The canonical basis and the quantum Frobenius morphism

Pierre Baumann

Abstract

The first goal of this paper is to study the amount of compatibility between two important constructions in the theory of quantized enveloping algebras, namely the canonical basis and the quantum Frobenius morphism. The second goal is to study orders with which the Kashiwara crystal $B(\infty)$ of a symmetrizable Kac-Moody algebra can be endowed; these orders are defined so that the transition matrices between bases naturally indexed by $B(\infty)$ are lower triangular.

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1 Introduction

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of a symmetrizable Kac-Moody algebra. Twenty years ago, with the help of the quantized enveloping algebra $U_q(\mathfrak{g})$, Lusztig and Kashiwara constructed a basis $\mathbf{B}$ in the enveloping algebra $U(\mathfrak{n}_+)$, called the canonical basis, whose properties make it particularly well suited to the study of integrable highest weight $\mathfrak{g}$-modules [35, 38, 23]. Subsequently, Kashiwara studied the combinatorics of $\mathbf{B}$ with his abstract notion of crystal [26], while Lusztig, during his investigation of the geometric problems raised by his construction of $\mathbf{B}$, was eventually led to the definition of a second basis, called the semicanonical basis [39].

The quantized enveloping algebra setting leads to other useful tools, as the quantum Frobenius morphism Fr and its splitting Fr'. Using these maps, Kumar and Littelmann algebraized the proofs that use Mehta and Ramanathan's Frobenius splitting for algebraic groups in positive characteristic [30]. Littelmann also used the quantum Frobenius splitting to define a basis in all simple $\mathfrak{g}$-modules from the combinatorics of LS-paths, completing thereby the program of standard monomial theory [34].

It is thus desirable to study the extent of compatibility between these two constructions, the canonical basis and the quantum Frobenius map. The best behavior would be that Fr and Fr' map a basis vector to a basis vector or to zero, in a way that admits a combinatorial characterization.

Lusztig observed that Fr commutes with the comultiplication. It is then tempting to use duality to understand the situation. More precisely, the graded dual of $U(\mathfrak{n}_+)$ can be identified with the algebra $\mathbb{Q}[N]$ of regular functions on N, the unipotent group with Lie algebra $\mathfrak{n}_+$, and the dual of the canonical basis behaves rather nicely with respect to the multiplication of $\mathbb{Q}[N]$ [8]. Unfortunately, rather nicely does not mean perfect agreement, as was shown by Leclerc [32]. We will see in Section 6.3 that Fr fails to be fully compatible with the canonical basis at the same spot where Leclerc found his counterexamples. This failure partially answers a question raised by McGerty ([40], Remark 5.10), asking whether his construction of Fr in the context of Hall algebras can be lifted to the level of perverse sheaves.

In view of the applications, the study of Fr' is perhaps even more important. An encouraging fact is that Fr' is compatible with $\mathbf{B}$ in small rank (type A_1, A_2, A_3 and B_2). Alas, in general, Fr' is not compatible with $\mathbf{B}$ (see Section 6.3).

One can however obtain a form of compatibility between Fr, Fr' and $\mathbf{B}$ by focusing on leading terms, that is, by neglecting terms that are smaller. This result is hardly more than an observation, but it invites us to study the orders with which $\mathbf{B}$ can be endowed. Since as a set $\mathbf{B}$ is just Kashiwara's cristal $B(-\infty)$, we will in fact investigate orders on $B(-\infty)$.

We will present two natural ways to order $B(-\infty)$. In the first method, one checks the values of the functions ε_i and φ_i; after stabilization by the crystal operations $\tilde{e}_i$ and $\tilde{f}_i$, one obtains an order $\leq_{\mathrm{str}}$ (Section 2.5).

The second method, which works only when $\mathfrak{g}$ is finite dimensional, relies on the notion of MV polytope [1, 21, 22]: to each $b \in B(-\infty)$ is associated a convex polytope $\mathrm{Pol}(b) \subseteq \mathfrak{h}^*$, and the containment of these polytopes defines an order $\leq_{\mathrm{pol}}$ on $B(-\infty)$ (Section 3.3).

The plan of this paper is as follows. In Section 2, we review the properties of the canonical basis that are important from a combinatorial viewpoint, following the methods set up by Kashiwara, Berenstein, Zelevinsky, and their coauthors. This leads us quite naturally to the definition of the order $\leq_{\mathrm{str}}$. In Section 3, we recall the definition of the MV polytope of an element of $B(-\infty)$ and explain how to numerically test whether two elements of $B(-\infty)$ are comparable w.r.t. the order $\leq_{\mathrm{pol}}$. In Section 4, we assume that the Cartan matrix of $\mathfrak{g}$ is symmetric and recall Lusztig's construction of the canonical and semicanonical bases; we relate $\leq_{\mathrm{pol}}$ to the degeneracy order between quiver representations (Proposition 4.1) and we show that the transition matrix between the canonical basis and the semicanonical basis is lower unitriangular, for both $\leq_{\mathrm{str}}$ and $\leq_{\mathrm{pol}}$ (Theorem 4.4). In Section 5, we revisit Leclerc's counterexamples in type A_5 and D_4; Theorem 5.2 provides the expansion on $\mathbf{B}$ of certain monomials in the Chevalley generators of $U(\mathfrak{n}_+)$. This result is used in Section 6 to show that the Frobenius morphism Fr and its splitting Fr' are not fully compatible with $\mathbf{B}$.

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2 Crystal operations

For all this paper, we fix a symmetrizable generalized Cartan matrix $A = (a_{i,j})$, with rows and columns indexed by a finite set I. We choose a $\mathbb{Q}$-vector space $\mathfrak{h}$, with a basis indexed by I. We denote by $(\alpha_i)_{i\in I}$ the dual basis in $\mathfrak{h}^*$ and we define elements $\alpha_i^\vee$ in $\mathfrak{h}$ by the equation $\langle\alpha_i^\vee, \alpha_j\rangle = a_{i,j}$. We fix a lattice $P \subseteq \mathfrak{h}^*$ such that

$$\{\alpha_i \mid i \in I\} \subseteq P \subseteq \{\lambda \in \mathfrak{h}^* \mid \forall i \in I,\ \langle\alpha_i^\vee, \lambda\rangle \in \mathbb{Z}\}.$$

We denote the $\mathbb{N}$-span of the simple roots α_i by Q_+.

2.1 Crystals

A crystal in the sense of Kashiwara [26] is a set B endowed with maps

$$\mathrm{wt} : B \to P, \quad \varepsilon_i, \varphi_i : B \to \mathbb{Z} \quad \text{and} \quad \tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\},$$

for each $i \in I$. The element 0 here is a ghost element added to B so that $\tilde{e}_i$ and $\tilde{f}_i$ are everywhere defined. One requires that $\langle\alpha_i^\vee, \mathrm{wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b)$ for each $b \in B$. The operators $\tilde{e}_i$ and $\tilde{f}_i$ are mutually converse partial bijections: $b'' = \tilde{e}_i b'$ if and only if $\tilde{f}_i b'' = b'$, and when these equalities hold,

$$\mathrm{wt}(b'') = \mathrm{wt}(b') + \alpha_i, \quad \varepsilon_i(b'') = \varepsilon_i(b') - 1 \quad \text{and} \quad \varphi_i(b'') = \varphi_i(b') + 1.$$

We say that a crystal B is lower normal if for each $b \in B$, the number $p = \varphi_i(b)$ is the largest integer $p \in \mathbb{N}$ such that $\tilde{f}_i^p b$ is defined. (In other words, $\tilde{f}_i^p b \in B$ and $\tilde{f}_i^{p+1} b = 0$.)

Let B be a lower normal crystal. For $i \in I$ and $b \in B$, we set $\tilde{f}_i^{\max} b = \tilde{f}_i^{\varphi_i(b)} b$. In addition, for a finite sequence $\mathbf{i} = (i_1, \ldots, i_\ell)$ of elements in I, we define maps $\Phi_{\mathbf{i}} : B \to \mathbb{N}^\ell$ and $\widetilde{F}_{\mathbf{i}} : B \to B$ in the following fashion. Given $b \in B$, we put $b_0 = b$, and for $k \in \{1, \ldots, \ell\}$, we set $n_k = \varphi_{i_k}(b_{k-1})$ and $b_k = \tilde{f}_{i_k}^{\max} b_{k-1}$. With these notations,

$$\Phi_{\mathbf{i}}(b) = (n_1, \ldots, n_\ell) \quad \text{and} \quad \widetilde{F}_{\mathbf{i}}(b) = b_\ell.$$

Of course, the datum of $\Phi_{\mathbf{i}}(b)$ and of $\widetilde{F}_{\mathbf{i}}(b)$ fully determines b. The map $\Phi_{\mathbf{i}}$ is usually called the string parametrization in direction $\mathbf{i}$ [8, 9, 25].

2.2 Bases of canonical type

Let $\mathbf{f}$ be the $\mathbb{Q}$-algebra generated by elements θ_i, for $i \in I$, submitted to the relations

$$\sum_{p+q=1-a_{i,j}} (-1)^p \frac{\theta_i^p}{p!} \theta_j \frac{\theta_i^q}{q!} = 0 \tag{1}$$

for all $i \neq j$ in I. This algebra $\mathbf{f}$ is naturally graded by Q_+ (gradation by the weight), the generator θ_i being of weight α_i; we write $\mathbf{f} = \bigoplus_{\nu\in Q_+} \mathbf{f}_\nu$. In addition, $\mathbf{f}$ has an antiautomorphism σ which fixes the θ_i. As usual, we introduce the divided powers $\theta_i^{(n)} = \theta_i^n/n!$.

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \widehat{\mathfrak{h}} \oplus \mathfrak{n}_+$ be the Kac-Moody algebra defined by the Cartan matrix A. By the Gabber-Kac Theorem [14], there are isomorphisms $x \mapsto x^\pm$ from $\mathbf{f}$ onto $U(\mathfrak{n}^\pm)$, that map the generators θ_i to the elements e_i or f_i of $\mathfrak{g}$.

A basis $\mathbf{B}$ of $\mathbf{f}$ is said to be of canonical type if it satisfies the conditions (i)–(v) below:

(i) The elements of $\mathbf{B}$ are weight vectors.

(ii) $1 \in \mathbf{B}$.

(iii) Each right ideal $\theta_i^p \mathbf{f}$ is spanned by a subset of $\mathbf{B}$.

(iv) In the bases induced by $\mathbf{B}$, the left multiplication by $\theta_i^{(p)}$ from $\mathbf{f}/\theta_i\mathbf{f}$ onto $\theta_i^p\mathbf{f}/\theta_i^{p+1}\mathbf{f}$ is given by a permutation matrix.

(v) $\mathbf{B}$ is stable by σ.

Let N be the unipotent group with Lie algebra $\mathfrak{n}_+$. The graded dual of $\mathbf{f}$ can be identified with the algebra $\mathbb{Q}[N]$ of regular functions on N. Berenstein, Zelevinsky, and their coauthors extensively studied bases of $\mathbb{Q}[N]$ that enjoy axioms dual to the conditions (i)–(iv) above. More precisely, the bases considered in [18, 43] (the so-called good bases) forget about condition (iv), the string bases of [8] satisfy stronger axioms inspired by the positivity properties of Lusztig's canonical basis in the quantized enveloping algebra $U_q(\mathfrak{n}_+)$, and the perfect bases of [7] relax the normalization constraint in condition (iv). Our definition above is of course directly inspired from these works.

Adopting this dual setting allows to exploit the multiplicative structure of the function algebra $\mathbb{Q}[N]$. We will however stick with $\mathbf{f}$, because the quantum Frobenius splitting Fr' is more easily defined on the enveloping algebra side.

2.3 Examples

Since A is symmetrizable, the half-quantum group $U_q^-(\mathfrak{g})$ has a canonical basis [36, 38, 23]. After specialization at $q = 1$, one gets a basis of $\mathbf{f}$ of canonical type, by Theorems 14.3.2 and 14.4.3 in [38], or by Theorem 7 in [23], Proposition 5.3.1 in [24] and Theorem 2.1.1 in [25].

If A is symmetric, then $\mathbf{f}$ can be endowed with the semicanonical basis [39]. Again, this is a basis of canonical type, by Theorems 3.1 and 3.8 and by the proof of Lemma 2.5 in [39].

If A is of finite type, then one can use the geometric Satake correspondence [19] and Mirković-Vilonen cycles [41] to define a basis of $\mathbf{f}$. It can be shown that this basis is of canonical type.

One major interest of bases of canonical type is that they induce bases in irreducible integrable highest weight representations of $\mathfrak{g}$. In more details, let M be an irreducible integrable highest weight representation of $\mathfrak{g}$, let m be a highest weight vector of V, and let λ be the weight of m. Then the map $a : x \mapsto x^- m$ from $\mathbf{f}$ to M is surjective with kernel $\sum_{i\in I} \mathbf{f}\theta_i^{n_i}$, where $n_i = 1 + \langle \alpha_i^\vee, \lambda\rangle$ (see [20], Corollary 10.4). If $\mathbf{B}$ is a basis of canonical type of $\mathbf{f}$, then $\ker a$ is spanned by a subset of $\mathbf{B}$, hence $a(\mathbf{B}) \setminus \{0\}$ is a basis of $\operatorname{im} a = M$. Moreover, the analysis done in [25], Section 3.2 shows that this basis is compatible with all Demazure submodules of M. In particular, the semicanonical basis is compatible with Demazure submodules, a result first obtained by Savage ([47], Theorem 7.1).

2.4 The crystal $B(-\infty)$

Each basis $\mathbf{B}$ of canonical type of $\mathbf{f}$ is endowed with the structure of a lower normal crystal, as follows. The application wt maps a vector $b \in \mathbf{B}$ to its weight. Let $i \in I$ and let $b \in \mathbf{B}$. We define $\varphi_i(b)$ as the largest $p \in \mathbb{N}$ such that $b \in \theta_i^p \mathbf{f}$ and we set $\varepsilon_i(b) = \varphi_i(b) - \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$. Thus, given $i \in I$ and $p \in \mathbb{N}$, the images of the elements $\{b \in \mathbf{B} \mid \varphi_i(b) = p\}$ form a basis of $\theta_i^p \mathbf{f}/\theta_i^{p+1}\mathbf{f}$. The partial bijections $\tilde{e}_i$ and $\tilde{f}_i$ are set up so that when $\varphi_i(b) = p$, the element $\tilde{f}_i^{\max} b$ is the element of $\mathbf{B}$ such that $\theta_i^{(p)} \, \tilde{f}_i^{\max} b \equiv b$ modulo $\theta_i^{p+1}\mathbf{f}$.

It turns out that any two bases of canonical type have the same underlying crystal. This fact is a particular case of a result by Berenstein and Kazhdan [7]. For the convenience of the reader, we now recall the proof.

Given a basis $\mathbf{B}$ of a vector space V, we denote by b^* the element of the dual basis that corresponds to $b \in \mathbf{B}$; in other words, $x = \sum_{b \in \mathbf{B}} \langle b^*, x \rangle \, b$ for all vectors $x \in V$.

Lemma 2.1 *Let V be a finite dimensional vector space and let $\mathbf{B}'$ and $\mathbf{B}''$ be two bases of V. For $a \in \{1,2\}$, let $\mathcal{C}_a$ be a partially ordered set and let $b'_a : \mathcal{C}_a \to \mathbf{B}'$ and $b''_a : \mathcal{C}_a \to \mathbf{B}''$ be bijections; thus both the rows and the columns of the transition matrix between $\mathbf{B}'$ and $\mathbf{B}''$ are indexed by $\mathcal{C}_a$. If the transition matrix between $\mathbf{B}'$ and $\mathbf{B}''$ is lower unitriangular with respect to both indexings $(\mathcal{C}_1, b'_1, b''_1)$ and $(\mathcal{C}_2, b'_2, b''_2)$, then $b''_1 \circ (b'_1)^{-1} = b''_2 \circ (b'_2)^{-1}$.*

Proof. Let $\Sigma = (b''_1)^{-1} \circ b''_2 \circ (b'_2)^{-1} \circ b'_1$. Suppose that Σ is not the identity permutation of $\mathcal{C}_1$. Let $m \in \mathcal{C}_1$ be a maximal element in a nontrivial cycle of Σ and let $n = (b'_2)^{-1} \circ b'_1(m)$. Then $\big\langle (b''_1(\Sigma(m)))^*, b'_1(m) \big\rangle = \big\langle (b''_2(n))^*, b'_2(n) \big\rangle = 1$, and therefore $\Sigma(m) \geq m$ in $\mathcal{C}_1$, by unitriangularity of the transition matrix. This contradicts the choice of m. We conclude that Σ is the identity. □

Lemma 2.2 *Let $\mathbf{B}$ be a basis of canonical type and let $b \neq 1$ be an element of $\mathbf{B}$. Then there exists $i \in I$ such that $\varphi_i(b) > 0$.*

Proof. For each $i \in I$, the right ideal $\theta_i \mathbf{f}$ is a spanned by a subset of $\mathbf{B}$, namely by $\mathbf{B}_{i;>0} = \mathbf{B} \cap \theta_i \mathbf{f} = \{b \in \mathbf{B} \mid \varphi_i(b) > 0\}$. The subspace $\sum_{i \in I} \theta_i \mathbf{f}$ is thus spanned by $\bigcup_{i \in I} \mathbf{B}_{i;>0}$. Any element of $\mathbf{B}$ in this subspace therefore belongs to $\bigcup_{i \in I} \mathbf{B}_{i;>0}$. □

Given an integer $\ell \geq 0$, we endow $\mathbb{N}^\ell$ with the lexicographic order $\leq_{\mathrm{lex}}$.

