] $g_i : \mathcal{O}(0) \to \mathcal{O}_{r_i[x_i]}$ [illegible]

$$0 \longrightarrow \mathcal{O}(m-n) \xrightarrow{\widetilde{f}} \mathcal{O}(0) \xrightarrow{\oplus g_i} \oplus_{i=1}^t \mathcal{O}_{r_i[x_i]} \longrightarrow 0$$

[illegible] U' [illegible] U'' [illegible] $\mathcal{O}(n)$ [illegible]

$$0 \longrightarrow \mathcal{O}(m) \xrightarrow{f} \mathcal{O}(n) \longrightarrow \oplus_{i=1}^t \mathcal{O}_{r_i[x_i]} \longrightarrow 0.$$

[illegible] f [illegible] $\mathcal{O}_{r_1[x_1]} \oplus \cdots \oplus \mathcal{O}_{r_t[x_t]}$ [illegible] $\operatorname{coker} f$ [illegible] $\{x\}$ [illegible] $t = 1$ [illegible] $P_1 = P$ [illegible]
□

Proposition 8 *Given a closed point x and an integer $r \geq 1$, let*

$$0 \longrightarrow \mathcal{O}(m) \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{O}_{(1^r)[x]} \longrightarrow 0$$

be a non-split short exact sequence of coherent sheaves. Then the middle term $\mathcal{F}$ is isomorphic to $\mathcal{O}_{(1^{r-1})[x]} \oplus \mathcal{O}(m + \deg x)$. If we write $f = h \oplus i$ in this decomposition, then $h = 0$ and $\operatorname{coker} i \simeq \mathcal{O}_{[x]}$.

Proof. an $\mathcal{F}$ a a n a $\mathcal{F}_0$ an a a y a $\mathcal{F}_1$ an 1 a f an g a $h \oplus i$ an $j \oplus \ell$ n n $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ h anno n i anno i n L a an a j n $\mathcal{F}_0$ a $\mathcal{O}_{(1^r)[x]}$ n n n a $\mathcal{O}_{(1^r)[x]}$ n n a y $\mathcal{O}_{\mathbf{P}^1(k),x}$ $(\mathcal{O}_{\mathbf{P}^1(k),x}/(\pi_x))^{\oplus r}$ n a a n n r $\mathcal{O}_{\mathbf{P}^1(k),x}/(\pi_x)$ a $\mathcal{F}_0$ $\mathcal{O}_{(1^s)[x]}$ $s \leq r$ n a ay an n L a 7 a a ℓ 0 $\mathcal{O}_{[x]}$

n n n j n an a $s = r$ y g n a $s = r - 1$ a $\operatorname{im} \ell \simeq \mathcal{O}_{[x]}$ an a $\mathcal{O}_{(1^r)[x]} = \operatorname{im} j \oplus \operatorname{im} \ell$ a y $g \circ f = 0$ n n a $j \circ h = 0$ an $\ell \circ i = 0$ S n j n $h = 0$ an $\operatorname{coker} i \simeq \mathcal{O}_{[x]}$ na y

$$\deg \mathcal{F}_1 = \deg \mathcal{O}(m) + \deg \mathcal{O}_{(1^r)[x]} - \deg \mathcal{F}_0 = m + r \deg x - s \deg x = m + \deg x,$$

a $\mathcal{F}_1 \simeq \mathcal{O}(m + \deg x)$ □

3 The Hall algebra of $\mathrm{Coh}(\mathbf{P}^1(\mathbf{F}_q))$

n n k n $\mathbf{F}_q$ q n an $\mathcal{A}$ an a y n a $\mathbf{P}^1(k)$ n n n a a $H(\mathcal{A})$ y n n S n a n n a n n C a an n n a y n a a $H(\mathcal{A}_{\{x\}})$ an $H(\mathcal{A}_{\text{tor}})$ a a a $H(\mathcal{A})$ a n a y n a an a a B $H(\mathcal{A})$ n n S n a an a n a a $U_q(\widehat{\mathfrak{sl}_2})$

3.1 Some Hall numbers for $\mathcal{A}$

a n n a y na a a

Lemma 9 *The number $\varphi(a, b)$ of pairs (J, L) consisting of coprime homogeneous polynomials in $\mathbf{F}_q[T, U]$ of degrees a and b, respectively, is given by*

$$\varphi(a,b) = \begin{cases} (q-1)(q^{a+b+1} - 1) & \text{if } a = 0 \text{ or } b = 0, \\ (q-1)(q^2 - 1)q^{a+b-1} & \text{if } a \geq 1 \text{ and } b \geq 1. \end{cases}$$

Proof. L S b a $(J, L) \in \mathbf{F}_q[T, U]$ n n n n n y n a a an b y a na y S $(q^{a+1}-1)(q^{b+1}-1)$ On an a n n S by a n a D J an L a d $d \leq \min(a, b)$ a $(q^{d+1}-1)/(q-1)$ b D a n n a a an a n

$$(q^{a+1}-1)(q^{b+1}-1) = \sum_{d=0}^{\min(a,b)} \frac{q^{d+1}-1}{q-1}\, \varphi(a-d, b-d).$$

a n by n n n $\min(a, b)$ $\square$

any n x $\mathbf{P}^1(\mathbf{F}_q)$ $q_x = q^{\deg x}$ b na a n $\mathcal{O}_{\mathbf{P}^1(\mathbf{F}_q),x}$ n a n an a a a n b a by $\lfloor a \rfloor$ n a n b a y $\mathcal{A}$

Theorem 10 *In the Hall algebra $H(\mathcal{A})$, one has the following relations:*

(i) $[\mathcal{O}(m)^{\oplus a}][\mathcal{O}(m)^{\oplus b}] = \left(\prod_{c=1}^{a} \frac{q^{b+c}-1}{q^c-1}\right)[\mathcal{O}(m)^{\oplus(a+b)}]$ *for every $m \in \mathbf{Z}$ and $a, b \in \mathbf{N}$.*

(ii) If $\mathcal{F} = \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_r)$ is a locally free sheaf, if $m \in \mathbf{Z}$ is strictly greater than $n_1, \ldots, n_r$, and if a is a non-negative integer, then $[\mathcal{F}][\mathcal{O}(m)^{\oplus a}] = [\mathcal{F} \oplus \mathcal{O}(m)^{\oplus a}]$.