Proposition 2.3 *Let $\mathbf{B}'$ and $\mathbf{B}''$ be two bases of canonical type of $\mathbf{f}$. Let $\mathbf{i} = (i_1, \ldots, i_\ell)$ be a finite sequence of elements of I.*

(i) Let $(b', b'') \in \mathbf{B}' \times \mathbf{B}''$. If $\langle (b'')^*, b' \rangle \neq 0$, then $\Phi_{\mathbf{i}}(b') \leq_{\mathrm{lex}} \Phi_{\mathbf{i}}(b'')$.

(ii) In (i), if moreover $\Phi_{\mathbf{i}}(b') = \Phi_{\mathbf{i}}(b'')$, then $\langle (b'')^*, b' \rangle = \langle (\widetilde{F}_{\mathbf{i}} b'')^*, \widetilde{F}_{\mathbf{i}} b' \rangle$.

(iii) For each $b'' \in \mathbf{B}''$, there is $b' \in \mathbf{B}'$ such that $\langle (b'')^*, b' \rangle \neq 0$ and $\Phi_{\mathbf{i}}(b') = \Phi_{\mathbf{i}}(b'')$.

Proof. The proof proceeds by induction on the length ℓ of $\mathbf{i}$. The result is trivial if $\ell = 0$. We now assume that $\ell > 0$ and that the result holds for the sequence $\mathbf{j} = (i_2, \dots, i_\ell)$.

Let $b' \in \mathbf{B}'$, let $n_1 = \varphi_{i_1}(b')$, and let $b'_1 = \tilde{f}_{i_1}^{\max} b'$. Let us write

$$b'_1 = \sum_{b''_1 \in \mathbf{B}''} \langle (b''_1)^*, b'_1 \rangle \, b''_1 \equiv \sum_{\substack{b''_1 \in \mathbf{B}'' \\ \varphi_{i_1}(b''_1) = 0}} \langle (b''_1)^*, b'_1 \rangle \, b''_1 \pmod{\theta_{i_1} \mathbf{f}}.$$

Multiplying on the left by $\theta_{i_1}^{(n_1)}$, we obtain

$$b' \equiv \theta_{i_1}^{(n_1)} b'_1 \equiv \sum_{\substack{b''_1 \in \mathbf{B}'' \\ \varphi_{i_1}(b''_1) = 0}} \langle (b''_1)^*, b'_1 \rangle \, \theta_{i_1}^{(n_1)} b''_1 \equiv \sum_{\substack{b''_1 \in \mathbf{B}'' \\ \varphi_{i_1}(b''_1) = 0}} \langle (b''_1)^*, b'_1 \rangle \, \tilde{e}_{i_1}^{n_1} b''_1 \pmod{\theta_{i_1}^{n_1+1} \mathbf{f}}.$$

Since $\theta_{i_1}^{n_1+1} \mathbf{f}$ is spanned by $\{ b'' \in \mathbf{B}'' \mid \varphi_{i_1}(b'') > n_1 \}$, we see that if $\langle (b'')^*, b' \rangle \neq 0$, then either $\varphi_{i_1}(b'') > n_1$, or $b'' = \tilde{e}_{i_1}^{n_1} b''_1$ with $\varphi_{i_1}(b''_1) = 0$ and $\langle (b''_1)^*, b'_1 \rangle = \langle (b'')^*, b' \rangle$. Assertions (i) and (ii) for b' and $\mathbf{i}$ thus readily follow from the corresponding assertions for b'_1 and $\mathbf{j}$.

Now let $b'' \in \mathbf{B}''$, let $n_1 = \varphi_{i_1}(b'')$, and let $b''_1 = \tilde{f}_{i_1}^{\max} b''$. By induction, there exists $b'_1 \in \mathbf{B}'$ such that $\langle (b''_1)^*, b'_1 \rangle \neq 0$ and $\Phi_{\mathbf{j}}(b'_1) = \Phi_{\mathbf{j}}(b''_1)$. We then have $0 \leq \varphi_{i_1}(b'_1) \leq \varphi_{i_1}(b''_1) = 0$. Let $b' = \tilde{e}_{i_1}^{n_1} b'_1$. Then $\Phi_{\mathbf{i}}(b') = \Phi_{\mathbf{i}}(b'')$, and also, by the reasoning used in the proof of (i), $\langle (b'')^*, b' \rangle = \langle (b''_1)^*, b'_1 \rangle \neq 0$. This shows (iii). □

Theorem 2.4 *Let $\mathbf{B}'$ and $\mathbf{B}''$ be two bases of canonical type of $\mathbf{f}$. Then there is a unique bijection $\Xi : \mathbf{B}' \to \mathbf{B}''$ such that $\Phi_{\mathbf{i}}(b') = \Phi_{\mathbf{i}}(\Xi(b'))$ for any finite sequence $\mathbf{i}$ of elements of I and any $b' \in \mathbf{B}'$. Moreover, Ξ is an isomorphism of crystals which commutes with the action of the involution σ.*

Proof. Proposition 2.3 generalizes in an obvious way to infinite sequences $\mathbf{i} = (i_1, i_2, \dots)$ of elements of I, because for any element b in a basis of canonical type, the sequence

$$(b, \tilde{f}_{i_1}^{\max} b, \tilde{f}_{i_2}^{\max} \tilde{f}_{i_1}^{\max} b, \dots)$$

eventually becomes constant for weight reasons. In this situation, if each $i \in I$ appears an infinite number of times in $\mathbf{i}$, then the limit of this sequence is necessarily equal to 1, by Lemma 2.2.

Let us fix such a sequence $\mathbf{i}$. Then for any basis $\mathbf{B}$ of canonical type, $\Phi_{\mathbf{i}}$ is an injective map from $\mathbf{B}$ to the set $\mathbb{N}^{(\infty)}$ of sequences of non-negative integers with finitely many nonzero terms. In addition, $\mathcal{C}_{\mathbf{i}} = \Phi_{\mathbf{i}}(\mathbf{B})$ does not depend on $\mathbf{B}$, by Proposition 2.3 (iii).

We can thus index any basis $\mathbf{B}$ of canonical type by $\mathcal{C}_{\mathbf{i}}$. Using this for two bases $\mathbf{B}'$ and $\mathbf{B}''$ of canonical type, we moreover deduce from Proposition 2.3 (i) and (ii) that the transition matrix is lower unitriangular if we endow $\mathcal{C}_{\mathbf{i}}$ with the lexicographic order on $\mathbb{N}^{(\infty)}$. From Lemma 2.1, we conclude that the bijection $\Xi : \mathbf{B}' \to \mathbf{B}''$ defined by the diagram

$$\begin{array}{ccc} \mathbf{B}' & \xrightarrow{\quad\Xi\quad} & \mathbf{B}'' \\ & {\scriptstyle \Phi_{\mathbf{i}}}\searrow \quad \swarrow{\scriptstyle \Phi_{\mathbf{i}}} & \\ & \mathcal{C}_{\mathbf{i}} & \end{array}$$

does not depend on $\mathbf{i}$. (Lemma 2.1 can be applied in our context because the weight spaces of $\mathbf{f}$ are finite dimensional.)

Lastly, we observe that the transition matrix is also lower unitriangular if we use the indexations $\sigma \circ \Phi_{\mathbf{i}}^{-1}$ from $\mathcal{C}_{\mathbf{i}}$ to $\mathbf{B}'$ and $\mathbf{B}''$, and so $\Xi = \sigma \circ \Xi \circ \sigma$, again by Lemma 2.1. □

The crystal common to all bases of canonical type is denoted by $B(-\infty)$.

Remark 2.5. Let $\mathbf{i}$ and $\mathcal{C}_{\mathbf{i}}$ be as in the proof of Theorem 2.4. To each sequence $\mathbf{n} = (n_1, n_2, \ldots)$ in $\mathcal{C}_{\mathbf{i}}$, define $\Theta_{\mathbf{i}}^{(\mathbf{n})} = \theta_{i_1}^{(n_1)} \theta_{i_2}^{(n_2)} \cdots$. Let $\mathbf{B}$ be a basis of canonical type of $\mathbf{f}$. Arguing as in the proof of Proposition 2.3, one shows that

$$\langle b^*, \Theta_{\mathbf{i}}^{(\mathbf{n})} \rangle = \begin{cases} 1 & \text{if } \mathbf{n} = \Phi_{\mathbf{i}}(b), \\ 0 & \text{if } \mathbf{n} \not\leq_{\mathrm{lex}} \Phi_{\mathbf{i}}(b). \end{cases}$$

It follows that the elements $\Theta_{\mathbf{i}}^{(\mathbf{n})}$ form a basis of $\mathbf{f}$. This result is due to Lakshmibai ([31], Theorems 6.5 and 6.6).

2.5 The string order on $B(-\infty)$

Given (b', b'') and (c', c'') in $B(-\infty)^2$, we write $(b', b'') \approx (c', c'')$ if one of the following three conditions holds:

- There is $i \in I$ such that $\varphi_i(b') = \varphi_i(b'')$ and $(c', c'') = (\tilde{e}_i b', \tilde{e}_i b'')$.
- There is $i \in I$ such that $\varphi_i(b') = \varphi_i(b'') > 0$ and $(c', c'') = \left(\tilde{f}_i b', \tilde{f}_i b''\right)$.
- $(c', c'') = (\sigma(b'), \sigma(b''))$.

Given $(b', b'') \in B(-\infty)^2$, we write $b' \leq_{\mathrm{str}} b''$ if b' and b'' have the same weight and if for any finite sequence of elementary moves

$$(b', b'') = (b'_0, b''_0) \approx (b'_1, b''_1) \approx \cdots \approx (b'_\ell, b''_\ell),$$

one has $\varphi_i(b'_\ell) \leq \varphi_i(b''_\ell)$ for all $i \in I$.

Proposition 2.6 *(i) The relation $\leq_{\mathrm{str}}$ is an order on $B(-\infty)$.*

(ii) The transition matrix between two bases of canonical type is lower unitriangular w.r.t. the order $\leq_{\mathrm{str}}$.

Proof. The relation $\leq_{\mathrm{str}}$ is obviously reflexive. Let us show that it is transitive. Suppose that $b' \leq_{\mathrm{str}} b''$ and $b'' \leq_{\mathrm{str}} b'''$ and let us consider a finite sequence

$$(b', b''') = (b'_0, b'''_0) \approx (b'_1, b'''_1) \approx \cdots \approx (b'_\ell, b'''_\ell).$$

By induction, we construct elements $b''_0 = b''$, b''_1, ..., b''_ℓ such that

$$(b', b'') = (b'_0, b''_0) \approx (b'_1, b''_1) \approx \cdots \approx (b'_\ell, b''_\ell)$$

and

$$(b'', b''') = (b''_0, b'''_0) \approx (b''_1, b'''_1) \approx \cdots \approx (b''_\ell, b'''_\ell).$$

More precisely, assuming that b''_{k-1} is constructed, the assumption $b' \leq_{\mathrm{str}} b''$ and $b'' \leq_{\mathrm{str}} b'''$ implies that $\varphi_i(b'_{k-1}) \leq \varphi_i(b''_{k-1}) \leq \varphi_i(b'''_{k-1})$ for all $i \in I$. Now:

- If $\varphi_i(b'_{k-1}) = \varphi_i(b'''_{k-1})$ and $(b'_k, b'''_k) = (\tilde{e}_i b'_{k-1}, \tilde{e}_i b'''_{k-1})$, then we note that $\varphi_i(b'_{k-1}) = \varphi_i(b''_{k-1}) = \varphi_i(b'''_{k-1})$, and we set $b''_k = \tilde{e}_i b''_{k-1}$.
- If $\varphi_i(b'_{k-1}) = \varphi_i(b'''_{k-1}) > 0$ and $(b'_k, b'''_k) = \big(\tilde{f}_i b'_{k-1}, \tilde{f}_i b'''_{k-1}\big)$, then we note that $\varphi_i(b'_{k-1}) = \varphi_i(b''_{k-1}) = \varphi_i(b'''_{k-1}) > 0$ and we set $b''_k = \tilde{f}_i b''_{k-1}$.
- If $(b'_k, b'''_k) = (\sigma(b'_{k-1}), \sigma(b'''_{k-1}))$, then we simply set $b''_k = \sigma(b''_{k-1})$.

Lastly, we note that if $b' \leq_{\mathrm{str}} b''$, then $\Phi_{\mathbf{i}}(b') \leq_{\mathrm{lex}} \Phi_{\mathbf{i}}(b'')$ for any sequence $\mathbf{i}$ of elements of I. If in addition $b'' \leq_{\mathrm{str}} b'$, then $\Phi_{\mathbf{i}}(b') = \Phi_{\mathbf{i}}(b'')$. The antisymmetry of $\leq_{\mathrm{str}}$ thus follows from the fact that the map $\Phi_{\mathbf{i}}$ separates the points of $B(-\infty)$, provided $\mathbf{i}$ has been chosen long enough (see the proof of Theorem 2.4). Assertion (i) is proved.

Assertion (ii) follows from arguments similar to those used in the proof of Proposition 2.3, the key observation being that given two bases of canonical type $\mathbf{B}'$ and $\mathbf{B}''$, for any $(b', c') \in (\mathbf{B}')^2$ and $(b'', c'') \in (\mathbf{B}'')^2$,

$$(b', b'') \approx (c', c'') \implies \langle (b'')^*, b' \rangle = \langle (c'')^*, c' \rangle.$$

□

Examples 2.7. (i) If the Cartan matrix A is of type A_3, then the order $\leq_{\mathrm{str}}$ is trivial: for any b' and b'' in $B(-\infty)$, the condition $b' \leq_{\mathrm{str}} b''$ implies $b' = b''$. This readily follows from Lemma 10.2 in [8] by induction on the weight.

Combining this result with Proposition 2.6 (ii), we see that in type A_3, $\mathbf{f}$ has only one basis of canonical type. (A similar uniqueness assertion had been obtained by Berenstein and Zelevinsky for string bases, see [8], Theorem 9.1.)

Similar results hold in type A_1 or A_2.

(ii) Consider now the type A_4 with the usual numbering of the vertices of the Dynkin diagram. One can check that for the elements

$$b' = \tilde{e}_3^4 \tilde{e}_2^4 \tilde{e}_1^4 \tilde{e}_4^4 \tilde{e}_3^4 \tilde{e}_2^4 1 \quad \text{and} \quad b'' = \tilde{e}_2 \tilde{e}_1^3 \tilde{e}_4 \tilde{e}_3^7 \tilde{e}_2^7 \tilde{e}_1 \tilde{e}_4^3 \tilde{e}_3 1,$$

one has $\varphi_i(b') < \varphi_i(b'')$ and $\varphi_i(\sigma(b')) < \varphi_i(\sigma(b''))$ for each $i \in I$; it follows that $b' <_{\mathrm{str}} b''$.

(iii) As a last example, we consider the type A_r. The word

$$\mathbf{i} = (1, 2, 1, 3, 2, 1, \ldots, r, r-1, \ldots, 2, 1)$$

is a reduced decomposition of the longest element in the Weyl group of $\mathfrak{g}$. The algebra $\mathbb{Q}[N]$ is a cluster algebra, and from the datum of $\mathbf{i}$, one can construct a seed in $\mathbb{Q}[N]$ by the process described in [17], Theorem 13.2. The cluster monomials built from this seed belong to the dual semicanonical basis ([17], Theorem 16.1), so are naturally indexed by a subset $C \subseteq B(-\infty)$. One can show that any element $b \in C$ is minimal w.r.t. the order $\leq_{\mathrm{str}}$. By Proposition 2.6 (ii), this minimality implies that the element indexed by b in the dual of a basis of canonical type is independent of the choice of this basis. In other words, the cluster monomials attached to our seed belong to the dual of any basis of canonical type. (Reineke had shown that they belong to the dual of the canonical basis, see Theorem 6.1 in [42].)

3 Inclusion of MV polytopes

In this section, the Cartan matrix A is supposed to be of finite type.

3.1 Lusztig data

Let W be the Weyl group of $\mathfrak{g}$; it acts on $\mathfrak{h}$ and on $\mathfrak{h}^*$ and is generated by the simple reflections s_i, for $i \in I$. We denote by $(\omega_i^\vee)_{i\in I}$ the basis of $\mathfrak{h}$ dual to the basis $(\alpha_i)_{i\in I}$ of $\mathfrak{h}^*$. A coweight $\gamma \in \mathfrak{h}$ is said to be a chamber coweight if it is W-conjugated to a $\omega_i^\vee$; we denote by Γ the set of all chamber coweights.

Let N be the number of positive roots; this is also the length of the longest element w_0 in W. We denote by $\mathscr{X}$ the set of all sequences $\mathbf{i} = (i_1, \ldots, i_N)$ such that $w_0 = s_{i_1} \cdots s_{i_N}$. An element $\mathbf{i} \in \mathscr{X}$ defines a sequence (β_k) of positive roots and a sequence (γ_k) of chamber coweights, for $1 \leq k \leq N$, as follows:

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}, \qquad \gamma_k = -s_{i_1} \cdots s_{i_k} \omega_{i_k}^\vee.$$

It is well known that (β_k) is an enumeration of the positive roots. One easily checks that $\langle \gamma_k, \beta_\ell \rangle$ is nonnegative if $k \geq \ell$, nonpositive if $k < \ell$, and is 1 if $k = \ell$.

Let v be an indeterminate. Let (d_i) be a family of positive integers such that the matrix $(d_i a_{i,j})$ is symmetric. For $n \in \mathbb{N}$ and $i \in I$, we set $[n]_i = (v^{d_i n} - v^{-d_i n})/(v^{d_i} - v^{-d_i})$ and $[n]_i! = [1]_i \cdots [n]_i$. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$; it is a $\mathbb{Q}(v)$-algebra generated by elements E_i, F_i and K_i, for $i \in I$, see for instance [9], Section 3.1. We define the divided powers of E_i by $E_i^{(n)} = E_i^n/[n]_i!$. Let $U_q(\mathfrak{n}_+)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements E_i, and let $x \mapsto \overline{x}$ be the automorphism of $\mathbb{Q}$-algebra of $U_q(\mathfrak{n}_+)$ such that $\overline{v} = v^{-1}$ and $\overline{E_i} = E_i$.

Let T_i be the automorphism of $U_q(\mathfrak{g})$ constructed by Lusztig and denoted by $T''_{i,-1}$ in [36]. For a fixed $\mathbf{i} \in \mathscr{X}$, it is known that when $\mathbf{n} = (n_1, \ldots, n_N)$ runs over $\mathbb{N}^N$, the monomials

$$E_{\mathbf{i}}^{(\mathbf{n})} = E_{i_1}^{(n_1)} \, T_{i_1}\big(E_{i_2}^{(n_2)}\big) \; \cdots \; (T_{i_1} \cdots T_{i_{N-1}})\big(E_{i_N}^{(n_N)}\big)$$

form a PBW basis of $U_q(\mathfrak{n}_+)$. In addition, for each $\mathbf{n} \in \mathbb{N}^N$, there is a unique bar-invariant element

$$b_{\mathbf{i}}(\mathbf{n}) = \sum_{\mathbf{m} \in \mathbb{N}^N} \zeta_{\mathbf{m}}^{\mathbf{n}} E_{\mathbf{i}}^{(\mathbf{m})}, \tag{2}$$

with $\zeta_{\mathbf{n}}^{\mathbf{n}} = 1$ and $\zeta_{\mathbf{m}}^{\mathbf{n}} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $\mathbf{m} \neq \mathbf{n}$. These elements $b_{\mathbf{i}}(\mathbf{n})$ form the canonical basis of $U_q(\mathfrak{n}_+)$, which does not depend on the choice of $\mathbf{i}$ [35].

After specialization at $v = 1$ and under the isomorphism $\mathbf{f} \cong U(\mathfrak{n}_+)$, this construction gives a basis of canonical type of $\mathbf{f}$. The map $\mathbf{n} \mapsto b_{\mathbf{i}}(\mathbf{n})$ can thus be regarded as a parameterization of $B(-\infty)$ by $\mathbb{N}^N$. The inverse bijection $B(-\infty) \to \mathbb{N}^N$ is called Lusztig datum in direction $\mathbf{i}$; we denote it by $b \mapsto \mathbf{n}_{\mathbf{i}}(b)$.

3.2 MV polytopes

To each $b \in B(-\infty)$, one associates its MV polytope $\mathrm{Pol}(b)$; this is a convex polytope in $\mathfrak{h}^*$, whose vertices belong to $Q_+ \cap (\mathrm{wt}(b) - Q_+)$. Moreover, the map $b \mapsto \mathrm{Pol}(b)$ is injective.

The polytope $\mathrm{Pol}(b)$ can be constructed in several ways: as the image by a moment map of a certain projective variety, called a Mirković-Vilonen cycle (whence the name 'MV polytope') [1, 21, 22], or as the Harder-Narasimhan polytope of a general representation of the preprojective algebra built from the Dynkin diagram of $\mathfrak{g}$ [4].

For our purpose however, the most relevant definition of $\mathrm{Pol}(b)$ uses the notion of Lusztig datum. Let $\mathbf{i} \in \mathscr{X}$. As before, we associate to $\mathbf{i}$ an enumeration (β_k) of the positive roots of $\mathfrak{g}$. Let $(n_1, \ldots, n_N) = \mathbf{n}_{\mathbf{i}}(b)$ be the Lusztig datum of b in direction $\mathbf{i}$. At first sight, the weight

$$\mathrm{wt}(b) - \sum_{t=1}^{k} n_t \beta_t$$

depends on $\mathbf{i}$ and k. One can however show that it depends only on $w = s_{i_1} \cdots s_{i_k}$, so it is legitimate to denote it by $\mu_w(b)$. Then $\mathrm{Pol}(b)$ can be defined as the convex hull of the weights $\mu_w(b)$, for all $w \in W$.

By [22], the normal fan of $\mathrm{Pol}(b)$ is a coarsening of the Weyl fan in $\mathfrak{h}$. The weight $\mu_w(b)$ is a vertex of $\mathrm{Pol}(b)$ and the normal cone to $\mathrm{Pol}(b)$ at $\mu_w(b)$ is wC_0, where

$$C_0 = \{\theta \in \mathfrak{h} \mid \forall i \in I,\ \langle \theta, \alpha_i \rangle > 0\}$$

is the dominant chamber. In particular, the datum of $\mathrm{Pol}(b)$ determines the weights $\mu_w(b)$.

The polytope $\mathrm{Pol}(b)$ can also be described by its facets. Specifically, one defines $M_\gamma(b)$ for each chamber coweight $\gamma \in \Gamma$ by the formula

$$M_{w\omega_i^\vee}(b) = \langle w\omega_i^\vee, \mu_w(b) \rangle$$

(one can check that the right-hand side depends only $w\omega_i^\vee$), and one has

$$M_{w\omega_i^\vee}(b) = \sup(w\omega_i^\vee)(\mathrm{Pol}(b)) \quad \text{and} \quad \mathrm{Pol}(b) = \{x \in \mathfrak{h}^* \mid \forall \gamma \in \Gamma,\ \langle \gamma, x \rangle \leq M_\gamma(b)\}.$$

The collection $(M_\gamma(b))$ is called the BZ datum of b.

3.3 The polytope order on $B(-\infty)$

Given $(b', b'') \in B(-\infty)^2$, we write $b' \leq_{\rm pol} b''$ if b' and b'' have the same weight and if $\operatorname{Pol}(b') \subseteq \operatorname{Pol}(b'')$. Obviously, the inclusion between the polytopes is equivalent to the set of equations $M_\gamma(b') \leq M_\gamma(b'')$, for all $\gamma \in \Gamma$. The relation $\leq_{\rm pol}$ is an order on $B(-\infty)$, because the map $b \mapsto \operatorname{Pol}(b)$ is injective.

This order can also be directly expressed in terms of Lusztig data. Specifically, let $\mathbf{i} \in \mathscr{X}$ and define the sequences of positive roots (β_k) and chamber coweights (γ_k) as in Section 3.1. For $\mathbf{n} = (n_1, \ldots, n_N)$ in $\mathbb{N}^N$, we set $|\mathbf{n}| = n_1\beta_1 + \cdots + n_N\beta_N$; this is the weight of the PBW monomial $E_{\mathbf{i}}^{(\mathbf{n})}$. Given another element $\mathbf{m} = (m_1, \ldots, m_N)$ in $\mathbb{N}^N$, we write $\mathbf{n} \leq_{\mathbf{i}} \mathbf{m}$ if $|\mathbf{m}| = |\mathbf{n}|$ and if

$$\sum_{t=1}^{k} \langle \gamma_k, \beta_t \rangle n_t \leq \sum_{t=1}^{k} \langle \gamma_k, \beta_t \rangle m_t$$

for each $k \in \{1, \ldots, N\}$. The relation $\leq_{\mathbf{i}}$ is an order on $\mathbb{N}^N$, less fine than the lexicographic order.

Proposition 3.1 *Let $(b', b'') \in B(-\infty)^2$. Then $b' \leq_{\rm pol} b''$ if and only if $\mathbf{n_i}(b') \leq_{\mathbf{i}} \mathbf{n_i}(b'')$ for all $\mathbf{i} \in \mathscr{X}$.*

Proof. Let $b \in B(-\infty)$ and let $\mathbf{i} \in \mathscr{X}$. As in Section 3.1, the word $\mathbf{i}$ defines a sequence (β_k) of positive roots and a sequence (γ_k) of chamber coweights. Write $(n_1, \ldots, n_N) = \mathbf{n_i}(b)$. Then for each $k \in \{1, \ldots, N\}$, we have

$$\sum_{t=1}^{k} \langle \gamma_k, \beta_t \rangle n_t - \langle \gamma_k, \operatorname{wt}(b) \rangle = \langle w\omega_{i_k}^\vee, \mu_w(b) \rangle = M_{w\omega_{i_k}^\vee}(b),$$

where $w = s_{i_1} \cdots s_{i_k}$.

Given $(b', b'') \in B(-\infty)^2$, the inequality $\mathbf{n_i}(b') \leq_{\mathbf{i}} \mathbf{n_i}(b'')$ is thus equivalent to $\operatorname{wt}(b') = \operatorname{wt}(b'')$ and $M_\gamma(b') \leq M_\gamma(b'')$ for all $\gamma \in \{s_{i_1} \cdots s_{i_k}\omega_{i_k}^\vee \mid 1 \leq k \leq N\}$. The lemma now follows from the fact that every chamber coweight can be written as $s_{i_1} \cdots s_{i_k}\omega_{i_k}^\vee$ for suitable $\mathbf{i}$ and k. □

Example 3.2. In type A_3, consider the elements $b' = (\tilde{e}_1\tilde{e}_3)\tilde{e}_2^2(\tilde{e}_1\tilde{e}_3)1$ and $b'' = \tilde{e}_2(\tilde{e}_1\tilde{e}_3)^2\tilde{e}_21$. Both elements are fixed by the involution σ. The polytope $\operatorname{Pol}(b')$ is the convex hull of

$$\{0, \alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3,$$
$$\alpha_1 + \alpha_2 + 2\alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, 2\alpha_1 + 2\alpha_2 + 2\alpha_3\}.$$

The polytope $\operatorname{Pol}(b'')$ is the convex hull of

$$\{0, \alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_3, 2\alpha_1 + \alpha_2, \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3,$$
$$2\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, 2\alpha_1 + 2\alpha_2 + 2\alpha_3\}.$$

From there, one easily checks that $\operatorname{Pol}(b') \subset \operatorname{Pol}(b'')$. Since b' and b'' have the same weight, we conclude that $b' \leq_{\rm pol} b''$. This example seems to have been first observed by Kamnitzer.

Remark 3.3. For any $b \in B(-\infty)$, the MV polytope $\mathrm{Pol}(\sigma(b))$ is the image of $\mathrm{Pol}(b)$ by the involution $x \mapsto \mathrm{wt}(b) - x$ ([21], Theorem 6.2). In addition, $\varphi_i(b)$ is the first component of $\mathbf{n_i}(b)$ whenever $\mathbf{i}$ begins by i.

It follows that $b' \leq_{\mathrm{pol}} b''$ if and only if $\sigma(b') \leq_{\mathrm{pol}} \sigma(b'')$, and that if $b' \leq_{\mathrm{pol}} b''$, then $\varphi_i(b') \leq \varphi_i(b'')$ for all $i \in I$. The order $\leq_{\mathrm{pol}}$ has thus some remote parentage with $\leq_{\mathrm{str}}$.

One can in fact define another order, weaker than $\leq_{\mathrm{str}}$ and $\leq_{\mathrm{pol}}$, in the following way: $b' \leq b''$ if b' and b'' have the same weight and if for any finite sequence of elementary moves

$$(b', b'') = (b'_0, b''_0) \approx (b'_1, b''_1) \approx \cdots \approx (b'_\ell, b''_\ell),$$

one has $b'_\ell \leq_{\mathrm{pol}} b''_\ell$. We believe that the order $\leq$ is trivial in type A_4; indeed, with the help of a computer running GAP [15, 13], we checked that in type A_4, the order $\leq$ is trivial in weights up to $10\alpha_1 + 16\alpha_2 + 16\alpha_3 + 10\alpha_4$.

3.4 Comparison between the canonical basis and PBW bases

We come back to the equation (2) that defines the coefficients of the transition matrix between the canonical basis and a PBW basis. Our aim in this section is to obtain a necessary condition so that $\zeta^{\mathbf{n}}_{\mathbf{m}} \neq 0$. Our result (Corollary 3.6) generalizes Theorem 9.13 (a) in [35] to arbitrary words $\mathbf{i} \in \mathscr{X}$, not necessarily compatible with a quiver orientation; it complements Proposition 5.1 in [12] and Theorem 3.13 (ii) in [6].

We choose $\mathbf{i} \in \mathscr{X}$. Then the PBW monomials $E^{(\mathbf{n})}_{\mathbf{i}}$ form a basis of $\mathbf{f}$. We can also consider the partial order $\leq_{\mathbf{i}}$ on $\mathbb{N}^N$.

Lemma 3.4 *Let $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{n}$ in $\mathbb{N}^N$ be such that $|\mathbf{m}_1| + |\mathbf{m}_2| = |\mathbf{n}|$. Let $\mathbf{p} \in \mathbb{N}^N$ be such that $E^{(\mathbf{p})}_{\mathbf{i}}$ appears with a nonzero coefficient in the expansion of $E^{(\mathbf{m}_1)}_{\mathbf{i}} E^{(\mathbf{m}_2)}_{\mathbf{i}}$ on the PBW basis. Let $0 \leq k \leq N$ and define $\mathbf{n}_1 \in \mathbb{N}^k \times \{0\}^{N-k}$ and $\mathbf{n}_2 \in \{0\}^k \times \mathbb{N}^{N-k}$ so that $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$. Then*

$$\big(\mathbf{m}_1 \geq_{\mathbf{i}} \mathbf{n}_1 \quad \text{and} \quad \mathbf{m}_2 \geq_{\mathbf{i}} \mathbf{n}_2\big) \implies \mathbf{p} \geq_{\mathbf{i}} \mathbf{n}.$$

In addition, if one of the inequalities $\mathbf{m}_1 \geq_{\mathbf{i}} \mathbf{n}_1$ and $\mathbf{m}_2 \geq_{\mathbf{i}} \mathbf{n}_2$ is strict, then $\mathbf{p} >_{\mathbf{i}} \mathbf{n}$.

Proof. We have to check the inequality

$$\sum_{t=1}^{\ell} \langle \gamma_\ell, \beta_t \rangle n_t \leq \sum_{t=1}^{\ell} \langle \gamma_\ell, \beta_t \rangle p_t \tag{3}$$

for all $\ell \in \{1, \ldots, N\}$. We decompose each element $\mathbf{q} \in \mathbb{N}^N$ as a sum $\mathbf{q}' + \mathbf{q}''$, with $\mathbf{q}' \in \mathbb{N}^\ell \times \{0\}^{N-\ell}$ and $\mathbf{q}'' \in \{0\}^\ell \times \mathbb{N}^{N-\ell}$; we then have $E^{(\mathbf{q})}_{\mathbf{i}} = E^{(\mathbf{q}')}_{\mathbf{i}} E^{(\mathbf{q}'')}_{\mathbf{i}}$. With this convention, $E^{(\mathbf{p})}_{\mathbf{i}}$ appears in the expansion of $E^{(\mathbf{m}'_1)}_{\mathbf{i}} E^{(\mathbf{m}''_1)}_{\mathbf{i}} E^{(\mathbf{m}'_2)}_{\mathbf{i}} E^{(\mathbf{m}''_2)}_{\mathbf{i}}$. There is thus $\mathbf{q} \in \mathbb{N}^N$ such that $E^{(\mathbf{q})}_{\mathbf{i}}$ appears in the expansion of $E^{(\mathbf{m}''_1)}_{\mathbf{i}} E^{(\mathbf{m}'_2)}_{\mathbf{i}}$ and $E^{(\mathbf{p})}_{\mathbf{i}}$ appears in the expansion of $E^{(\mathbf{m}'_1)}_{\mathbf{i}} E^{(\mathbf{q}')}_{\mathbf{i}} E^{(\mathbf{q}'')}_{\mathbf{i}} E^{(\mathbf{m}''_2)}_{\mathbf{i}}$. The convexity property of PBW bases (Lemma 1 in [33], Proposition 7 in [5], or Theorem 2.3 in [51]) imply then that $|\mathbf{p}'| = |\mathbf{m}'_1| + |\mathbf{q}'|$ and $|\mathbf{p}''| = |\mathbf{q}''| + |\mathbf{m}''_2|$.

Assume first that $\ell \leq k$. Since $\mathbf{n}_1 \leq_{\mathbf{i}} \mathbf{m}_1$, we have

$$\langle \gamma_\ell, |\mathbf{n}_1'| \rangle \leq \langle \gamma_\ell, |\mathbf{m}_1'| \rangle.$$

A fortiori,

$$\langle \gamma_\ell, |\mathbf{n}'| \rangle = \langle \gamma_\ell, |\mathbf{n}_1'| \rangle \leq \langle \gamma_\ell, |\mathbf{m}_1'| + |\mathbf{q}'| \rangle = \langle \gamma_\ell, |\mathbf{p}'| \rangle,$$

which is exactly (3).