(iii) If $m < n$, then

$$[\mathcal{O}(n)][\mathcal{O}(m)] = q^{n-m+1}[\mathcal{O}(m) \oplus \mathcal{O}(n)] + \sum_{a=1}^{\lfloor (n-m)/2 \rfloor} (q^2-1)\, q^{n-m-1}\, [\mathcal{O}(m+a) \oplus \mathcal{O}(n-a)].$$

(iv) If $\mathcal{F}$ is a locally free sheaf and $\mathcal{G}$ is a torsion sheaf, then $[\mathcal{F}][\mathcal{G}] = [\mathcal{F} \oplus \mathcal{G}]$.

(v) If $\mathcal{F}$ and $\mathcal{G}$ are torsion sheaves with disjoint supports, then $[\mathcal{F}][\mathcal{G}] = [\mathcal{F} \oplus \mathcal{G}]$.

(vi) If x is a closed point, r is a positive integer, and $n \in \mathbf{Z}$, then

$$[\mathcal{O}_{(1^r)[x]}][\mathcal{O}(n)] = [\mathcal{O}(n+\deg x) \oplus \mathcal{O}_{(1^{r-1})[x]}] + q_x^r\, [\mathcal{O}(n) \oplus \mathcal{O}_{(1^r)[x]}].$$

Proof. S n n n $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(m))$ an by C a y any a n

$$0 \longrightarrow \mathcal{O}(m)^{\oplus b} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}(m)^{\oplus a} \longrightarrow 0$$

n a y an $[\mathcal{O}(m)^{\oplus a}][\mathcal{O}(m)^{\oplus b}]$ n $H(\mathcal{A})$ a a a $[\mathcal{O}(m)^{\oplus(a+b)}]$ a n n a n b S n $\mathrm{End}_{\mathcal{A}}(\mathcal{O}(m)) \simeq \mathbf{F}_q$ by L a a n b a n b b a n n b n a a n n $a+b$ $\mathbf{F}_q$ na y $\left(\prod_{c=1}^{a} \frac{q^{b+c}-1}{q^c-1}\right)$

y C... $\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{O}(n_i), \mathcal{O}(m))$ ... any ...

$$0 \longrightarrow \mathcal{O}(m)^{\oplus a} \xrightarrow{f} \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow 0$$

... $[\mathcal{F}][\mathcal{O}(m)^{\oplus a}]$ ... $H(\mathcal{A})$ ... $[\mathcal{F} \oplus \mathcal{O}(m)^{\oplus a}]$. L... $\mathcal{G} = \mathcal{F} \oplus \mathcal{O}(m)^{\oplus a}$ ... $f = h \oplus i$ ... $h = 0$ ... $\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(n_i))$ ... i ... $f : \mathcal{O}(m)^{\oplus a} \to \mathcal{G}$ ... $|\operatorname{Aut}_{\mathcal{A}}(\mathcal{O}(m)^{\oplus a})|$ ... 1

y ... any ...

$$0 \longrightarrow \mathcal{O}(m) \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{O}(n) \longrightarrow 0$$

... $\mathcal{F} \simeq \mathcal{O}(m) \oplus \mathcal{O}(n)$ ... $1 \leq a \leq \lfloor (n-m)/2 \rfloor$ ... $\mathcal{F} \simeq \mathcal{O}(m+a) \oplus \mathcal{O}(n-a)$

... $f = h \oplus i$ and $g = j \oplus \ell$ ... $h \in \operatorname{End}_{\mathcal{A}}(\mathcal{O}(m))$, $i, j \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(n))$ and $\ell \in \operatorname{End}_{\mathcal{A}}(\mathcal{O}(n))$. S... $\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(n), \mathcal{O}(m)) = 0$ by L... f ... h ... ℓ ... i ... j ... $-\ell \circ i \circ h^{-1}$ ... (f, g) ... $\operatorname{Aut}_{\mathcal{A}}(\mathcal{O}(m)) \times \operatorname{Aut}_{\mathcal{A}}(\mathcal{O}(n)) \times \operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(n))$ and ... $|\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}(m), \mathcal{O}(n))| = q^{n-m+1}$ ... $q^{n-m+1}[\mathcal{O}(m) \oplus \mathcal{O}(n)]$ ...

... $g : \mathcal{F} \to \mathcal{O}(n)$ ... $\ker g \simeq \mathcal{O}(m)$ ... $(q-1)(q^2-1)q^{n-m-1}$ by ... L... 9. S... $|\operatorname{Aut}_{\mathcal{A}}(\mathcal{O}(n))| = q - 1$ ... $\phi^{[\mathcal{F}]}_{[\mathcal{O}(n)],[\mathcal{O}(m)]}$ ... $(q^2-1)q^{n-m-1}$ ... $(q^2-1)\, q^{n-m-1}\, [\mathcal{O}(m+a) \oplus \mathcal{O}(n-a)]$ ...

... $\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ and $\operatorname{Hom}_{\mathcal{A}}(\mathcal{G}, \mathcal{F})$ ...

... L... L... 7 and ... 8 ...

□

Application 11. L... x ... $\mathbf{P}^1(\mathbf{F}_q)$ and $n \in \mathbf{Z}$ ...

$$[\mathcal{O}_{[x]}][\mathcal{O}(n)^{\oplus 2}] = q_x[\mathcal{O}(n) \oplus \mathcal{O}(n + \deg x)] + q_x\left(1 - \frac{1}{q}\right) \sum_{a=1}^{\lfloor (\deg x)/2 \rfloor} [\mathcal{O}(n+a) \oplus \mathcal{O}(n + \deg x - a)] + q_x[\mathcal{O}(n)^{\oplus 2} \oplus \mathcal{O}_{[x]}].$$

... $\deg x \geq 2$ ... $\phi^{[\mathcal{O}(n+a) \oplus \mathcal{O}(n+\deg x - a)]}_{[\mathcal{O}_{[x]}],[\mathcal{O}(n)^{\oplus 2}]}$ ... $q^{\deg x - 1}(q-1)$ ... $1 \leq a \leq \deg x - 1$ ... 8 ...

If $P \in \mathbf{F}_q[T]$ is an irreducible polynomial of degree $d \geq 2$ and if $1 \leq a \leq d-1$, then there are exactly $q^{d-1}(q-1)^2$ quadruples $(H, I, J, L) \in \mathbf{F}_q[T]^4$ consisting of polynomials of degree a, $d-a$, $a-1$, $d-a-1$, respectively, such that $HI - JL = P$.

3.2 The Hall subalgebras $H(\mathcal{A}_{\{x\}})$ and $H(\mathcal{A}_{\text{tor}})$

… $H(\mathcal{A})$ … $[\mathcal{O}_{r[x]}]$ … $[\mathcal{O}_{\lambda[x]}]$ … $[\mathcal{O}_{(1^r)[x]}]$ … x … $\mathbf{P}^1(\mathbf{F}_q)$ … $\mathcal{A}_{\{x\}}$ … $\mathcal{A}$ … k … $H(\mathcal{A}_{\{x\}})$ … $\mathcal{A}_{\{x\}}$ … $\mathcal{A}$ … $H(\mathcal{A}_{\{x\}})$ … $H(\mathcal{A})$ … $\widetilde{\mathbf{Z}}$ … [redacted] … $H(\mathcal{A}_{\{x\}})$ … $K(\mathcal{A}_{\{x\}})$ …

… $\widehat{h}_{r,x} = \sum_{|\lambda|=r}[\mathcal{O}_{\lambda[x]}]$ … r … $\widetilde{\mathbf{Z}} = \mathbf{Z}[v, v^{-1}]/(v^2-q)$ … Λ … $h_r \in \Lambda$ … $e_r \in \Lambda$ … $P_\lambda(t) \in \Lambda[t]$ … $H(\mathcal{A}_{\{x\}})$ …

Proposition 12 …

(i) There is a ring isomorphism $\Psi_x : H(\mathcal{A}_{\{x\}}) \to \Lambda$ that sends the elements $\widehat{h}_{r,x}$, $[\mathcal{O}_{(1^r)[x]}]$, and $[\mathcal{O}_{\lambda[x]}]$ of $H(\mathcal{A}_{\{x\}})$, respectively, to the elements h_r, $q_x^{-r(r-1)/2}e_r$ and $q_x^{-n(\lambda)}P_\lambda(q_x^{-1})$ of Λ, respectively, for any integer $r \geq 1$ and any partition λ.

(ii) The $\widetilde{\mathbf{Z}}$-algebra $H(\mathcal{A}_{\{x\}})$ is a polynomial algebra on the set $\{\widehat{h}_{r,x} \mid r \geq 1\}$, as well as on the set $\{[\mathcal{O}_{(1^r)[x]}] \mid r \geq 1\}$. The family $([\mathcal{O}_{r[x]}])_{r\geq 1}$ consists of algebraically independent elements and generates the $\mathbf{Q}[v]/(v^2-q)$-algebra $H(\mathcal{A}_{\{x\}}) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Proof. … $\mathcal{A}_{\{x\}}$ … $\mathcal{O}_{\mathbf{P}^1(\mathbf{F}_q),x}$ … $H(\mathcal{A}_{\{x\}})$ …

… Λ … $\widetilde{\mathbf{Z}}$ … h_r … e_r … $r \geq 1$ … □

... $H(\mathcal{A}_{\{x\}})[[s]]$ by

$$\widehat{H}_x(s) = 1 + \sum_{r\geq 1} \widehat{h}_{r,x}\, s^r = \sum_{\beta \in \mathrm{Iso}(\mathcal{A}_{\{x\}})} \beta\, s^{\deg \beta / \deg x},$$

$$\widehat{E}_x(s) = 1 + \sum_{r\geq 1} q_x^{r(r-1)/2}\, [\mathcal{O}_{(1^r)[x]}]\, s^r,$$

$$\widehat{Q}_x(s) = 1 + \sum_{r\geq 1} (1 - q_x^{-1})\, v^{r \deg x}\, [\mathcal{O}_{r[x]}]\, s^r.$$

Lemma 13 *(i) The following relations hold in* $H(\mathcal{A}_{\{x\}})[[s]]$*:*

$$\widehat{H}_x(s)\, \widehat{E}_x(-s) = 1 \quad \textit{and} \quad \widehat{Q}_x(s) = \frac{\widehat{H}_x(s\, v^{\deg x})}{\widehat{H}_x(s/v^{\deg x})}.$$

(ii) In $H(\mathcal{A}_{\{x\}})[[s]]$*, one has*

$$\widehat{Q}_x(s) = \sum_{r\geq 0} \left|\mathrm{Aut}_{\mathcal{A}}(\mathcal{O}_{r[x]})\right|\, v^{-r\deg x}\, [\mathcal{O}_{r[x]}]\, s^r.$$

Proof. ... and ... in $\Lambda[[s]]$ by

$$H(s) = 1 + \sum_{r\geq 1} h_r\, s^r \qquad \text{and} \qquad E(s) = 1 + \sum_{r\geq 1} e_r\, s^r.$$

... and ... in $\Lambda[[s]]$

$$H(s)E(-s) = 1 \quad \text{and} \quad 1 + \sum_{r\geq 1} (1 - q_x^{-1})\, (s\, v^{\deg x})^r\, P_{(r)}(q_x^{-1}) = \frac{H(s\, v^{\deg x})}{H(s\, v^{\deg x}/q_x)}.$$

... by Ψ_x ... $\left|\mathrm{Aut}_{\mathcal{A}}(\mathcal{O}_{r[x]})\right| = q_x^r(1 - q_x^{-1})$ ... any $r \geq 1$ ... □