Now assume that $\ell > k$. Since $\mathbf{n}_2 \leq_{\mathbf{i}} \mathbf{m}_2$, we have

$$\langle \gamma_\ell, |\mathbf{n}_2'| \rangle \leq \langle \gamma_\ell, |\mathbf{m}_2'| \rangle.$$

Using $|\mathbf{m}_2| = |\mathbf{n}_2|$ and $|\mathbf{p}''| = |\mathbf{q}''| + |\mathbf{m}_2''|$, we get

$$\langle \gamma_\ell, |\mathbf{n}''| \rangle = \langle \gamma_\ell, |\mathbf{n}_2''| \rangle \geq \langle \gamma_\ell, |\mathbf{m}_2''| \rangle \geq \langle \gamma_\ell, |\mathbf{p}''| \rangle.$$

This last relation is equivalent to (3), because $|\mathbf{n}| = |\mathbf{p}|$. □

We define coefficients $\omega_{\mathbf{m}}^{\mathbf{n}} \in \mathbb{Q}(q)$ by the expansion

$$\overline{E_{\mathbf{i}}^{(\mathbf{n})}} = \sum_{\mathbf{m} \in \mathbb{N}^N} \omega_{\mathbf{m}}^{\mathbf{n}} E_{\mathbf{i}}^{(\mathbf{m})}$$

on the PBW basis. It is known that these coefficients actually belong to $\mathbb{Z}[v, v^{-1}]$. The following proposition generalizes equations 9.12 (a) and (b) in [35].

Proposition 3.5 *We have $\omega_{\mathbf{n}}^{\mathbf{n}} = 1$; moreover, $\omega_{\mathbf{m}}^{\mathbf{n}} \neq 0$ implies $\mathbf{m} \geq_{\mathbf{i}} \mathbf{n}$.*

Proof. Let $\mathbf{n} \in \mathbb{N}^N$. Suppose first that $\mathbf{n}$ has several nonzero entries. We can then find $k \in \{1, \ldots, N-1\}$ such that one of the first k entries and one of the last $N-k$ entries of $\mathbf{n}$ is nonzero. Writing $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$ with $\mathbf{n}_1 \in \mathbb{N}^k \times \{0\}^{N-k}$ and $\mathbf{n}_2 \in \{0\}^k \times \mathbb{N}^{N-k}$, we obviously have $E_{\mathbf{i}}^{(\mathbf{n})} = E_{\mathbf{i}}^{(\mathbf{n}_1)} E_{\mathbf{i}}^{(\mathbf{n}_2)}$ and $\overline{E_{\mathbf{i}}^{(\mathbf{n})}} = \overline{E_{\mathbf{i}}^{(\mathbf{n}_1)}}\, \overline{E_{\mathbf{i}}^{(\mathbf{n}_2)}}$. Using Lemma 3.4, we can then easily conclude by induction on $|\mathbf{n}|$.

Now suppose that $\mathbf{n}$ has a single nonzero entry. Then $\mathbf{n}$ is the smallest element of its weight in $\mathbb{N}^N$. We can thus write

$$b_{\mathbf{i}}(\mathbf{n}) = E_{\mathbf{i}}^{(\mathbf{n})} + \sum_{\mathbf{m} >_{\mathbf{i}} \mathbf{n}} \zeta_{\mathbf{m}}^{\mathbf{n}} E_{\mathbf{i}}^{(\mathbf{m})},$$

and therefore

$$\overline{E_{\mathbf{i}}^{(\mathbf{n})}} = E_{\mathbf{i}}^{(\mathbf{n})} + \sum_{\mathbf{m} >_{\mathbf{i}} \mathbf{n}} \left(\zeta_{\mathbf{m}}^{\mathbf{n}} E_{\mathbf{i}}^{(\mathbf{m})} - \overline{\zeta_{\mathbf{m}}^{\mathbf{n}} E_{\mathbf{i}}^{(\mathbf{m})}} \right).$$

The first case of our reasoning shows that only monomials $E_{\mathbf{i}}^{(\mathbf{p})}$ with $\mathbf{p} \geq_{\mathbf{i}} \mathbf{m} >_{\mathbf{i}} \mathbf{n}$ appear in the expansion of $\overline{E_{\mathbf{i}}^{(\mathbf{m})}}$, which concludes the proof. □

The transition matrix $(\zeta_{\mathbf{m}}^{\mathbf{n}})$ between the PBW basis and the canonical basis can be computed from the matrix $(\omega_{\mathbf{m}}^{\mathbf{n}})$ by the Kazhdan-Lusztig algorithm ([35], Section 7.11). Proposition 3.5 then implies:

Corollary 3.6 *The coefficients of the transition matrix between the PBW basis and the canonical basis satisfy*

$$\zeta_{\mathbf{m}}^{\mathbf{n}} \neq 0 \implies \mathbf{m} \geq_{\mathbf{i}} \mathbf{n}.$$

4 Further examples

We now consider the case where the generalized Cartan matrix A is symmetric. Using representations of quivers, Lusztig constructed two bases of canonical type of $\mathbf{f}$, which he called the canonical and the semicanonical bases. In this section, we study what information the orders $\leq_{\mathrm{str}}$ and $\leq_{\mathrm{pol}}$ convey about these constructions. Our main results are Proposition 4.1 and Theorem 4.4.

4.1 Background on Lusztig's constructions

We adopt the notation of [36]. The Dynkin diagram is a graph with vertex set I; between two vertices i and j, there are $-a_{i,j}$ edges. Since the graph is without loops, each edge has two endpoints. Orienting an edge is recognizing one of these endpoints as the tail and the other one as the head. We denote by H the set of oriented edges; the tail of $h \in H$ is denoted by h' and its head by h''. In addition, there is a fixed point free involution $h \mapsto \overline{h}$ that exchanges tails and heads. An orientation of our graph is a subset $\Omega \subseteq H$ such that $(\Omega, \overline{\Omega})$ is a partition of H.

A dimension-vector is a function $\nu \in \mathbb{N}^I$; we identify such a ν with the weight $\sum_{i\in I} \nu(i)\alpha_i$ in Q_+. Given a dimension-vector ν, we denote by S_ν the set of all pairs consisting of a sequence $\mathbf{i} = (i_1, \ldots, i_m)$ of elements of I and a sequence $\mathbf{a} = (a_1, \ldots a_m)$ of natural numbers such that $\nu = \sum_{k=1}^m a_k \alpha_{i_k}$. Each $(\mathbf{i}, \mathbf{a}) \in S_\nu$ defines an element $\Theta_{\mathbf{i}}^{(\mathbf{a})} = \theta_{i_1}^{(a_1)} \cdots \theta_{i_m}^{(a_m)}$ in $\mathbf{f}_\nu$.

Let $\mathbf{k}$ be an algebraically closed field. To an I-graded $\mathbf{k}$-vector space $\mathbf{V} = \bigoplus_{i\in I} \mathbf{V}_i$, we associate its dimension-vector $\nu : i \mapsto \dim \mathbf{V}_i$. Given $\mathbf{V}$, let $G_{\mathbf{V}} = \prod_{i\in I} \mathbf{GL}(\mathbf{V}_i)$ and let

$$\mathbf{E_V} = \bigoplus_{h\in H} \mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}).$$

The group $G_{\mathbf{V}}$ acts on the vector space $\mathbf{E_V}$ in a natural fashion. An element $x = (x_h)$ in $\mathbf{E_V}$ is said to be nilpotent if there exists an $N \geq 1$ such that the composition $x_{h_N} \cdots x_{h_1} : \mathbf{V}_{h'_1} \to \mathbf{V}_{h''_N}$ is zero for each path $(h_1, \ldots, h_N)$ in the oriented graph. Lastly, given an orientation Ω, let $\mathbf{E}_{\mathbf{V},\Omega}$ be the subspace of $\mathbf{E_V}$ consisting of all vectors (x_h) such that $x_h = 0$ whenever $h \in H \setminus \Omega$.

Fix an I-graded $\mathbf{k}$-vector space $\mathbf{V}$ of dimension-vector ν and an orientation Ω. Let $(\mathbf{i}, \mathbf{a}) \in S_\nu$. By definition, a flag of type $(\mathbf{i}, \mathbf{a})$ is a decreasing filtration $\mathbf{V} = \mathbf{V}^0 \supseteq \mathbf{V}^1 \supseteq \cdots \supseteq \mathbf{V}^m = 0$ of I-graded vector spaces such that $\mathbf{V}^{k-1}/\mathbf{V}^k$ has dimension vector $a_k \alpha_{i_k}$. Let $\mathcal{F}_{\mathbf{i},\mathbf{a}}$ be the set of all flags of type $(\mathbf{i}, \mathbf{a})$ and let $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$ be the set of all pairs $(x, \mathbf{V}^\bullet) \in \mathbf{E}_{\mathbf{V},\Omega} \times \mathcal{F}_{\mathbf{i},\mathbf{a}}$ such that each $\mathbf{V}^k$ is stable by the action of x. Let $\pi_{\mathbf{i},\mathbf{a}} : \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}} \to \mathbf{E}_{\mathbf{V},\Omega}$ be the first projection and set $L_{\mathbf{i},\mathbf{a};\Omega} = (\pi_{\mathbf{i},\mathbf{a}})_! 1$, where 1 is the trivial local system on $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$. By the Decomposition Theorem, $L_{\mathbf{i},\mathbf{a};\Omega}$ is a semisimple complex.

Let $\mathcal{P}_{\mathbf{V},\Omega}$ be the set of isomorphism classes of simple perverse sheaves L such that $L[d]$ appears as a direct summand of the sheaf $L_{\mathbf{i},\mathbf{a};\Omega}$, for some $(\mathbf{i}, \mathbf{a}) \in S_\nu$ and some $d \in \mathbb{Z}$. Let $\mathcal{Q}_{\mathbf{V},\Omega}$ be the smallest full subcategory of the category of bounded complexes of constructible sheaves on $\mathbf{E}_{\mathbf{V},\Omega}$ that contains the sheaves $L_{\mathbf{i},\mathbf{a};\Omega}$ and that is stable by direct sums, direct summands, and shifts. Lastly, let $\mathcal{K}_{\mathbf{V},\Omega}$ be the abelian group with one generator (L) for each isomorphism class of object in $\mathcal{Q}_{\mathbf{V},\Omega}$, and with relations $(L[1]) = (L)$ and $(L) = (L') + (L'')$ whenever L is isomorphic to $L' \oplus L''$. (Thus our $\mathcal{K}_{\mathbf{V},\Omega}$ is the specialization at $v = 1$ of the $\mathcal{K}_{\mathbf{V},\Omega}$ defined in Section 10.1 of [36].)

Given $\nu \in Q_+$, the groups $\mathcal{K}_{\mathbf{V},\Omega}$, for $\mathbf{V}$ of dimension-vector ν, can be canonically identified. We thus obtain an abelian group $\mathcal{K}_{\nu,\Omega}$ equipped with isomorphisms $\mathcal{K}_{\nu,\Omega} \cong \mathcal{K}_{\mathbf{V},\Omega}$ for any $\mathbf{V}$ of dimension-vector ν. With this notation, $\mathcal{P}_{\mathbf{V},\Omega}$ gives rise to a $\mathbb{Z}$-basis of $\mathcal{K}_{\nu,\Omega}$ which does not depend on $\mathbf{V}$. We set $\mathcal{K}_\Omega = \bigoplus_{\nu \in Q_+} \mathcal{K}_{\nu,\Omega}$; this is a free $\mathbb{Z}$-module endowed with a canonical basis.

We endow $\mathcal{K}_\Omega$ with the structure of an associative Q_+-graded algebra and we define an isomorphism of algebras $\lambda_\Omega : \mathbf{f} \to \mathcal{K}_\Omega$ as in Sections 10.2 and 10.16 of [36]. Then $\lambda_\Omega\big(\Theta_{\mathbf{i}}^{(\mathbf{a})}\big) = (L_{\mathbf{i},\mathbf{a};\Omega})$ for each $(\mathbf{i}, \mathbf{a}) \in S_\nu$.

By Theorem 10.17 in [36], the inverse image of the canonical basis of $\mathcal{K}_\Omega$ by λ_Ω does not depend on Ω. This inverse image is called the canonical basis of $\mathbf{f}$; it is a basis of canonical type. By Section 2.4, there is thus a unique isomorphism of crystals from $B(-\infty)$ onto the canonical basis; following Kashiwara, we denote this isomorphism by G. In the sequel, given $b \in B(-\infty)$ and $\mathbf{V}$ of dimension-vector $\mathrm{wt}(b)$, we denote by $L_{b,\Omega}$ the element in $\mathcal{P}_{\mathbf{V},\Omega}$ such that $(L_{b,\Omega}) = \lambda_\Omega(G(b))$ in $\mathcal{K}_{\mathbf{V},\Omega}$.

For an orientation Ω and for $h \in H$, we set $\varepsilon_\Omega(h) = 1$ if $h \in \Omega$ and $\varepsilon_\Omega(h) = -1$ otherwise. Given an I-graded $\mathbf{k}$-vector space $\mathbf{V}$, let $\Lambda_{\mathbf{V},\Omega}$ be the set of all nilpotent elements $x = (x_h)$ in $\mathbf{E}_\mathbf{V}$ such that for each $i \in I$,

$$\sum_{\substack{h \in H \\ h''=i}} \varepsilon_\Omega(h) x_h x_{\overline{h}} = 0.$$

This set $\Lambda_{\mathbf{V},\Omega}$ is called the nilpotent variety.

Up to a canonical bijection, the set of irreducible components of $\Lambda_{\mathbf{V},\Omega}$ depends only on the dimension-vector ν of $\mathbf{V}$, and not on $\mathbf{V}$ or Ω ([36], Sections 12.14 and 12.15). We denote this set by $\mathbf{Z}_\nu$ and we set $\mathbf{Z} = \bigsqcup_{\nu \in Q_+} \mathbf{Z}_\nu$. Then $\mathbf{Z}$ can be endowed with the structure of a crystal isomorphic to $B(-\infty)$ ([28], Theorem 5.3.2). Given $b \in B(-\infty)$ and $\mathbf{V}$ of dimension-vector $\mathrm{wt}(b)$, we denote by $\Lambda_{b,\Omega}$ the irreducible component of $\Lambda_{\mathbf{V},\Omega}$ that corresponds to b.

Up to a canonical isomorphism, the space of $\mathbb{Q}$-valued, $G_\mathbf{V}$-invariant, constructible fonctions on $\Lambda_{\mathbf{V},\Omega}$ depends only on the dimension-vector ν of $\mathbf{V}$; we denote it by $\widetilde{M}(\nu)$. In Section 12 of [36], Lusztig endows $\widetilde{M} = \bigoplus_{\nu \in Q_+} \widetilde{M}(\nu)$ with the structure of an algebra and constructs an injective homomorphism $\kappa : \mathbf{f} \to \widetilde{M}$ (this morphism is denoted by γ in [36] and by κ in [39]). The map κ is defined so that for each $(\mathbf{i}, \mathbf{a}) \in S_\nu$, the value of $\kappa\big(\Theta_{\mathbf{i}}^{(\mathbf{a})}\big)$ at a point $x \in \Lambda_{\mathbf{V},\Omega}$ is the Euler characteristic of the set of all x-stable flags of type $(\mathbf{i}, \mathbf{a})$ in $\mathbf{V}$.

Each irreducible component X of $\Lambda_{\mathbf{V},\Omega}$ contains a dense open subset X_0 such that any function in $\kappa(\mathbf{f}_\nu)$ is constant on X_0. We denote by $\delta_X : \mathbf{f}_\nu \to \mathbb{Q}$ the linear form obtained by composing κ with the evaluation at a point of X_0. By Section 12.14 in [36] or by Theorem 2.7 in [39], the elements δ_X, for $X \in \mathbf{Z}_\nu$, form a basis of the dual of $\mathbf{f}_\nu$. Gathering the corresponding dual bases in $\mathbf{f}_\nu$ for all $\nu \in Q_+$, we get a basis of $\mathbf{f}$. This is the semicanonical basis, and it is of canonical type. There is thus a unique isomorphism of crystals from $B(-\infty)$ onto the semicanonical basis; we denote this isomorphism by S. Comparing the constructions in [28] and in [39], one checks that for any $b \in B(-\infty)$, the dual vector to $S(b)$ is the δ_X with $X = \Lambda_{b,\Omega}$. In other words, the indexations of the semicanonical basis and of $\mathbf{Z}$ by $B(-\infty)$ agree.

4.2 Condition for singular supports

In the context of Section 4.1, the trace duality allows us to regard $\mathbf{E_V}$ as the cotangent space of $\mathbf{E}_{\mathbf{V},\Omega}$. Lusztig proved ([36], Corollary 13.6) that the singular support of a complex in $\mathscr{Q}_{\mathbf{V},\Omega}$ is the union of irreducible components of $\Lambda_{\mathbf{V},\Omega}$. In this context, Kashiwara and Saito ([28], Lemma 8.2.1) proved that

$$\Lambda_{b'',\Omega} \subseteq SS(L_{b',\Omega}) \implies b' \leq_{\mathrm{str}} b''.$$

4.3 Degeneracy order between quiver representations

We continue with the case where A is a symmetric Cartan matrix and assume in addition that A is of finite type. We fix $\nu \in Q_+$ and an I-graded $\mathbf{k}$-vector space $\mathbf{V}$ of dimension-vector ν. For each orientation Ω and each element $b \in B(-\infty)$ of weight ν, there is a $G_{\mathbf{V}}$-orbit $\mathscr{O}_{b,\Omega} \subseteq \mathbf{E}_{\mathbf{V},\Omega}$ such that the perverse sheaf $L_{b,\Omega}$ is the intersection cohomology sheaf on $\overline{\mathscr{O}_{b,\Omega}}$ w.r.t. the trivial local system on $\mathscr{O}_{b,\Omega}$.