Remark 14. ... $\mathcal{A}_{\{x\}}$ ... $H(\mathcal{A}_{\{x\}})$ ... $K(\mathcal{A}_{\{x\}})$ ... $H(\mathcal{A}_{\{x\}}) \otimes_{\widetilde{\mathbf{Z}}} H(\mathcal{A}_{\{x\}})$ ... $H(\mathcal{A}_{\{x\}})$ ... $\widetilde{\mathbf{Z}}$ ... 8 ... $H(\mathcal{A}_{\{x\}})$ ...
[redacted] Λ ... $\widetilde{\mathbf{Z}}$ ... Ψ_x ... $H(\mathcal{A}_{\{x\}})$ ... $\widehat{h}_{r,x}$ ... Λ ... $\Psi_x(\widehat{h}_{r,x}) = h_r$ ... $(H(\mathcal{A}_{\{x\}}) \otimes_{\widetilde{\mathbf{Z}}} H(\mathcal{A}_{\{x\}}))[[s^{\deg x}]]$

$$\begin{aligned}
\Delta(\widehat{H}_x(s^{\deg x})) &= \sum_{\beta\in \mathrm{Iso}(\mathcal{A}_{\{x\}})} s^{\deg\beta}\, \Delta(\beta)\\
&= \sum_{\alpha,\beta,\gamma\in \mathrm{Iso}(\mathcal{A}_{\{x\}})} s^{\deg\beta}\, \frac{g_\alpha g_\gamma}{g_\beta}\, \phi^{\beta}_{\alpha\gamma}\, (\alpha\otimes\gamma)\\
&= \sum_{\alpha,\gamma\in \mathrm{Iso}(\mathcal{A}_{\{x\}})} \left(\sum_{\beta} \frac{g_\alpha g_\gamma}{g_\beta}\, \phi^{\beta}_{\alpha\gamma}\right) (s^{\deg\alpha}\alpha)\otimes(s^{\deg\gamma}\gamma)\\
&= \sum_{\alpha,\gamma\in \mathrm{Iso}(\mathcal{A}_{\{x\}})} (s^{\deg\alpha}\alpha)\otimes(s^{\deg\gamma}\gamma)\\
&= \widehat{H}_x(s^{\deg x})\otimes \widehat{H}_x(s^{\deg x}).
\end{aligned}$$

[illegible] $\Delta(\widehat{h}_{r,x}) = \sum_{s=0}^{r} \widehat{h}_{s,x}\otimes\widehat{h}_{r-s,x}$ in $H(\mathcal{A}_{\{x\}})\otimes_{\widetilde{\mathbf{Z}}} H(\mathcal{A}_{\{x\}})$ [illegible] h_r [illegible] Λ [illegible]

[illegible] $\mathcal{A}_{\mathrm{tor}}$ [illegible] $\mathcal{A}$ [illegible] $H(\mathcal{A}_{\mathrm{tor}})$ [illegible] $H(\mathcal{A})$ [illegible] $\mathcal{A}_{\mathrm{tor}}$ [illegible] $\mathcal{A}_{\mathrm{tor}}$ [illegible] $\mathcal{A}_{\{x\}}$ [illegible] $H(\mathcal{A}_{\mathrm{tor}})$ [illegible] $\widetilde{\mathbf{Z}}$ [illegible] $H(\mathcal{A}_{\{x\}})$ [illegible] $*$ [illegible] $H(\mathcal{A}_{\mathrm{tor}})$ [illegible] $K(\mathcal{A}_{\mathrm{tor}})$ [illegible]

[illegible] $\widehat{h}_r$, $\widehat{e}_r$ and $\widehat{q}_r$ [illegible] $H(\mathcal{A}_{\mathrm{tor}})$ [illegible] $r\geq 1$ by [illegible]

$$\widehat{H}(s) = 1+\sum_{r\geq 1}\widehat{h}_r s^r = \prod_{x\in \mathbf{P}^1(\mathbf{F}_q)} \widehat{H}_x(s^{\deg x}),$$

$$\widehat{E}(s) = 1+\sum_{r\geq 1}\widehat{e}_r s^r = \prod_{x\in \mathbf{P}^1(\mathbf{F}_q)} \widehat{E}_x(-s^{\deg x}),$$

$$\widehat{Q}(s) = 1+\sum_{r\geq 1}\widehat{q}_r s^r = \prod_{x\in \mathbf{P}^1(\mathbf{F}_q)} \widehat{Q}_x(s^{\deg x}).$$

[illegible] $H(\mathcal{A}_{\mathrm{tor}})[[s]]$ [illegible] $\mathbf{P}^1(\mathbf{F}_q)$ [illegible]

Lemma 15 *(i) One has the relations*

$$\widehat{H}(s)\,\widehat{E}(s) = 1 \quad \textit{and} \quad \widehat{Q}(s) = \frac{\widehat{H}(sv)}{\widehat{H}(s/v)},$$

or equivalently, for each $r \geq 1$,

$$\widehat{h}_r + \sum_{s=1}^{r-1} \widehat{h}_s \, \widehat{e}_{r-s} + \widehat{e}_r = 0,$$

$$(q^r - 1)\, \widehat{h}_r = v^r \widehat{q}_r + \sum_{s=1}^{r-1} v^{r-s}\, \widehat{h}_s \, \widehat{q}_{r-s}.$$

(ii) The three families $(\widehat{h}_r)_{r\geq 1}$, $(\widehat{e}_r)_{r\geq 1}$, and $(\widehat{q}_r)_{r\geq 1}$ consist of algebraically independent elements.

Proof. [illegible] $H(\mathcal{A}_{\rm tor})$ [illegible] $H(\mathcal{A}_{\{x\}})$ [illegible] $x \neq \infty$ [illegible] $H(\mathcal{A}_{\rm tor})$ [illegible] $\widehat{h}_{r,\infty}$ [illegible] $\widehat{h}_r - \widehat{h}_{r,\infty}$ [illegible] $\Gamma[\widehat{h}_{1,\infty}, \cdots, \widehat{h}_{r-1,\infty}]$ [illegible] $\widehat{h}_r$ [illegible] □

3.3 A subalgebra of $H(\mathcal{A})$

[illegible] $H(\mathcal{A})$ [illegible] $H(\mathcal{A}_{\rm tor})$ [illegible] B_1 [illegible] B [illegible] $H(\mathcal{A})$ [illegible] $U_q(\widehat{\mathfrak{sl}}_2)$

[illegible] $H(\mathcal{A})$ [illegible] $\widetilde{\mathbf{Z}}$ [illegible] B [illegible] $\widetilde{\mathbf{Z}}$ [illegible] $\widetilde{\mathbf{Z}}$ [illegible] R [illegible] R [illegible] $B \otimes_{\widetilde{\mathbf{Z}}} R$ [illegible] $B_{(R)}$

[illegible] $[a] = (v^a - v^{-a})/(v - v^{-1})$ [illegible] $a \in \mathbf{Z}$ [illegible] $[a]! = \prod_{i=1}^{a}[i]$ [illegible] $a \geq 1$ [illegible] $[0]! = 1$ [illegible] $[a]$ [illegible] $[a]!$ [illegible] v

[illegible]

Lemma 16 *(i) For all $m, n \in \mathbf{Z}$, one has*

$$[\mathcal{O}(m+1)] * [\mathcal{O}(n)] - v^2\, [\mathcal{O}(n)] * [\mathcal{O}(m+1)] = v^2\, [\mathcal{O}(m)] * [\mathcal{O}(n+1)] - [\mathcal{O}(n+1)] * [\mathcal{O}(m)].$$

(ii) If $n_1 < \cdots < n_r$ is an increasing sequence of integers and if $c_1, \ldots, c_r$ is a sequence of positive integers, then one has

$$[\mathcal{O}(n_1)]^{*c_1} * \cdots * [\mathcal{O}(n_r)]^{*c_r} = \left(\prod_{i=1}^{r} q^{c_i(c_i-1)/2}[c_i]!\right) v^{\sum_{1\leq i<j\leq r}(n_j-n_i+1)c_ic_j} \left[\bigoplus_{i=1}^{r} \mathcal{O}(n_i)^{\oplus c_i}\right].$$

Proof. [illegible]

$$
\begin{aligned}
[\mathcal{O}(n_i)]^{*c_i} &= v^{c_i(c_i-1)/2}\,[\mathcal{O}(n_i)]^{c_i} \\
&= v^{c_i(c_i-1)/2}\left(\prod_{a=1}^{c_i}\frac{q^a-1}{q-1}\right)[\mathcal{O}(n_i)^{\oplus c_i}] \\
&= q^{c_i(c_i-1)/2}\,[c_i]!\,[\mathcal{O}(n_i)^{\oplus c_i}],
\end{aligned}
$$

[illegible]

$$
\begin{aligned}
&[\mathcal{O}(n_1)]^{*c_1} * \cdots * [\mathcal{O}(n_r)]^{*c_r} \\
&\quad = v^{\sum_{1\le i<j\le r}(n_j-n_i+1)c_ic_j}\,[\mathcal{O}(n_1)]^{*c_1}\cdots[\mathcal{O}(n_r)]^{*c_r} \\
&\quad = \left(\prod_{i=1}^{r} q^{c_i(c_i-1)/2}[c_i]!\right) v^{\sum_{1\le i<j\le r}(n_j-n_i+1)c_ic_j}\,[\mathcal{O}(n_1)^{\oplus c_1}]\cdots[\mathcal{O}(n_r)^{\oplus c_r}] \\
&\quad = \left(\prod_{i=1}^{r} q^{c_i(c_i-1)/2}[c_i]!\right) v^{\sum_{1\le i<j\le r}(n_j-n_i+1)c_ic_j}\left[\bigoplus_{i=1}^{r}\mathcal{O}(n_i)^{\oplus c_i}\right],
\end{aligned}
$$

[illegible] □

[illegible] C [illegible] $\underline{c} = (c_n)_{n\in\mathbf{Z}}$ [illegible] $\underline{c} \in C$ [illegible]

$$X_{\underline{c}} = \prod_{n\in\mathbf{Z}}[\mathcal{O}(n)]^{*c_n},$$

[illegible] $*$ [illegible] $\mathbf{Z}$ [illegible] $\widetilde{\mathbf{Z}}$ [illegible] $H(\mathcal{A})$ [illegible] B_1

Proposition 17 *(i) B_1 is a subalgebra of $H(\mathcal{A})$.*

(ii) If R is a $\widetilde{\mathbf{Z}}$-algebra containing $\mathbf{Q}$, then the family $(X_{\underline{c}})_{c\in C}$ is a basis of the R-module $(B_1)_{(R)}$.

(iii) The multiplication in the Ringel algebra $H(\mathcal{A})$ induces an isomorphism of $\widetilde{\mathbf{Z}}$-modules from $B_1 \otimes_{\widetilde{\mathbf{Z}}} H(\mathcal{A}_{\mathrm{tor}})$ onto $H(\mathcal{A})$.

Proof. [illegible] $\mathcal{A}_{\mathrm{lf}}$ [illegible] $\mathcal{A}$ [illegible] $*$ [illegible] $H(\mathcal{A})$ [illegible] $\mathcal{F}$ [illegible] $\mathcal{F}_0$ [illegible] $\mathcal{F}_1$ [illegible] $\mathcal{A}$ [illegible] $\mathcal{F}$ [illegible] $\mathcal{F}_0$ [illegible] $\mathcal{F}_1$ [illegible] □

Remark 18. A ... $H(\mathcal{A}_{\mathrm{tor}})$ ... $H(\mathcal{A})$ ... $a \in H(\mathcal{A}_{\mathrm{tor}})$ ... by $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ ... $H(\mathcal{A}_{\mathrm{tor}})$ ... $H(\mathcal{A})$ ... $\mathrm{ad} : H(\mathcal{A}_{\mathrm{tor}}) \to \mathrm{End}_{\widetilde{\mathbf{Z}}}(H(\mathcal{A}))$ ... by

$$a * x = \sum_{(a)} \big(\mathrm{ad}(a_{(1)}) \cdot x\big) * a_{(2)},$$

... $a \in H(\mathcal{A}_{\mathrm{tor}})$ and $x \in H(\mathcal{A})$ ... B_1 ... $H(\mathcal{A}_{\mathrm{tor}})$ ... $H(\mathcal{A})$ ... A ... $H(\mathcal{A})$ ... $H(\mathcal{A}_{\mathrm{tor}})$ by ... $H(\mathcal{A}_{\mathrm{tor}})$ ... B_1

... B_1 and ... $H(\mathcal{A}_{\mathrm{tor}})$

Lemma 19 *For $n \in \mathbf{Z}$ and $r \geq 1$, one has*

$$\widehat{h}_r * [\mathcal{O}(n)] = [r+1]\ [\mathcal{O}(n+r)]\ +\ \sum_{s=0}^{r-1} [s+1]\ [\mathcal{O}(n+s)] * \widehat{h}_{r-s}.$$

Proof. ... $\widehat{E}_x(s)$ and $\widehat{H}(s)$ ...

$$X(t) = \sum_{n \in \mathbf{Z}} [\mathcal{O}(n)]\ t^n$$

and ...