Proposition 4.1 *Let b' and b'' be two elements of weight ν in $B(-\infty)$. If $b' \leq_{\mathrm{pol}} b''$, then $\overline{\mathscr{O}_{b',\Omega}} \supseteq \mathscr{O}_{b'',\Omega}$ for all orientations Ω.*

Proof. Fix an orientation Ω. For μ and ν in Q_+, define

$$\langle \mu, \nu \rangle_\Omega = \sum_{i \in I} \mu_i \nu_i - \sum_{h \in \Omega} \mu_{h'} \nu_{h''}.$$

The oriented graph $Q = (I, \Omega)$ is a Dynkin quiver. Given a positive root α, we denote by $M(\alpha)$ the indecomposable $\mathbf{k}Q$-module of dimension-vector α. Ringel ([45], p. 59) has shown that

$$\begin{aligned} \dim \operatorname{Hom}_{\mathbf{k}Q}(M(\alpha), M(\beta)) &= \max\bigl(0, \langle \alpha, \beta \rangle_\Omega\bigr), \\ \dim \operatorname{Ext}^1_{\mathbf{k}Q}(M(\alpha), M(\beta)) &= \max\bigl(0, -\langle \alpha, \beta \rangle_\Omega\bigr). \end{aligned} \tag{4}$$

Choose $\mathbf{i} \in \mathscr{X}$ adapted to Ω ([35], Section 4.7 and Proposition 4.12 (b)). As in Section 3.1, the word $\mathbf{i}$ defines a sequence (β_k) of positive roots and a sequence (γ_k) of chamber coweights. By Proposition 7.4 in [3], we have, for any $k \in \{1, \ldots, N\}$,

$$\langle \gamma_k, ? \rangle = \langle \beta_k, ? \rangle_{\overline{\Omega}} = \langle ?, \beta_k \rangle_\Omega.$$

It follows that for any k and ℓ in $\{1, \ldots, N\}$, we have

$$\dim \operatorname{Hom}_{\mathbf{k}Q}(M(\beta_\ell), M(\beta_k)) = \max\bigl(0, \langle \gamma_k, \beta_\ell \rangle\bigr) = \begin{cases} \langle \gamma_k, \beta_l \rangle & \text{if } k \geq \ell, \\ 0 & \text{if } k < \ell. \end{cases}$$

For $b \in B(-\infty)$ of weight ν, the $G_{\mathbf{V}}$-orbit $\mathscr{O}_{b,\Omega}$ is the set of all $x \in \mathbf{E}_{\mathbf{V},\Omega}$ such that

$$(\mathbf{V}, x) \cong M(\beta_1)^{\oplus n_1} \oplus \cdots \oplus M(\beta_N)^{\oplus n_N}$$

as $\mathbf{k}Q$-modules, where $(n_1, \ldots, n_N)$ is the Lusztig datum of b in direction $\mathbf{i}$ ([35], Sections 4.15–4.16).

Assume that $b' \leq_{\mathrm{pol}} b''$. We then have $\mathbf{n_i}(b') \leq_{\mathbf{i}} \mathbf{n_i}(b'')$. This inequality is equivalent to the fact that for each $k \in \{1, \ldots, N\}$,

$$\dim \mathrm{Hom}_{\mathbf{k}Q}((\mathbf{V}, x'), M(\beta_k)) \leq \dim \mathrm{Hom}_{\mathbf{k}Q}((\mathbf{V}, x''), M(\beta_k)),$$

where $x' \in \mathscr{O}_{b',\Omega}$ and $x'' \in \mathscr{O}_{b'',\Omega}$. The inclusion $\overline{\mathscr{O}_{b',\Omega}} \supseteq \mathscr{O}_{b'',\Omega}$ now follows from Riedtmann's criterion [44, 10]. □

Remark 4.2. (i) The converse statement to Proposition 4.1 is true in type A. This comes from the fact that in this case, any chamber coweight can be written as $s_{i_1} \cdots s_{i_k} \omega_{i_k}^{\vee}$, where $\mathbf{i} \in \mathscr{X}$ is compatible with an orientation (see the proof of Proposition A.2 in [4]).

(ii) Let us go back to the problem studied in Section 4.2. Let b' and b'' in $B(-\infty)$ have the same weight. Given an orientation Ω, it is known that $\Lambda_{b'',\Omega}$ is the closure of the conormal bundle to $\mathcal{O}_{b'',\Omega}$ (see [29], Section 5.3 or [3], Section 7.3). Therefore, in order that $\Lambda_{b'',\Omega} \subseteq SS(L_{b',\Omega})$, it is necessary that $\mathcal{O}_{b'',\Omega} \subseteq \overline{\mathcal{O}_{b',\Omega}}$. Since the condition $\Lambda_{b'',\Omega} \subseteq SS(L_{b',\Omega})$ does not depend on Ω ([28], Theorem 6.2.1), we must in fact have $\overline{\mathcal{O}_{b',\Omega}} \supseteq \mathcal{O}_{b'',\Omega}$ for all orientations Ω. In type A, this means that $b' \leq_{\mathrm{pol}} b''$ by the previous remark. We do not know if this result extends to the other types.

4.4 Comparison between the canonical and the semicanonical bases

We continue to assume that the Cartan matrix A is symmetric and of finite type. As both the canonical and the semicanonical bases are of canonical type, the transition matrix between them is lower unitriangular w.r.t. the order $\leq_{\mathrm{str}}$ (see Proposition 2.6). Our aim now is to compare these bases w.r.t. the order $\leq_{\mathrm{pol}}$.

Our method is to use a PBW basis as an intermediary. Conditions on the transition matrix between the canonical basis and a PBW basis were obtained in Corollary 3.6. We now focus on the transition matrix between the semicanonical basis and a PBW basis.

Lemma 4.3 *Let $\mathbf{i} \in \mathscr{X}$, $\mathbf{n} \in \mathbb{N}^N$ and $b \in B(-\infty)$. If $S(b)$ appears with a nonzero coefficient in the expansion of $E_{\mathbf{i}}^{(\mathbf{n})}$ on the semicanonical basis, then $\mathbf{n_i}(b) \geq_{\mathbf{i}} \mathbf{n}$.*

Proof. We can of course assume that $E_{\mathbf{i}}^{(\mathbf{n})}$ and b have the same weight; call it ν. As before, $\mathbf{i}$ defines a sequence (β_k) of positive roots and a sequence (γ_k) of chamber coweights.

The coefficient of $S(b)$ in the expansion of an element $u \in \mathbf{f}_\nu$ on the semicanonical basis is equal to $\delta_x(u) = \kappa(u)(x)$, where x is a general point in $\Lambda_{b,\Omega}$. By definition of the algebra structure on $\widetilde{M}$, if this number is nonzero for $u = E_{\mathbf{i}}^{(\mathbf{n})}$, then there is a filtration

$$\mathbf{V} = \mathbf{V}_0 \supseteq \mathbf{V}_1 \supseteq \cdots \supseteq \mathbf{V}_{N-1} \supseteq \mathbf{V}_N = 0$$

by x-stable I-graded vector spaces such that the dimension-vector of $\mathbf{V}_{k-1}/\mathbf{V}_k$ is $n_k\beta_k$.

Theorem 6.3 in [3] tells us how to extract the BZ datum of b from the pair $(\mathbf{V}, x)$, viewed as a representation of the preprojective algebra Π. Specifically, if we introduce the Π-modules $N(\gamma)$ as in Section 3.4 of [3], then $M_\gamma(b) = \dim \operatorname{Hom}_\Pi(N(\gamma), (\mathbf{V}, x))$.

Let $k \in \{1, \ldots, N\}$ and set $w = s_{i_1} \cdots s_{i_k}$. Observing that $(\mathbf{V}_k, x)$ is a Π-submodule of $(\mathbf{V}, x)$ and using Proposition 4.3 in [3], we get

$$\langle w\omega_{i_k}^\vee, \mu_w(b)\rangle = M_{w\omega_{i_k}^\vee}(b) \geq \dim \operatorname{Hom}_\Pi(N(w\omega_{i_k}^\vee), (\mathbf{V}_k, x)) \geq \langle w\omega_{i_k}^\vee, \nu_k\rangle,$$

where ν_k is the dimension-vector of $\mathbf{V}_k$. Substituting $\nu_k = \operatorname{wt}(b) - \sum_{t=1}^k n_t\beta_t$ and $w\omega_{i_k}^\vee = -\gamma_k$, we get exactly the inequality asked for in the definition of $\mathbf{n_i}(b) \geq_{\mathbf{i}} \mathbf{n}$. □

Theorem 4.4 *The transition matrix between the canonical and the semicanonical bases is lower unitriangular w.r.t. the order* $\leq_{\mathrm{pol}}$.

Proof. Proposition 3.1, Corollary 3.6 and Lemma 4.3 readily imply that the transition matrix is lower triangular w.r.t. the order $\leq_{\mathrm{pol}}$. In addition, Proposition 2.6 (ii) guarantees that the diagonal coefficients are equal to 1. □

Remark 4.5. The proof of Proposition 2.6 relies on the observation that the moves $\approx$ preserve the coefficients of the transition matrix between two bases of canonical type. Using Theorem 4.4, we then see that the transition matrix between the canonical and the semicanonical bases is also lower triangular w.r.t. the order $\leq$ defined in Remark 3.3, and obtained by "stabilizing" $\leq_{\mathrm{pol}}$ under the moves $\approx$. In addition, this transition matrix is also invariant under Saito's crystal reflections (see [2], equation (3)); we can thus weaken again our order by introducing a further move. (The author borrowed this idea from Kashiwara and Saito, see Lemma 8.2.2 in [28].)

5 A study in types A_5 and D_4

In [28], Kashiwara and Saito discovered a situation where $\Lambda_{b'',\Omega} \subseteq SS(L_{b',\Omega})$ for two elements $b' \neq b''$ of $B(-\infty)$, in the notation of Section 4.2. In [16], Geiss, Leclerc and Schröer made a more detailed investigation of this situation, and observed that for these elements, the element $S(b'')$ do occur in the expansion of the canonical basis element $G(b')$ on the semicanonical basis. By Proposition 2.6 (ii) and Theorem 4.4, it follows that $b' \leq_{\mathrm{str}} b''$ and $b' \leq_{\mathrm{pol}} b''$.

Geiss, Leclerc and Schröer explain that these phenomena are related to the fact that the algebra $\mathbb{Q}[N]$ has a tame cluster type, namely $E_8^{(1,1)}$, and that the counterexample is located precisely at one of the imaginary indecomposable roots of this elliptic root system. Leclerc explained to the author that the other imaginary root gives rise to a similar counterexample, and that the story can be repeated word for word in type D_4.

Our aim here is to add a small piece to the almost complete description of the situation given in [16]: we will compute explicitly the elements $G(b')$ and $G(b'')$ above, in a way that will allow us in the next section to compute the image of $G(b')$ by the quantum Frobenius map and the quantum Frobenius splitting.

5.1 Statement of the results

Our results can be stated in an uniform way for four situations, numbered (I)–(IV). In each case, and for each $p \geq 1$, we define a product $\widetilde{E}_p$ of crystal operators and elements ξ_p and η_p in $\mathbf{f}$, as in the following table. The first two cases are in type A_5, with the vertices of the Dynkin diagram sequentially numbered. The last two cases are in type D_4, where the index of the central node is 2.

I	Type A_5 $\widetilde{E}_p = (\tilde{e}_2\tilde{e}_4)^p\ (\tilde{e}_1\tilde{e}_3^2\tilde{e}_5)^p\ (\tilde{e}_2\tilde{e}_4)^p$ $\xi_p = \big(\theta_2^{(p)}\theta_4^{(p)}\big)\ \big(\theta_1^{(p)}\theta_3^{(2p)}\theta_5^{(p)}\big)\ \big(\theta_2^{(p)}\theta_4^{(p)}\big)$ $\eta_p = \big(\theta_2^{(p)}\theta_4^{(p)}\big)\ \big(\theta_1^{(p)}\theta_3^{(2p)}\theta_5^{(p)}\big)\ \big(\theta_2^{(2p)}\theta_4^{(2p)}\big)\ \big(\theta_1^{(p)}\theta_3^{(2p)}\theta_5^{(p)}\big)\ \big(\theta_2^{(p)}\theta_4^{(p)}\big)$
II	Type A_5 $\widetilde{E}_p = (\tilde{e}_1\tilde{e}_3^2\tilde{e}_5)^p\ (\tilde{e}_2\tilde{e}_4)^{3p}\ (\tilde{e}_1\tilde{e}_3^2\tilde{e}_5)^p$ $\xi_p = \big(\theta_1^{(p)}\theta_3^{(2p)}\theta_5^{(p)}\big)\ \big(\theta_2^{(3p)}\theta_4^{(3p)}\big)\ \big(\theta_1^{(p)}\theta_3^{(2p)}\theta_5^{(p)}\big)$ $\eta_p = \big(\theta_1^{(p)}\theta_3^{(2p)}\theta_5^{(p)}\big)\ \big(\theta_2^{(3p)}\theta_4^{(3p)}\big)\ \big(\theta_1^{(2p)}\theta_3^{(4p)}\theta_5^{(2p)}\big)\ \big(\theta_2^{(3p)}\theta_4^{(3p)}\big)\ \big(\theta_1^{(p)}\theta_3^{(2p)}\theta_5^{(p)}\big)$
III	Type D_4 $\widetilde{E}_p = \tilde{e}_2^p\ (\tilde{e}_1\tilde{e}_3\tilde{e}_4)^p\ \tilde{e}_2^p$ $\xi_p = \theta_2^{(p)}\ \big(\theta_1^{(p)}\theta_3^{(p)}\theta_4^{(p)}\big)\ \theta_2^{(p)}$ $\eta_p = \theta_2^{(p)}\ \big(\theta_1^{(p)}\theta_3^{(p)}\theta_4^{(p)}\big)\ \theta_2^{(2p)}\ \big(\theta_1^{(p)}\theta_3^{(p)}\theta_4^{(p)}\big)\ \theta_2^{(p)}$
IV	Type D_4 $\widetilde{E}_p = (\tilde{e}_1\tilde{e}_3\tilde{e}_4)^p\ \tilde{e}_2^{3p}\ (\tilde{e}_1\tilde{e}_3\tilde{e}_4)^p$ $\xi_p = \big(\theta_1^{(p)}\theta_3^{(p)}\theta_4^{(p)}\big)\ \theta_2^{(3p)}\ \big(\theta_1^{(p)}\theta_3^{(p)}\theta_4^{(p)}\big)$ $\eta_p = \big(\theta_1^{(p)}\theta_3^{(p)}\theta_4^{(p)}\big)\ \theta_2^{(3p)}\ \big(\theta_1^{(2p)}\theta_3^{(2p)}\theta_4^{(2p)}\big)\ \theta_2^{(3p)}\ \big(\theta_1^{(p)}\theta_3^{(p)}\theta_4^{(p)}\big)$

For each $(r, s) \in \mathbb{N}^2$, we set $b_{r,s} = \widetilde{E}_{r+s}\widetilde{E}_s 1$; this is an element in $B(-\infty)$.

Proposition 5.1 *Let $r \in \mathbb{N}$. Then $b_{r,0}$ is maximal in $B(-\infty)$ w.r.t. the order $\leq_{\mathrm{str}}$. Further, if $\mathbf{B}$ is a basis of canonical type of $\mathbf{f}$, then ξ_r is the element of $\mathbf{B}$ indexed by $b_{r,0}$ in the bijection $B(-\infty) \cong \mathbf{B}$.*

Proposition 5.1 is proved in Section 5.2.

Recall that $G(b_{r,s})$ and $S(b_{r,s})$ denote the elements indexed by $b_{r,s}$ in the canonical and semicanonical bases of $\mathbf{f}$. Proposition 5.1 tells us that $\xi_p = G(b_{p,0}) = S(b_{p,0})$. We now look for similar expansions of η_p on the two bases, canonical and semicanonical.

Theorem 5.2 *(i) Let $(r', s', r'', s'') \in \mathbb{N}^4$. Then*

$$b_{r',s'} \leq_{\text{str}} b_{r'',s''} \iff b_{r',s'} \leq_{\text{pol}} b_{r'',s''} \iff \big(r' + 2s' = r'' + 2s'' \;\; \textit{and} \;\; r' \leq r''\big).$$

(ii) For each $p \in \mathbb{N}$,

$$\eta_p = G(b_{0,p}) + G(b_{2,p-1}) + G(b_{4,p-2}) + \cdots + G(b_{2p,0}).$$

(iii) For each $(r, s) \in \mathbb{N}^2$,

$$\langle S(b_{2r,s})^*, \eta_{r+s} \rangle = \binom{2r}{r}.$$

The proof of Theorem 5.2 occupies Sections 5.3–5.6.

Remark 5.3. Recall the notation of Section 4.1. Let $\nu \in Q_+$, let Ω be an orientation, let $\mathbf{V}$ be an I-graded $\mathbf{k}$-vector space of dimension-vector ν, and let $(\mathbf{i}, \mathbf{a}) \in S_\nu$. By Theorem 2.2 in [46], the characteristic cycle of $L_{\mathbf{i},\mathbf{a};\Omega}$ is the projection on $T^*\mathbf{E}_{\mathbf{V},\Omega}$ of the intersection in the ambient space $T^*(\mathcal{F}_{\mathbf{i},\mathbf{a}} \times \mathbf{E}_{\mathbf{V},\Omega})$ of $\mathcal{F}_{\mathbf{i},\mathbf{a}} \times T^*\mathbf{E}_{\mathbf{V},\Omega}$ with the conormal bundle of $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a}}$. (In this recipe, $\mathcal{F}_{\mathbf{i},\mathbf{a}}$ is identified to the zero section of $T^*\mathcal{F}_{\mathbf{i},\mathbf{a}}$ and intersection means the intersection product in homology or in algebraic geometry.)

Let $p \in \mathbb{N}$ and take $(\mathbf{i}, \mathbf{a})$ such that $\Theta_{\mathbf{i}}^{(\mathbf{a})} = \eta_p$. Looking over a general point of $\Lambda_{b_{2r,s}}$, where $r + s = p$, the intersection is transversal and consists of $\binom{2r}{r}$ points. (These are the same points as those occurring in the proof of Theorem 5.2 (iii).) The multiplicity of $[\Lambda_{b_{2r,s}}]$ in the characteristic cycle of $L_{\mathbf{i},\mathbf{a};\Omega}$ is therefore equal to this binomial coefficient, up to a sign.

Take $p = 1$. The multiplicity of $[\Lambda_{b_{2,0}}]$ in the characteristic cycle of $L_{\mathbf{i},\mathbf{a};\Omega}$ is thus ± 2. A similar calculation shows that the multiplicity of $[\Lambda_{b_{2,0}}]$ in the characteristic cycle of $L_{b_{2,0},\Omega}$ is equal to ± 1. In addition, the proof of Theorem 5.2 (ii) gives $L_{\mathbf{i},\mathbf{a};\Omega} = L_{b_{0,1},\Omega} \oplus L_{b_{2,0},\Omega}$. We conclude that $\Lambda_{b_{2,0}}$ is contained in the singular support of $L_{b_{0,1},\Omega}$.

The elements b and b' of Kashiwara and Saito ([28], Section 7.2) are our elements $b_{0,1}$ and $b_{2,0}$ in case I. The arguments above thus provide a new proof of Theorem 7.2.1 in [28].

5.2 Proof of Proposition 5.1

Though each case requires its own set of calculations, we will content ourselves with treating the case I.

We fix $r \in \mathbb{N}$. Standard tools for crystal combinatorics allow to compute the Lusztig data of $b_{r,0}$ w.r.t. the reduced words

$$\mathbf{i} = (2, 4, 1, 3, 5, 2, 4, 1, 3, 5, 2, 4, 1, 3, 5),$$
$$\mathbf{j} = (1, 3, 5, 2, 4, 1, 3, 5, 2, 4, 1, 3, 5, 2, 4).$$

One finds

$$\mathbf{n}_{\mathbf{i}}(b_{r,0}) = (r, r, 0, 0, 0, 0, 0, 0, 0, 0, r, r, 0, 0, 0),$$
$$\mathbf{n}_{\mathbf{j}}(b_{r,0}) = (0, 0, 0, r, r, r, 0, 0, 0, 0, 0, 0, 0, r, r).$$

One deduces that

$$\varphi_i(b_{r,0}) = \varphi_i(\sigma(b_{r,0})) = \begin{cases} r & \text{if } i \in \{2,4\}, \\ 0 & \text{if } i \in \{1,3,5\}. \end{cases}$$

We claim that an element $b \in B(-\infty)$ such that $\mathrm{wt}(b) = \mathrm{wt}(b_{r,0})$ and such that

$$\varphi_i(b) \geq \varphi_i(b_{r,0}) \quad \text{and} \quad \varphi_i(\sigma(b)) \geq \varphi_i(\sigma(b_{r,0})) \tag{5}$$

for each $i \in \{1, \ldots, 5\}$ is necessarily equal to $b_{r,0}$.

To prove this claim, pick such a b and set $t = \varphi_2(b)$, $u = \varphi_2(\sigma(b))$, $b' = \sigma\big(\tilde{f}_2^{\max}\sigma(b)\big)$. By Proposition 5.3.1 (1) in [28], we have

$$t = \max\big(\varphi_2(b'), u + \langle \alpha_2^\vee, \mathrm{wt}(b') \rangle\big) = \max\big(\varphi_2(b'), r - u\big).$$

Since $t \geq r$ and $u \geq r$, this forces $\varphi_2(b') = t$. Therefore, in its expansion on the basis of simple roots, the α_2-coordinate of $\mathrm{wt}(b')$ is thus at least t, so $2r - u \geq t$. This forces $t = u = r$. By symmetry, we have $\varphi_4(b) = \varphi_4(\sigma(b)) = r$. Further, the crystal operations at vertex 2 commute with the crystal operations at vertex 4, so the element $b'' = \sigma\big(\tilde{f}_4^r\sigma(b')\big)$ satisfies $\varphi_2(b'') = \varphi_4(b'') = r$.

Now $b''' = \tilde{f}_1^r\tilde{f}_3^{2r}\tilde{f}_5^r 1$ is the unique element in $B(-\infty)$ with weight $r(\alpha_1 + 2\alpha_3 + \alpha_5)$. We thus necessarily have $b''' = \tilde{f}_2^r\tilde{f}_4^r b''$, and we conclude that $b = \sigma \circ \big(\tilde{f}_2^r\tilde{f}_4^r\big) \circ \sigma \circ \big(\tilde{f}_2^r\tilde{f}_4^r\big)(b''')$. This reasoning holds in particular for $b_{r,0}$, whence our claim that $b = b_{r,0}$.

From this, it immediately follows that $b_{r,0}$ is maximal in $B(-\infty)$ w.r.t. the order $\leq_{\mathrm{str}}$. Therefore, all bases of canonical type share the same element at this spot, by Proposition 2.6 (ii). It remains to prove that this element is ξ_r.

Let us adopt the setup of the definition of the semicanonical basis. Pick an orientation of the Dynkin diagram and an I-graded $\mathbf{k}$-vector space $\mathbf{V}$ of dimension-vector $\mathrm{wt}(b_{r,0})$. Let $b \in B(-\infty)$. By construction (see Section 4.1), the value of the constructible function $\kappa(\xi_r)$ at the general point x of $\Lambda_{b,\Omega}$ is nonzero only if $\mathbf{V}$ contains x-stable subspaces $\mathbf{W}'$ and $\mathbf{W}''$ such that $\mathbf{W}'$ and $\mathbf{V}/\mathbf{W}''$ have dimension-vector $r(\alpha_2 + \alpha_4)$. This latter condition is equivalent to (5), hence to $b = b_{r,0}$; when it is fulfilled, $\mathbf{W}'$ and $\mathbf{W}''$ are unique and satisfy $\mathbf{W}' \subseteq \mathbf{W}''$, which leads to $\kappa(\xi_r)(x) = 1$.

To sum up, $\langle S(b)^*, \xi_r \rangle$ is equal to 1 if $b = b_{r,0}$ and to 0 otherwise. We conclude that $\xi_r = S(b_{r,0})$, as announced.

5.3 Proof of Theorem 5.2 (i)

Again, we restrict our attention to the case I.

We set $\nu = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$.

We fix $(r,s) \in \mathbb{N}^2$. The weight of $b_{r,s}$ is $(r + 2s)\nu$. As in Section 5.2, a direct computation gives the Lusztig data of $b_{r,s}$ w.r.t. the reduced word $\mathbf{i}$:

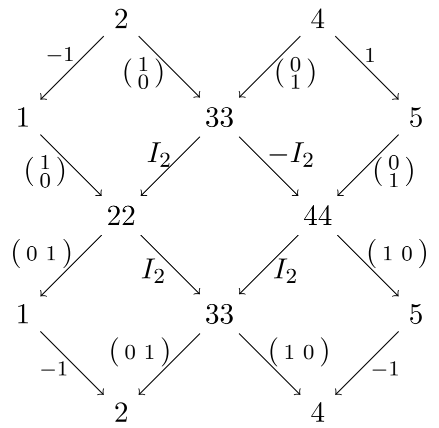
$$\mathbf{n}_{\mathbf{i}}(b_{r,s}) = (r+s, r+s, 0, 0, 0, s, s, 0, 0, 0, r+s, r+s, 0, 0, 0).$$

Let Ω be the orientation

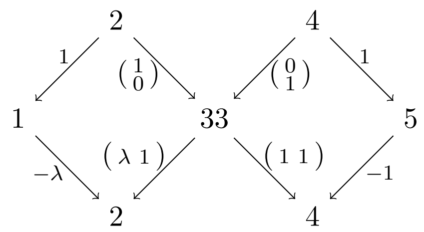
$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$$

of the Dynkin diagram and let Λ_Ω be the preprojective algebra it defines. A Λ_Ω-module is the datum of a pair $(\mathbf{V}, x)$, where $\mathbf{V}$ is an I-graded $\mathbf{k}$-vector space and $x \in \Lambda_{\mathbf{V},\Omega}$.

The category of Λ_Ω-modules is tame and is described in full details in [16]. We refer to this paper for more information. As usual, we denote by S_i the simple Λ_Ω-module of dimension-vector α_i. Let P and Q_λ be the following Λ_Ω-modules, where λ is a parameter in $\mathbf{k} \setminus \{0, 1\}$.



P



Q_λ

In these pictures, as customary, a digit i denotes a basis vector in the vector space $\mathbf{V}_i$, and the small matrices that adorn the arrows indicate the action of the arrows h. The module P is projective and injective. Each module Q_λ lies at the mouth of an homogeneous tube, whence $\operatorname{Ext}^1(Q_\lambda, Q_\mu) = 0$ for $\lambda \neq \mu$.

We fix an I-graded $\mathbf{k}$-vector space $\mathbf{W}$ of dimension-vector ν. Since the module P is projective, it is rigid: the closure of the orbit $\{x \in \Lambda_{\mathbf{W}^2,\Omega} \mid (\mathbf{W}^2, x) \cong P\}$ is an irreducible component of $\Lambda_{\mathbf{W}^2,\Omega}$. The orbit of each module Q_λ has codimension 1 in $\Lambda_{\mathbf{W},\Omega}$; therefore, the closure of the union of these orbits is an irreducible component of $\Lambda_{\mathbf{W},\Omega}$.

Let $(r, s) \in \mathbb{N}^2$. By Section 2.6 in [16], the closure of the set of all points $x \in \Lambda_{\mathbf{W}^{r+2s},\Omega}$ such that $(\mathbf{W}^{r+2s}, x)$ is isomorphic to a module of the form

$$M_{r,s} = Q_{\lambda_1} \oplus \cdots \oplus Q_{\lambda_r} \oplus \underbrace{P \oplus \cdots \oplus P}_{s \text{ times}}$$

is an irreducible component $\Lambda_{b,\Omega}$ of $\Lambda_{\mathbf{W}^{r+2s},\Omega}$.

Observing that the word $\mathbf{i}$ is adapted to the orientation Ω, we can determine the Lusztig datum of b in direction $\mathbf{i}$ by looking at the multiplicities in a Krull-Remak-Schmidt decomposition of the restriction of $M_{r,s}$ to the quiver (I, Ω) (see Section 7.3 in [3] for the full justification of this procedure). We find

$$\mathbf{n_i}(b) = (r+s, r+s, 0, 0, 0, s, s, 0, 0, 0, r+s, r+s, 0, 0, 0) = \mathbf{n_i}(b_{r,s}),$$

from where we conclude that $b = b_{r,s}$.

Using the definition of the crystal structure on irreducible components of the nilpotent varieties, one then deduces that

$$b_{r,s} = \sigma(b_{r,s}) \quad \text{and} \quad \varphi_i(b_{r,s}) = \begin{cases} r+s & \text{if } i \in \{2,4\}, \\ 0 & \text{if } i \in \{1,3,5\}. \end{cases}$$

Similarly, for $(t_1, t_3, t_5) \in \mathbb{N}^3$ and $b = \tilde{e}_1^{t_1} \tilde{e}_3^{t_3} \tilde{e}_5^{t_5} b_{r,s}$, a general point in $\Lambda_{b,\Omega}$ is the datum of the structural maps of a Λ_Ω-module isomorphic to something of the form

$$Q_{\lambda_1} \oplus \cdots \oplus Q_{\lambda_r} \oplus P^{\oplus s} \oplus S_1^{\oplus t_1} \oplus S_3^{\oplus t_3} \oplus S_5^{\oplus t_5},$$

whence

$$\tilde{e}_1^{t_1} \tilde{e}_3^{t_3} \tilde{e}_5^{t_5} b_{r,s} = \sigma\big(\tilde{e}_1^{t_1} \tilde{e}_3^{t_3} \tilde{e}_5^{t_5} b_{r,s}\big) \quad \text{and} \quad \varphi_i\big(\tilde{e}_1^{t_1} \tilde{e}_3^{t_3} \tilde{e}_5^{t_5} b_{r,s}\big) = \begin{cases} r+s & \text{if } i \in \{2,4\}, \\ t_i & \text{if } i \in \{1,3,5\}. \end{cases}$$

Given $(r', s', r'', s'') \in \mathbb{N}^4$, the equivalence

$$b_{r',s'} \leq_{\text{str}} b_{r'',s''} \iff \big(r' + 2s' = r'' + 2s'' \ \text{ and } \ r' \leq r''\big)$$

now follows from the definition of $\leq_{\text{str}}$.

If $b_{r',s'} \leq_{\text{pol}} b_{r'',s''}$, then $b_{r',s'}$ and $b_{r'',s''}$ have the same weight, whence $r' + 2s' = r'' + 2s''$, and $\varphi_i(b_{r',s'}) \leq \varphi_i(b_{r'',s''})$ for all i, whence $r' \leq r''$. The converse implication

$$\big(r' + 2s' = r'' + 2s'' \ \text{ and } \ r' \leq r''\big) \implies b_{r',s'} \leq_{\text{pol}} b_{r'',s''}$$

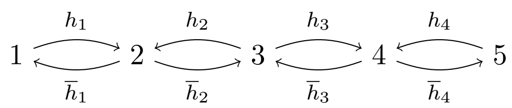
follows from the two following facts: first, $\operatorname{Pol}(b_{0,1}) \subset \operatorname{Pol}(b_{2,0})$; second, for any $(r,s) \in \mathbb{N}^2$, $\operatorname{Pol}(b_{r,s}) = r\operatorname{Pol}(b_{1,0}) + s\operatorname{Pol}(b_{0,1})$, where the sum is the Minkowski sum of convex bodies. The first fact can be shown either by a direct computation, or as a consequence of Theorem 4.4, once noticed that

$$\langle S(b_{2,0})^*, G(b_{0,1})\rangle = \langle S(b_{2,0})^*, \eta_1\rangle - \langle S(b_{2,0})^*, G(b_{2,0})\rangle = \binom{2}{1} - 1 \neq 0.$$

The second fact follows from the construction of $\operatorname{Pol}(b)$ given in [3] or from Remark 3.5 (ii) in [4].

5.4 Proof of Theorem 5.2 (iii)

We keep the notation of the previous section. We label the oriented edges in H as follows.



From the datum $(\mathbf{V}, x)$ of a Λ_Ω-module, we will define four maps

$$y : \mathbf{V}_1 \oplus \mathbf{V}_5 \xrightarrow{x_{h_1} \oplus x_{h_4}} \mathbf{V}_2 \oplus \mathbf{V}_4, \qquad z : \mathbf{V}_3 \xrightarrow{\binom{x_{h_2}}{x_{h_3}}} \mathbf{V}_2 \oplus \mathbf{V}_4,$$

$$\overline{y} : \mathbf{V}_2 \oplus \mathbf{V}_4 \xrightarrow{x_{\overline{h}_1} \oplus x_{\overline{h}_4}} \mathbf{V}_1 \oplus \mathbf{V}_5, \qquad \overline{z} : \mathbf{V}_2 \oplus \mathbf{V}_4 \xrightarrow{(x_{\overline{h}_2}\ x_{\overline{h}_3})} \mathbf{V}_3.$$

In the case of P, both maps y and z are injective and both maps $\overline{y}$ and $\overline{z}$ are surjective. The subspace $\operatorname{im} y \cap \operatorname{im} z$ has dimension-vector $\alpha_2 + \alpha_4$ (it is linearly spanned by the 2 and the 4 on the fifth line of the picture of P). The subspace $\ker \overline{y} + \ker \overline{z}$ has dimension-vector $3(\alpha_2 + \alpha_4)$ (it is spanned by the 2 and the 4 on the third and fifth lines of the picture of P). In addition, $(y\overline{y})(\ker \overline{y} + \ker \overline{z}) = \operatorname{im} y \cap \operatorname{im} z$ and $(y\overline{y})^{-1}(\operatorname{im} y \cap \operatorname{im} z) = \ker \overline{y} + \ker \overline{z}$.

In the case of Q_λ, both maps y and z are injective and both maps $\overline{y}$ and $\overline{z}$ are surjective. We have $\operatorname{im} y \cap \operatorname{im} z = \ker \overline{y} + \ker \overline{z}$; this subspace has dimension-vector $\alpha_2 + \alpha_4$ (it is linearly spanned by the 2 and the 4 on the third line of the picture of Q_λ). This subspace is also the kernel as well as the image of $y\overline{y}$.