$$\begin{aligned}
\widehat{E}_x(-s^{\deg x}) * X(t) &= \sum_{n \in \mathbf{Z},\, r \geq 0} (-1)^r\ s^{r \deg x}\ t^n\ q_x^{r(r-1)/2}\ [\mathcal{O}_{(1^r)[x]}] * [\mathcal{O}(n)] \\
&= \sum_{n \in \mathbf{Z},\, r \geq 0} (-1)^r\ s^{r \deg x}\ t^n\ q_x^{r(r-1)/2} \\
&\qquad \times \Big([\mathcal{O}(n)] * [\mathcal{O}_{(1^r)[x]}] + v^{(1-2r)\deg x}\ [\mathcal{O}(n + \deg x)] * [\mathcal{O}_{(1^{r-1})[x]}]\Big) \\
&= X(t) * \widehat{E}_x(-s^{\deg x})\ \Big(1 - (s/tv)^{\deg x}\Big).
\end{aligned}$$

... and L...

$$\widehat{H}(s) * X(t) = X(t) * \widehat{H}(s) \prod_{x \in \mathbf{P}^1(\mathbf{F}_q)} \frac{1}{1 - (s/tv)^{\deg x}},$$

... $1/(1 - (s/tv)^{\deg x})$ ... s/tv ... $\mathbf{Z}[[s]]$ ...

$$\prod_{x \in \mathbf{P}^1(\mathbf{F}_q)} \frac{1}{1 - s^{\deg x}} = \frac{1}{(1-s)(1-qs)},$$

... $\mathbf{P}^1(\mathbf{F}_q)$ ... $\mathbf{P}^1(\mathbf{F}_q)$ ...

$$\widehat{H}(s) * X(t) = X(t) * \widehat{H}(s)\, \frac{1}{(1-s/tv)(1-sv/t)},$$

... □

Remark 20. ... $\psi_r \in H(\mathcal{A}_{\rm tor})$ ...

$$\widehat{Q}(s) = 1 + \sum_{r\geq 1} v^{-r}\,\psi_r\, s^r.$$

On ...

$$\widehat{Q}(s) * X(t) = X(t) * \widehat{Q}(s)\, \frac{1-s/tq}{1-sq/t}.$$

... B_0 ... $H(\mathcal{A}_{\rm tor})$ ... $(\widehat{h}_r)_{r\geq 1}$ ... B ... $H(\mathcal{A})$ ... B_0 ... B_1 ... D ... $\underline{d} = (d_r)_{r\geq 1}$ ... $\underline{d} \in D$

$$\widehat{h}_{\underline{d}} = \prod_{r\geq 1} \widehat{h}_r^{d_r}, \quad \widehat{e}_{\underline{d}} = \prod_{r\geq 1} \widehat{e}_r^{d_r}, \quad \text{and} \quad \widehat{q}_{\underline{d}} = \prod_{r\geq 1} \widehat{q}_r^{d_r}.$$

Proposition 21 *Let R be a field of characteristic 0 which is also a $\widetilde{\mathbf{Z}}$-algebra.*

(i) The algebra $B_{(R)}$ is generated by the elements $[\mathcal{O}(n)]$ for $n \in \mathbf{Z}$ and the elements $\widehat{h}_r$ for $r \geq 1$.

(ii) The families $(X_{\underline{c}} * \widehat{h}_{\underline{d}})_{(\underline{c},\underline{d})\in C\times D}$, $(X_{\underline{c}} * \widehat{e}_{\underline{d}})_{(\underline{c},\underline{d})\in C\times D}$, and $(X_{\underline{c}} * \widehat{q}_{\underline{d}})_{(\underline{c},\underline{d})\in C\times D}$ are three bases of the R-module $B_{(R)}$.

Proof. ... $*$ ... $\widetilde{\mathbf{Z}}$ ... $B_1 \otimes_{\widetilde{\mathbf{Z}}} B_0$ ... $B_1 * B_0$ ... $B_1 * B_0$ ... $H(\mathcal{A})$ ... B ... $\widetilde{\mathbf{Z}}$ ... B_0 ... B_1 ... $*$ ... $H(\mathcal{A})_{(R)}$ ... R ... $(B_1)_{(R)} \otimes_R (B_0)_{(R)}$ ... $B_{(R)}$

... $(B_0)_{(R)}$ ... $\{\widehat{h}_r \mid r \geq 1\}$ $\{\widehat{e}_r \mid r \geq 1\}$ ... $\{\widehat{q}_r \mid r \geq 1\}$ ... $(\widehat{h}_{\underline{d}})_{\underline{d}\in D}$ $(\widehat{e}_{\underline{d}})_{\underline{d}\in D}$ and $(\widehat{q}_{\underline{d}})_{\underline{d}\in D}$ ... R ... $(B_0)_{(R)}$ ... □

Remark 22. ... B ... $H(\mathcal{A})$ ... B_0 ... $H(\mathcal{A}_{\rm tor})$ ...

4 Link with the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$

… $H(\mathcal{A})$ … $U_q(\widehat{\mathfrak{sl}_2})$ … q … $U_q(\widehat{\mathfrak{sl}_2})$ … q … R … 0 … v … q … R … $U_q(\widehat{\mathfrak{sl}_2})$ … V^+ … R … V^+ … R … $B_{(R)}$ … $U_q(\widehat{\mathfrak{sl}_2})$ …

4.1 Definition of $U_q(\widehat{\mathfrak{sl}_2})$

… 7 … $U_q(\widehat{\mathfrak{sl}_2})$ … R … $K^{\pm 1}$, $C^{\pm 1/2}$, h_r … $r \in \mathbf{Z} \setminus \{0\}$ … $x_n^\pm$ … $n \in \mathbf{Z}$ …

$$K\, K^{-1} = K^{-1}\, K = 1,$$

$$C^{1/2}\, C^{-1/2} = C^{-1/2}\, C^{1/2} = 1,$$

$C^{1/2}$ …

$$[K, h_r] = 0 \qquad r \in \mathbf{Z} \setminus \{0\}$$

$$K\, x_n^\pm = v^{\pm 2} x_n^\pm\, K \qquad n \in \mathbf{Z}$$

$$[h_r, h_s] = \delta_{r,-s}\, \frac{[2r]}{r}\, \frac{C^r - C^{-r}}{v - v^{-1}} \qquad r, s \in \mathbf{Z} \setminus \{0\}$$

$$[h_r, x_n^\pm] = \pm \frac{[2r]}{r}\, C^{\mp |r|/2} x_{n+r}^\pm \qquad n, r \in \mathbf{Z},\ r \neq 0$$

$$x_{m+1}^\pm\, x_n^\pm - v^{\pm 2} x_n^\pm\, x_{m+1}^\pm = v^{\pm 2} x_m^\pm\, x_{n+1}^\pm - x_{n+1}^\pm\, x_m^\pm \qquad m, n \in \mathbf{Z} \tag{7}$$

$$[x_m^+, x_n^-] = \frac{C^{(m-n)/2}\psi_{m+n}^+ - C^{(n-m)/2}\psi_{m+n}^-}{v - v^{-1}} \qquad m, n \in \mathbf{Z}$$