Let $(r, s) \in \mathbb{N}^2$. Set $p = r + s$ and $\mathbf{V} = \mathbf{W}^p$. Let $(\mathbf{j}, \mathbf{a}) \in S_{p\nu}$ be so that $\eta_p = \Theta_{\mathbf{j}}^{(\mathbf{a})}$. A flag of type $(\mathbf{j}, \mathbf{a})$ in $\mathbf{V}$ is the datum of subspaces $\mathbf{X}_i \subseteq \mathbf{V}_i$ for $i \in \{1, 3, 5\}$ and of 2-steps filtrations $0 \subseteq \mathbf{X}'_i \subseteq \mathbf{X}''_i \subseteq \mathbf{V}_i$ for $i \in \{2, 4\}$ of suitable dimension, namely

$$\dim \mathbf{X}_1 = \dim \mathbf{X}'_2 = \dim \mathbf{X}'_4 = \dim \mathbf{X}_5 = p, \quad \dim \mathbf{X}_3 = 2p, \quad \dim \mathbf{X}''_2 = \dim \mathbf{X}''_4 = 3p.$$

Let x be a general point in $\Lambda_{b_{2r,s},\Omega}$. Thus, $(\mathbf{V}, x)$ is isomorphic to a Λ_Ω-module of the form

$$Q_{\lambda_1} \oplus \cdots \oplus Q_{\lambda_{2r}} \oplus \underbrace{P \oplus \cdots \oplus P}_{s \text{ times}}$$

where $\lambda_1, \ldots, \lambda_{2r}$ are distinct. In agreement with this decomposition, we write $\mathbf{V}$ as a direct sum $\mathbf{V}^{(Q)} \oplus \mathbf{V}^{(P)}$.

We look for x-stable flags of type $(\mathbf{j}, \mathbf{a})$ in $\mathbf{V}$. Since the maps y and z are injective, we must have $y(\mathbf{X}_1 \oplus \mathbf{X}_5) = z(\mathbf{X}_3) = \mathbf{X}'_2 \oplus \mathbf{X}'_4$, for dimension reasons. Likewise, the surjectivity of $\overline{y}$ and $\overline{z}$ implies that $\overline{y}^{-1}(\mathbf{X}_1 \oplus \mathbf{X}_5) = \overline{z}^{-1}(\mathbf{X}_3) = \mathbf{X}''_2 \oplus \mathbf{X}''_4$. Noticing that $y\overline{y}$ must map $\mathbf{X}''_2 \oplus \mathbf{X}''_4$ to $\mathbf{X}'_2 \oplus \mathbf{X}'_4$, we get

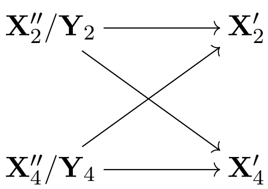
$$\begin{aligned}(y\overline{y})(\ker \overline{y} + \ker \overline{z}) \subseteq \mathbf{X}'_2 \oplus \mathbf{X}'_4 \subseteq \operatorname{im} y \cap \operatorname{im} z, \\ \ker \overline{y} + \ker \overline{z} \subseteq \mathbf{X}''_2 \oplus \mathbf{X}''_4 \subseteq (y\overline{y})^{-1}(\operatorname{im} y \cap \operatorname{im} z).\end{aligned}$$

These conditions imply that $\mathbf{X}'_2$, $\mathbf{X}''_2$, $\mathbf{X}'_4$ and $\mathbf{X}''_4$ are compatible with the decomposition $\mathbf{V} = \mathbf{V}^{(Q)} \oplus \mathbf{V}^{(P)}$ and determine the intersections of these four subspaces with $\mathbf{V}^{(P)}$. For our problem of finding the x-stable flags of type $(\mathbf{j}, \mathbf{a})$, we can thus neglect $\mathbf{V}^{(P)}$. Consequently, we now focus on $\mathbf{V}^{(Q)}$ and implicitly restrict the maps x_h, y, z, $\overline{y}$ and $\overline{z}$ to this subspace. We also simplify the notation by writing $\mathbf{X}'_2$ instead of $\mathbf{X}'_2 \cap \mathbf{V}^{(Q)}$, and similarly for $\mathbf{X}''_2$, $\mathbf{X}'_4$ and $\mathbf{X}''_4$.

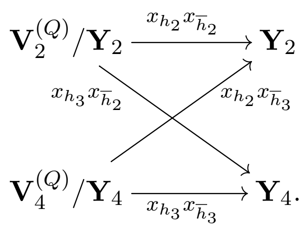
After this renaming, $\dim \mathbf{X}'_2 = \dim \mathbf{X}'_4 = r$ and $\dim \mathbf{X}''_2 = \dim \mathbf{X}''_4 = 3r$. Let $\mathbf{Y}_2 \oplus \mathbf{Y}_4$ denote the subspace $\operatorname{im} y \cap \operatorname{im} z$ in $\mathbf{V}^{(Q)}$. Then $\dim \mathbf{Y}_2 = \dim \mathbf{Y}_4 = 2r$. We must have

$$\begin{aligned}\mathbf{X}'_2 \subseteq \mathbf{Y}_2 \subseteq \mathbf{X}''_2, \\ \mathbf{X}'_4 \subseteq \mathbf{Y}_4 \subseteq \mathbf{X}''_4, \\ (y\overline{y})(\mathbf{X}''_2 \oplus \mathbf{X}''_4) \subseteq \mathbf{X}'_2 \oplus \mathbf{X}'_4, \\ (z\overline{z})(\mathbf{X}''_2 \oplus \mathbf{X}''_4) \subseteq \mathbf{X}'_2 \oplus \mathbf{X}'_4.\end{aligned}$$

Recalling the preprojective relations $x_{h_1}x_{\overline{h}_1} + x_{h_2}x_{\overline{h}_2} = 0$ and $x_{h_3}x_{\overline{h}_3} + x_{h_4}x_{\overline{h}_4} = 0$, we can rephrase this in terms of representations of a tame quiver:



is a subrepresentation of



Choosing adequate bases in $\mathbf{V}_2^{(Q)}$ and $\mathbf{V}_4^{(Q)}$, the linear maps on the right diagram are given by the identity matrix of size $2r$, except for the top horizontal arrow, which is represented by a diagonal matrix with coefficients $\lambda_1, \ldots, \lambda_{2r}$. Subrepresentations of the required dimension are obtained by taking r among the $2r$ eigenspaces of the composed map

$$(x_{h_3}x_{\overline{h}_2})^{-1}(x_{h_3}x_{\overline{h}_3})(x_{h_2}x_{\overline{h}_3})^{-1}(x_{h_2}x_{\overline{h}_2}).$$

All in all, we have $\binom{2r}{r}$ x-stable flags of type $(\mathbf{j}, \mathbf{a})$ in $\mathbf{V}$. The Euler characteristic of this set of flags is thus equal to $\binom{2r}{r}$, and also to $\kappa\big(\Theta_{\mathbf{j}}^{(\mathbf{a})}\big)(x) = \langle S(b_{2r,s})^*, \eta_p\rangle$ (see Section 4.1).

5.5 Analysis of extensions of quiver representations

Again, we consider the Dynkin diagram of type A_5. The reduced word

$$\mathbf{i} = (2, 4, 1, 3, 5, 2, 4, 1, 3, 5, 2, 4, 1, 3, 5)$$

is adapted to the following orientation Ω, in the sense of [35], Section 4.7.

$$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$$

Our interest lies in the representation theory over $\mathbf{k}$ of the quiver $Q = (I, \Omega)$. The dimension-vector map induces a bijection from the set of isomorphism classes of indecomposable representations of Q onto the set of positive roots (Gabriel's theorem). For each positive root β, we pick an indecomposable $\mathbf{k}Q$-module $M(\beta)$ of dimension-vector β. These modules can be organized in an Auslander-Reiten quiver, as follows.

$$\begin{array}{ccccccccc}
 & M(\alpha_{1,2}) & & M(\alpha_{3,4}) & & M(\alpha_5) \\
\nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\
M(\alpha_2) & & M(\alpha_{1,4}) & & M(\alpha_{3,5}) & \\
\searrow & & \nearrow \quad \searrow & & \nearrow \quad \searrow & \\
 & M(\alpha_{2,4}) & & M(\alpha_{1,5}) & & M(\alpha_3) \\
\nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\
M(\alpha_4) & & M(\alpha_{2,5}) & & M(\alpha_{1,3}) & \\
\searrow & & \nearrow \quad \searrow & & \nearrow \quad \searrow & \\
 & M(\alpha_{4,5}) & & M(\alpha_{2,3}) & & M(\alpha_1)
\end{array}$$

Here, we wrote $\alpha_{i,j}$ for $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. The enumeration $(\beta_1, \ldots, \beta_{15})$ of the positive roots defined by $\mathbf{i}$, namely

$$(\,\alpha_2,\ \alpha_4,\ \alpha_{1,2},\ \alpha_{2,4},\ \alpha_{4,5},\ \alpha_{1,4},\ \alpha_{2,5},\ \alpha_{3,4},\ \alpha_{1,5},\ \alpha_{2,3},\ \alpha_{3,5},\ \alpha_{1,3},\ \alpha_5,\ \alpha_3,\ \alpha_1\,),$$

can be read from this Auslander-Reiten quiver, proceeding columnwise from the top left corner to the bottom right corner. The simple $\mathbf{k}Q$-modules are of course the modules $M(\alpha_i)$, for $i \in I$; as customary, we denote them by S_i.

Consider the following lists of $\mathbf{k}Q$-modules.

t	M_t	N_t
1	$M(\alpha_2)$	0
2	$M(\alpha_{1,2})$	$M(\alpha_1)$
3	$M(\alpha_{1,2}) \oplus M(\alpha_{2,4})$	$M(\alpha_{1,4})$
4	$M(\alpha_{1,2}) \oplus M(\alpha_{2,5})$	$M(\alpha_{1,5})$
5	$M(\alpha_{1,2}) \oplus M(\alpha_{2,3})$	$M(\alpha_{1,3})$
6	$M(\alpha_{2,4})$	$M(\alpha_{3,4})$
7	$M(\alpha_{1,4})$	$M(\alpha_1) \oplus M(\alpha_{3,4})$
8	$M(\alpha_{1,4}) \oplus M(\alpha_{2,5})$	$M(\alpha_{1,5}) \oplus M(\alpha_{3,4})$
9	$M(\alpha_{1,4}) \oplus M(\alpha_{2,3})$	$M(\alpha_{1,3}) \oplus M(\alpha_{3,4})$
10	$M(\alpha_{2,5})$	$M(\alpha_{3,5})$
11	$M(\alpha_{1,5})$	$M(\alpha_1) \oplus M(\alpha_{3,5})$
12	$M(\alpha_{1,5}) \oplus M(\alpha_{2,3})$	$M(\alpha_{1,3}) \oplus M(\alpha_{3,5})$
13	$M(\alpha_{2,3})$	$M(\alpha_3)$
14	$M(\alpha_{1,3})$	$M(\alpha_1) \oplus M(\alpha_3)$

Proposition 5.4 *Let $0 \to S_2 \xrightarrow{f} M \to N \to 0$ be a short exact sequence of $\mathbf{k}Q$-modules. Then there exists $t \in \{1, \ldots, 14\}$ and a $\mathbf{kQ}$-module L such that $M \cong M_t \oplus L$ and $N \cong N_t \oplus L$.*

Proof. Let $K = \{1, 3, 4, 6, 7, 9, 10, 12\}$. In the $\mathbf{k}Q$-module $M(\beta_k)$, the vector space attached to the vertex 2 is nonzero if and only if $k \in K$. In this case, it is one dimensional, spanned by a vector e_k.

We endow K with an order by saying that $k \succ \ell$ is there is a path of positive length from $M(\beta_k)$ to $M(\beta_\ell)$ in the Auslander-Reiten quiver. Thus for instance $\alpha_{2,4} \succ \alpha_{1,3}$ but $\alpha_{1,2} \not\succ \alpha_{2,3}$.

Writing M as a direct sum of indecomposable modules, we find an isomorphism

$$g : M \xrightarrow{\simeq} \bigoplus_{k=1}^{15} M(\beta_k) \otimes W_k,$$

where the vector spaces W_k account for the multiplicities. Then $(g \circ f)(e_1)$ has the form $\sum_{k \in K} e_k \otimes w_k$, where $w_k \in W_k$.

Using the explicit form of the modules $M(\beta_k)$, one sees by a case-by-case analysis that it is possible to modify g so as to cancel the elements w_ℓ each time there is a $k \succ \ell$ such that $w_k \neq 0$. After this modification, the elements in the set $K' = \{k \in K \mid w_k \neq 0\}$ are pairwise incomparable. This leaves a list of fourteen possibilities for K' (note that K' cannot be empty).

For $k \in K'$, choose a complementary subspace W'_k in W_k to the line $\mathbf{k}w_k$, and for $k \notin K'$, set $W'_k = W_k$. Let

$$L = \bigoplus_{k=1}^{15} M(\beta_k) \otimes W'_k.$$

Then $M \cong M_t \oplus L$ for a certain $t \in \{1, \ldots, 14\}$. In this isomorphism, the image of f is contained in M_t, so the cokernel N of f is isomorphic to $N_t \oplus L$. $\square$

One can of course analyze in a similar fashion the extensions by S_4, using for instance the diagram automorphism.

To a $\mathbf{k}Q$-module M, we associate the row vector $\mathbf{n}(M) = (n_1, \ldots, n_{15})$, where n_k is the multiplicity of $M(\beta_k)$ in a Krull-Remak-Schmidt decomposition of M. Then $M \mapsto \mathbf{n}(M)$ induces a bijection from the set of isomorphism classes of $\mathbf{k}Q$-modules onto $\mathbb{N}^{15}$.

Consider the following elements in $\mathbb{N}^{15}$.

$$\begin{array}{lcl}
\mathbf{w}_1 &=& (\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 0,\ 1,\ 1\ ) \\
\mathbf{w}_2 &=& (\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 0,\ 1,\ 1,\ 0\ ) \\
\mathbf{w}_3 &=& (\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 0,\ 1,\ 0\ ) \\
\mathbf{w}_4 &=& (\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ -1,\ 0,\ 0,\ -1,\ 0,\ 1,\ 1\ ) \\
\mathbf{w}_5 &=& (\ 0,\ 0,\ 1,\ 1,\ 0,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 0,\ 1,\ 1\ ) \\
\mathbf{w}_6 &=& (\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ -1,\ 0,\ 0,\ 0,\ -1,\ 0,\ 1,\ 1\ ) \\
\mathbf{w}_7 &=& (\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ -1,\ -1,\ 0,\ 0,\ -1,\ 0,\ 1,\ 1\ ) \\
\mathbf{w}_8 &=& (\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ -1,\ -1,\ -1,\ 0,\ 1,\ 1,\ 0\ ) \\
\mathbf{w}_9 &=& (\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ -1,\ 0,\ 1,\ 1,\ 0\ ) \\
\mathbf{w}_{10} &=& (\ 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ -1,\ 0,\ 0,\ 0,\ -1,\ 0,\ 1,\ 1,\ 0\ ) \\
\mathbf{w}_{11} &=& (\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ -1,\ 0,\ -1,\ 0,\ 1,\ 1,\ 0\ ) \\
\mathbf{w}_{12} &=& (\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 0,\ 0,\ 1,\ 0\ ) \\
\mathbf{w}_{13} &=& (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ -1,\ 0,\ 1,\ 0,\ 0\ ) \\
\mathbf{w}_{14} &=& (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ -1,\ -1,\ 0,\ 1,\ 0\ ) \\
\mathbf{w}_{15} &=& (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ -1,\ 0,\ 0,\ 1\ ) \\
\mathbf{w}_{16} &=& (\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ -1,\ 0,\ 0,\ 0,\ -1,\ 0,\ 1,\ 0\ ) \\
\mathbf{w}_{17} &=& (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ -1,\ -1,\ 0,\ 0,\ 1,\ 0\ ) \\
\mathbf{x} &=& (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ -1,\ -2,\ -1\ ) \\
\mathbf{x}' &=& (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ -1,\ -1\ ) \\
\mathbf{y} &=& (\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0\ ) \\
\mathbf{z} &=& (\ 1,\ 1,\ 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0\ )
\end{array}$$

Corollary 5.5 *Let $p \in \mathbb{N}$ and let M et N be two $\mathbf{k}Q$-modules. If there is a short exact sequence*

$$0 \to S_2^{\oplus p} \oplus S_4^{\oplus p} \to M \to N \to 0,$$

then $\mathbf{n}(M) - \mathbf{n}(N) - p\mathbf{x}$ *belongs to the* $\mathbb{N}$*-span of the vectors* $\mathbf{w}_k$.

Proof. With the notation of Proposition 5.4, it suffices to check that each $\mathbf{n}(M_t) - \mathbf{n}(N_t) - \mathbf{x}'$ can be written as a (possibly empty) sum of vectors $\mathbf{w}_k$. □

5.6 Proof of Theorem 5.2 (ii)

As before, we focus on the case I.

Let $\nu = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$ and let $p \in \mathbb{N}$. Define $(\mathbf{j}, \mathbf{a}) \in S_{p\nu}$ so that $\eta_p = \Theta_{\mathbf{j}}^{(\mathbf{a})}$. Let $\mathbf{V}$ be an I-graded $\mathbf{k}$-vector space of dimension-vector $2p\nu$ and recall the notation set up in Section 4.1. To show Theorem 5.2 (ii), it suffices to prove the equation

$$L_{\mathbf{j},\mathbf{a};\Omega} = L_{b_{0,p},\Omega} \oplus L_{b_{2,p-1},\Omega} \oplus L_{b_{4,p-2},\Omega} \oplus \cdots \oplus L_{b_{2p,0},\Omega}. \tag{6}$$

To prove (6), we will show that the map $\pi_{\mathbf{j},\mathbf{a}}$ is semismall with respect to the stratification given by the $G_{\mathbf{V}}$-orbits on $\mathbf{E}_{\mathbf{V},\Omega}$ and identify the relevant strata. The precise statement is given in Proposition 5.6 below.