… $\psi_{\pm r}^\pm$ …

$$\sum_{r \geq 0} \psi_{\pm r}^\pm\, s^{\pm r} = K^{\pm 1} \exp\left(\pm (v - v^{-1}) \sum_{r \geq 1} h_{\pm r}\, s^{\pm r} \right)$$

… $r \geq 0$ … $r \leq -1$ …

… $U_q(\widehat{\mathfrak{sl}_2})$ … $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ … $A_1^{(1)}$ …

L V^+ b ba b a $U_q(\widehat{\mathfrak{sl}_2})$ n a d by n x_n^+ and $h_r\,C^{r/2}$ $n \in \mathbf{Z}$ and $r \geq 1$ a S n n

Theorem 23 *The R-algebras $B_{(R)}$ and V^+ are isomorphic.*

4.2 Structure of $U_q(\widehat{\mathfrak{sl}_2})$

n S n d n n $\widetilde{\psi}^{\pm}_{\pm r}$ $r \geq 1$ by

$$1 \pm \sum_{r \geq 1} (v - v^{-1})\, \widetilde{\psi}^{\pm}_{\pm r}\, s^{\pm r} \;=\; \exp\left(\pm (v - v^{-1}) \sum_{r \geq 1} h_{\pm r}\, C^{\pm r/2}\, s^{\pm r} \right). \qquad 8$$

L dn by

- $N^{\pm}$ ba b a n a d by n $x_n^{\pm}$ $n \in \mathbf{Z}$
- H ba b a n a d by n $K^{\pm 1}$ $C^{\pm 1/2}$ and h_r $r \in \mathbf{Z} \setminus \{0\}$
- $H^{\pm}$ ba b a n a d by n $\widetilde{\psi}^{\pm}_{\pm r}$ $r \geq 1$
- H^0 ba b a n a d by n $K^{\pm 1}$ and $C^{\pm 1/2}$

Proposition 24

(i) The multiplication induces a linear isomorphism $N^- \otimes_R H \otimes_R N^+ \to U_q(\widehat{\mathfrak{sl}_2})$.

(ii) The multiplication induces a linear isomorphism $H^- \otimes_R H^0 \otimes_R H^+ \to H$.

(iii) The generators $\widetilde{\psi}^+_r$ $(r \geq 1)$ of the algebra H^+ are algebraically independent.

(iv) The family of products $\left(\prod_{n \in \mathbf{Z}} (x_n^+)^{c_n}\right)_{\underline{c} \in C}$, performed in the ascending order of $\mathbf{Z}$, is a basis of N^+.

Proof. a d n n a a $N^- \otimes_R H \otimes_R N^+ \to U_q(\widehat{\mathfrak{sl}_2})$ n d d by a n $U_q(\widehat{\mathfrak{sl}_2})$ d n n a n $U_q(\widehat{\mathfrak{sl}_2})$ y a y a a $H^- \otimes_R H^0 \otimes_R H^+ \to H$ n d d by a n $U_q(\widehat{\mathfrak{sl}_2})$ a a b a $H^{\pm}$ n a d by a n n $\widetilde{\psi}^{\pm}_{\pm r}$ $r \geq 1$ a n a $\left(\prod_{r \geq 1} (\widetilde{\psi}^{\pm}_{\pm r})^{d_r}\right)$ $\underline{d} \in D$ an R a $H^{\pm}$ S a y a y n $(K^a C^{b/2})_{(a,b) \in \mathbf{Z}^2}$ an R a H^0 na y an a y n d n a d $\left(\prod_{n \in \mathbf{Z}} (x^{\pm}_{\pm n})^{c_n}\right)$ d n a n d n d $\mathbf{Z}$ and $\underline{c} \in C$ ah R a $N^{\pm}$ C n n y n

$$M(a, b, \underline{c}', \underline{c}'', \underline{d}', \underline{d}'') = \left(\prod_{n \in \mathbf{Z}} (x^-_{-n})^{c'_n} \right) \left(\prod_{r \geq 1} (\widetilde{\psi}^-_{-r})^{d'_r} \right) K^a C^{b/2} \left(\prod_{r \geq 1} (\widetilde{\psi}^+_r)^{d''_r} \right) \left(\prod_{n \in \mathbf{Z}} (x^+_n)^{c''_n} \right),$$

$a, b \in \mathbf{Z}$ $\underline{c}', \underline{c}'' \in C$ and $\underline{d}', \underline{d}'' \in D$ an R a $U_q(\widehat{\mathfrak{sl}_2})$

[illegible] $U_q(\widehat{\mathfrak{sl}_2})$ [illegible] T [illegible] R [illegible] $U_q(\widehat{\mathfrak{sl}_2})$ [illegible]

$$T(x_n^\pm) = x_{n\mp1}^\pm, \quad T(K^{\pm1}) = K^{\pm1}C^{\mp1}, \quad T(C^{\pm1/2}) = C^{\pm1/2}, \quad T(h_r) = h_r,$$

[illegible] $n, r \in \mathbf{Z}$ [illegible] $r \neq 0$ [illegible] T [illegible] $U_q(\widehat{\mathfrak{sl}_2})$ [illegible] $\mathfrak{sl}_2$. On [illegible] $U_q(\widehat{\mathfrak{sl}_2})$ [illegible]

$$\left\{M(a,b,\underline{c}',\underline{c}'',\underline{d}',\underline{d}'') \;\middle|\; a,b \in \mathbf{Z},\ \underline{c}',\underline{c}'' \in C,\ \underline{d}',\underline{d}'' \in D,\ n<0 \Rightarrow c_n' = c_n'' = 0\right\}$$

[illegible] R [illegible] T [illegible] R [illegible] $(M(a,b,\underline{c}',\underline{c}'',\underline{d}',\underline{d}''))_{(a,b,\underline{c}',\underline{c}'',\underline{d}',\underline{d}'')\in\mathbf{Z}^2\times C^2\times D^2}$ [illegible] $U_q(\widehat{\mathfrak{sl}_2})$ [illegible] □

L[illegible] H^+ [illegible] $\mathbf{U}^+(0)$ [illegible] $\widetilde{P}_r$ and P_r [illegible] H^+ [illegible] $r \geq 1$ by [illegible]

$$\widetilde{P}(s) = 1 + \sum_{r\geq1} \widetilde{P}_r s^r \;=\; \exp\left(\sum_{r\geq1} \frac{h_r\, C^{r/2}}{[r]}\, s^r\right), \qquad (9)$$

$$P(s) = 1 + \sum_{r\geq1} P_r s^r \;=\; \exp\left(-\sum_{r\geq1} \frac{h_r\, C^{r/2}}{[r]}\, s^r\right). \qquad ([illegible])$$

[illegible] $\underline{c} = (c_n)_{n\in\mathbf{Z}} \in C$ and $\underline{d} = (d_r)_{r\geq1} \in D$ [illegible]

$$x_{\underline{c}}^+ = \prod_{n\in\mathbf{Z}} (x_n^+)^{c_n}, \quad \widetilde{P}_{\underline{d}} = \prod_{r\geq1} \widetilde{P}_r^{d_r}, \quad P_{\underline{d}} = \prod_{r\geq1} P_r^{d_r}, \quad \text{and} \quad \widetilde{\psi}_{\underline{d}}^+ = \prod_{r\geq1} (\widetilde{\psi}_r^+)^{d_r}.$$

Proposition 25 *(i) The algebra V^+ is generated by the elements x_n^+ and $\widetilde{P}_r$, for $n \in \mathbf{Z}$ and $r \geq 1$.*

(ii) The families $(x_{\underline{c}}^+ \widetilde{P}_{\underline{d}})_{(\underline{c},\underline{d})\in C\times D}$, $(x_{\underline{c}}^+ P_{\underline{d}})_{(\underline{c},\underline{d})\in C\times D}$, and $(x_{\underline{c}}^+ \widetilde{\psi}_{\underline{d}}^+)_{(\underline{c},\underline{d})\in C\times D}$ are three bases of the R-vector space V^+.

Proof. [illegible] V^+ [illegible] $U_q(\widehat{\mathfrak{sl}_2})$ [illegible] x_n^+ and $h_r C^{r/2}$ [illegible] $n \in \mathbf{Z}$ and $r \geq 1$. [illegible] $\widetilde{P}_r$ [illegible] (9) and [illegible] $[r]$ [illegible] R [illegible] any $r \geq 1$.

[illegible] 8 [illegible] $n \geq 0$ and $r \geq 1$ [illegible]

$$\widetilde{P}_r\, x_n^+ = [r+1]\, x_{n+r}^+ \;+\; \sum_{s=0}^{r-1} [s+1]\, x_{n+s}^+\, \widetilde{P}_{r-s}. \qquad ([illegible])$$

... T ... any $n \in \mathbf{Z}$ ... $N^+ \otimes_R H^+$ ... V^+

... 8 and 9 ... $H^+[[s]]$

$$\widetilde{P}(s)\, P(s) = 1 \quad \text{and} \quad \frac{\widetilde{P}(sv)}{\widetilde{P}(s/v)} = 1 + \sum_{r \geq 1} (v - v^{-1})\, \widetilde{\psi}_r^+\, s^r,$$

... $r \geq 1$

$$\widetilde{P}_r + \sum_{s=1}^{r-1} \widetilde{P}_s P_{r-s} + P_r = 0,$$

$$[r]\, \widetilde{P}_r = \widetilde{\psi}_r^+ + \sum_{s=1}^{r-1} v^{-s}\, \widetilde{P}_s\, \widetilde{\psi}_{r-s}^+.$$

... H^+ ... $\{\widetilde{\psi}_r \mid r \geq 1\}$ $\{\widetilde{P}_r \mid r \geq 1\}$ ... $\{P_r \mid r \geq 1\}$ ... $(\widetilde{P}_{\underline{d}})_{\underline{d} \in D}$ $(P_{\underline{d}})_{\underline{d} \in D}$ and $(\widetilde{\psi}_{\underline{d}}^+)_{\underline{d} \in D}$ ... R ... H^+ ... $\square$

... $[\mathcal{O}(n)]$ ... x_n^+ $\widehat{h}_r$ ... $\widetilde{P}_r$ $\widehat{e}_r$ ... P_r and $\widehat{q}_r$ ... $(v - v^{-1})\widetilde{\psi}_r^+$ ... 7 and ...

4.3 Concluding remarks

...

- ... $[\mathcal{O}(n_1)^{\oplus c_1} \oplus \cdots \oplus \mathcal{O}(n_r)^{\oplus c_r}]$ ... $H(\mathcal{A})$ ... v ...

$$\left(\frac{1}{[c_1]!}[\mathcal{O}(n_1)]^{*c_1}\right) * \cdots * \left(\frac{1}{[c_r]!}[\mathcal{O}(n_r)]^{*c_r}\right);$$

- ... $\widehat{h}_r$ ... $r \geq 1$ ... H^+

... $H(\mathcal{A})$ ... B ... q ...

n P_r $\widetilde{P}_r$ and $\widetilde{\psi}_r$ ay an $U_q(\widehat{\mathfrak{sl}_2})$ na y n a a on n d n na $U_q(\widehat{\mathfrak{sl}_2})$ d n and n n on a n a V ba L n a $U_q(\widehat{\mathfrak{sl}_2})$ a n 8 9 and nn a a a a an $B_{(R)} \to V^+$ n y a and a d n a a ba yn a n a y n n a L yn a a d a a ba $H(\mathcal{A}_{\{x\}})$ and a a d h $\mathbf{P}^1(\mathbf{F}_q)$

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