We regard a flag of type $(\mathbf{j}, \mathbf{a})$ in $\mathbf{V}$ as a 5-step filtration

$$\mathbf{V} = \mathbf{V}^0 \supseteq \mathbf{V}^1 \supseteq \mathbf{V}^2 \supseteq \mathbf{V}^3 \supseteq \mathbf{V}^4 \supseteq \mathbf{V}^5 = 0, \tag{7}$$

such that the dimension-vector of $\mathbf{V}^{k-1}/\mathbf{V}^k$ is $p(\alpha_2 + \alpha_4)$, $p(\alpha_1 + 2\alpha_3 + \alpha_5)$, or $2p(\alpha_2 + \alpha_4)$ according to whether $k \in \{1, 5\}$, $k \in \{2, 4\}$, or $k = 3$.

To a row vector $\mathbf{n} = (n_1, \ldots, n_{15})$ in $\mathbb{N}^{15}$, we associate the weight $|\mathbf{n}| = n_1\beta_1 + \cdots + n_{15}\beta_{15}$. This is the dimension-vector of any $\mathbf{k}Q$-module M such that $\mathbf{n}(M) = \mathbf{n}$.

A point $(x, \mathbf{V}^\bullet)$ in $\widetilde{\mathcal{F}}_{\mathbf{j},\mathbf{a}}$ yields $\mathbf{k}Q$-modules $M_{k,\ell} = (\mathbf{V}^k/\mathbf{V}^\ell, x)$, for all $k < \ell$. Given $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{N}^{15}$ such that $|\mathbf{u}| = 2p\nu$ and $|\mathbf{v}| = p\nu$, we denote by $\widetilde{\mathcal{F}}^{\mathbf{u},\mathbf{v}}$ the set of all $(x, \mathbf{V}^\bullet) \in \widetilde{\mathcal{F}}_{\mathbf{j},\mathbf{a}}$ such that $\mathbf{n}(M_{1,3}) = \mathbf{v}$ and $\mathbf{n}(M_{0,5}) = \mathbf{u}$.

Likewise, for $\mathbf{u} \in \mathbb{N}^{15}$ such that $|\mathbf{u}| = 2p\nu$, we denote by $\mathscr{O}_{\mathbf{u}}$ the set of all $x \in \mathbf{E}_{\mathbf{V},\Omega}$ such that $\mathbf{n}((\mathbf{V}, x)) = \mathbf{u}$. This is a $G_{\mathbf{V}}$-orbit in $\mathbf{E}_{\mathbf{V},\Omega}$. If $b \in B(-\infty)$ has Lusztig datum $\mathbf{u}$ in direction $\mathbf{i}$, then $L_{b,\Omega}$ is the intersection cohomology sheaf on $\overline{\mathscr{O}_{\mathbf{u}}}$.

In this fashion, we partition $\widetilde{\mathcal{F}}_{\mathbf{j},\mathbf{a}}$ and $\mathbf{E}_{\mathbf{V},\Omega}$ into locally closed pieces. The first projection $\pi_{\mathbf{j},\mathbf{a}} : \widetilde{\mathcal{F}} \to \mathbf{E}_{\mathbf{V},\Omega}$ restricts to a map $\pi^{\mathbf{u},\mathbf{v}} : \widetilde{\mathcal{F}}^{\mathbf{u},\mathbf{v}} \to \mathscr{O}_{\mathbf{u}}$.

Proposition 5.6 *Assume that* $\widetilde{\mathcal{F}}^{\mathbf{u},\mathbf{v}} \neq \varnothing$. *Then the map* $\pi^{\mathbf{u},\mathbf{v}}$ *is a locally trivial fibration with a smooth and connected fiber. Moreover, for* $x \in \mathscr{O}_{\mathbf{u}}$,

$$2\dim(\pi^{\mathbf{u},\mathbf{v}})^{-1}(x) + \dim \mathscr{O}_{\mathbf{u}} \leq \dim \widetilde{\mathcal{F}}_{\mathbf{j},\mathbf{a}}, \tag{8}$$

with equality if and only if $\mathbf{u} = 2(p - s)\mathbf{y} + s\mathbf{z}$ *and* $\mathbf{v} = \mathbf{u} - p\mathbf{y}$ *for some* $s \in \{0, \ldots, p\}$.

The Lusztig datum in direction $\mathbf{i}$ of the element $b_{r,s}$ is $r\mathbf{y} + s\mathbf{z}$ (see Section 5.3), so for $\mathbf{u} = 2(p-s)\mathbf{y} + s\mathbf{z}$, the intersection cohomology sheaf on $\overline{\mathscr{O}_\mathbf{u}}$ with coefficients in the trivial local system is $L_{b_{2(p-s),s},\Omega}$. In view of Proposition 5.6, and since each stratum $\mathscr{O}_\mathbf{u}$ is simply connected, the Decomposition Theorem for semismall maps then implies equation (6).

The remainder of this section is devoted to the proof of Proposition 5.6. We have to compute the difference Δ between the right and left-hand sides of (8) and to show that $\Delta \geq 0$. The assertion about the smoothness and the connectedness of the fiber of $\pi^{\mathbf{u},\mathbf{v}}$ will be proved on the way.

Lemma 1.6 (c) in [36] gives $\dim \widetilde{\mathcal{F}}_{\mathbf{j},\mathbf{a}} = 40p^2$. In addition, $\dim \mathbf{E}_{\mathbf{V},\Omega} = 48p^2$.

The formulas (4) (see Section 4.3) give a recipe to compute the matrices H and E with entries

$$h_{k,\ell} = \dim \operatorname{Hom}_{\mathbf{k}Q}(M(\beta_k), M(\beta_\ell)),$$
$$e_{k,\ell} = \dim \operatorname{Ext}^1_{\mathbf{k}Q}(M(\beta_k), M(\beta_\ell)).$$

We can then find the dimension of any $G_\mathbf{V}$-orbit contained in $\mathbf{E}_{\mathbf{V},\Omega}$: by Lemma 1 in [11], §3, we have

$$\dim \mathbf{E}_{\mathbf{V},\Omega} - \dim \mathscr{O}_\mathbf{u} = \mathbf{u}\, E\, \mathbf{u}^T,$$

where the superscript T denotes the matrix transposition.

Now we fix $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{N}^{15}$ such that $|\mathbf{u}| = 2p\nu$ and $|\mathbf{v}| = p\nu$ and we fix $x \in \mathscr{O}_\mathbf{u}$.

If there exists a flag $\mathbf{V}^\bullet$ such that $(x, \mathbf{V}^\bullet) \in \widetilde{\mathcal{F}}^{\mathbf{u},\mathbf{v}}$, then we can consider the $\mathbf{k}Q$-modules $M_{k,\ell} = (\mathbf{V}^k/\mathbf{V}^\ell, x)$. They are related by

$$M_{0,5} \cong M_{1,5} \oplus \big(S_2^{\oplus p} \oplus S_4^{\oplus p}\big) \quad \text{and} \quad M_{0,4} \cong M_{1,3} \oplus \big(S_1^{\oplus p} \oplus S_2^{\oplus p} \oplus S_3^{\oplus 2p} \oplus S_4^{\oplus p} \oplus S_5^{\oplus p}\big),$$

because S_2 and S_4 are projective and S_1, S_3 and S_5 are injective. Setting

$$\begin{aligned} \mathbf{s} &= (\,1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,),\\ \mathbf{t} &= (\,1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 2,\ 1\,), \end{aligned}$$

we thus have $\mathbf{n}(M_{1,5}) = \mathbf{u} - p\mathbf{s}$ and $\mathbf{n}(M_{0,4}) = \mathbf{v} + p\mathbf{t}$. Further, Corollary 5.5 asserts that $\mathbf{u} - \mathbf{n}(M_{0,4}) - p\mathbf{x}$ belongs to the $\mathbb{N}$-span of the vectors $\mathbf{w}_k$. Denoting by W the matrix whose lines are the vectors $\mathbf{w}_k$, we deduce the existence of a row vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{17})$ in $\mathbb{N}^{17}$ such that $\mathbf{u} - \mathbf{v} - p\mathbf{y} = \boldsymbol{\tau}\, W$.

The choice of a flag $\mathbf{V}^\bullet$ in the fiber $(\pi^{\mathbf{u},\mathbf{v}})^{-1}(x)$ can be decomposed in three steps: first the choice of $\mathbf{V}^1$, then the choice of $\mathbf{V}^3$, and finally the choice of $\mathbf{V}^2$ and $\mathbf{V}^4$.

We thus begin with $\mathbf{V}^1$. The vector space $\mathbf{V}^1_2$ has codimension p in $\mathbf{V}_2$ and contains the image of x_{h_1} and x_{h_2}. To choose it therefore amounts to choose a codimension p subspace in a space of dimension u_1; in other words, to a point in a Grassmannian of dimension $p(u_1 - p)$. Likewise the choice of $\mathbf{V}^1_4$ amounts to the choice of a point in a Grassmannian of dimension $p(u_2 - p)$. (If p is larger than u_1 or u_2, then the fiber $(\pi^{\mathbf{u},\mathbf{v}})^{-1}(x)$ is empty.) The choice of $\mathbf{V}^1$ therefore contributes a smooth connected variety of dimension $p(u_1 + u_2) - 2p^2$ to the fiber.

Now suppose that $\mathbf{V}^1$ has been chosen, and pick a $\mathbf{k}Q$-module N such that $\mathbf{n}(N) = \mathbf{v}$. An x-stable I-graded $\mathbf{k}$-vector subspace $\mathbf{V}^3$ defines a $\mathbf{k}Q$-module $M_{1,3} = (\mathbf{V}^1/\mathbf{V}^3, x)$. The datum of a $\mathbf{V}^3$ such that $\mathbf{n}(M_{1,3}) = \mathbf{v}$ is equivalent to the datum of a surjective morphism $M_{1,5} \to N$, up to composition with an automorphism of N. Surjective morphisms form an open dense subset in $\mathrm{Hom}_{\mathbf{k}Q}(M_{1,5}, N)$ (or do not exist at all, in which case the fiber $(\pi^{\mathbf{u},\mathbf{v}})^{-1}(x)$ is empty). Therefore the choice of $\mathbf{V}^3$ contributes a smooth connected variety of dimension

$$\dim \mathrm{Hom}_{\mathbf{k}Q}(M_{1,5}, N) - \dim \mathrm{Hom}_{\mathbf{k}Q}(N, N) = (\mathbf{u} - p\mathbf{s})\, H\, \mathbf{v}^T - \mathbf{v}\, H\, \mathbf{v}^T$$

to the fiber.

Lastly, we notice that the choice of $\mathbf{V}^1$ and $\mathbf{V}^3$ fully determines $\mathbf{V}^2$ and $\mathbf{V}^4$:

$$\mathbf{V}_i^2 = \begin{cases} \mathbf{V}_i^3 & \text{if } i \in \{1,3,5\}, \\ \mathbf{V}_i^1 & \text{if } i \in \{2,4\}, \end{cases} \qquad \mathbf{V}_i^4 = \begin{cases} 0 & \text{if } i \in \{1,3,5\}, \\ \mathbf{V}_i^3 & \text{if } i \in \{2,4\}. \end{cases}$$

In other words, the third step does not contribute further to the fiber.

Taking every contribution into account, we find that the difference between the two sides of (8) is equal to

$$\Delta = -4p^2 + \mathbf{u}\, E\, \mathbf{u}^T - 2p\, \mathbf{u}\, \mathbf{s}^T - 2(\mathbf{u} - p\mathbf{s} - \mathbf{v})\, H\, \mathbf{v}^T.$$

In this expression, we substitute $\mathbf{u} = \mathbf{v} + p\mathbf{y} + \boldsymbol{\tau}\, W$. Then Δ is a polynomial of degree 2 in the variables $\mathbf{v}$, p, and $\boldsymbol{\tau}$. Our aim is to show that $\Delta \geq 0$ under the assumptions that the variables are nonnegative and that $|\mathbf{v}| = p\nu$.

The equation $|\mathbf{v}| = p\nu$ is equivalent to the system

$$\begin{aligned} p - (v_3 + v_6 + v_9 + v_{12} + v_{15}) &= 0, \\ 2p - (v_1 + v_3 + v_4 + v_6 + v_7 + v_9 + v_{10} + v_{12}) &= 0, \\ 2p - (v_4 + v_6 + v_7 + v_8 + v_9 + v_{10} + v_{11} + v_{12} + v_{14}) &= 0, \\ 2p - (v_2 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{11}) &= 0, \\ p - (v_5 + v_7 + v_9 + v_{11} + v_{13}) &= 0. \end{aligned}$$

We denote the left-hand sides of these equations by $L_1, \ldots, L_5$, from top to bottom.

Let us write $\Delta = \Delta_0 + \Delta_1 + \Delta_2$, where Δ_d collects the terms of degree d in the variables τ_k. Then

$$\Delta_0 = \mathbf{v}\, E\, \mathbf{v}^T + p\left[\mathbf{y}\,(E + E^T) - 2\mathbf{s} - 2(\mathbf{y} - \mathbf{s})\, H\right] \mathbf{v}^T + p^2(-4 + \mathbf{y}\, E\, \mathbf{y}^T - 2\, \mathbf{y}\, \mathbf{s}^T).$$

If we add

$$L_1(L_1 - L_2 + v_{15}) + L_2(L_2 + v_1 + v_{12}) + L_3(L_3 - L_2 - L_4 + v_{14}) + L_4(L_4 + v_2 + v_{11}) + L_5(L_5 - L_4 + v_{13})$$

to Δ_0, we get, after a lengthy but straightforward calculation

$$\begin{aligned} &(p - v_1)(v_{12} - p) + (p - v_2)(v_{11} - p) \\ &+ (v_3 + v_4 + v_6)(v_6 + v_8 + v_9) + (v_4 + v_5 + v_7)(v_7 + v_9 + v_{10}) \\ &+ (v_3^2 - v_3 v_8 + v_8^2) + (v_4^2 - v_4 v_9 + v_9^2) + (v_5^2 - v_5 v_{10} + v_{10}^2). \end{aligned}$$

The first term is

$$\left(\frac{v_{12}-v_1}{2}\right)^2 - \left(p - \frac{v_1+v_{12}}{2}\right)^2 \equiv \left(\frac{v_{12}-v_1}{2}\right)^2 - \left(\frac{v_3+v_4+v_6}{2} + \frac{v_7+v_9+v_{10}}{2}\right)^2 \pmod{L_2}$$

and the second term is

$$\left(\frac{v_{11}-v_2}{2}\right)^2 - \left(p - \frac{v_2+v_{11}}{2}\right)^2 \equiv \left(\frac{v_{11}-v_2}{2}\right)^2 - \left(\frac{v_4+v_5+v_7}{2} + \frac{v_6+v_8+v_9}{2}\right)^2 \pmod{L_4}.$$

Modulo the relations L_i, Δ_0 is thus congruent to

$$\begin{aligned}\widetilde{\Delta}_0 = {} & \left(\frac{v_{12}-v_1}{2}\right)^2 + \left(\frac{v_{11}-v_2}{2}\right)^2 \\ & - \left(\frac{v_3+v_4+v_6}{2} + \frac{v_7+v_9+v_{10}}{2}\right)^2 - \left(\frac{v_4+v_5+v_7}{2} + \frac{v_6+v_8+v_9}{2}\right)^2 \\ & + (v_3+v_4+v_6)(v_6+v_8+v_9) + (v_4+v_5+v_7)(v_7+v_9+v_{10}) \\ & + \left(\frac{v_3-v_8}{2} + \frac{v_4-v_9}{2}\right)^2 + \left(\frac{v_4-v_9}{2} + \frac{v_5-v_{10}}{2}\right)^2 + \left(\frac{v_3-v_8}{2} - \frac{v_5-v_{10}}{2}\right)^2 \\ & + \left(\frac{v_3-v_8}{2} - \frac{v_4-v_9}{2} + \frac{v_5-v_{10}}{2}\right)^2 + \left(\frac{v_3+v_8}{2}\right)^2 + \left(\frac{v_4+v_9}{2}\right)^2 + \left(\frac{v_5+v_{10}}{2}\right)^2.\end{aligned}$$

A final transformation yields

$$\begin{aligned}\widetilde{\Delta}_0 = {} & \left(\frac{v_{12}-v_1}{2}\right)^2 + \left(\frac{v_{11}-v_2}{2}\right)^2 \\ & + \frac{1}{2}\left(v_6 - v_7 + \frac{v_3-v_5+v_8-v_{10}}{2}\right)^2 + \frac{1}{2}\left(\frac{v_3-v_8}{2} - \frac{v_5-v_{10}}{2}\right)^2 \\ & + \left(\frac{v_3-v_8}{2} - \frac{v_4-v_9}{2} + \frac{v_5-v_{10}}{2}\right)^2 + \left(\frac{v_3+v_8}{2}\right)^2 + \left(\frac{v_4+v_9}{2}\right)^2 + \left(\frac{v_5+v_{10}}{2}\right)^2.\end{aligned}$$

We now turn to

$$\Delta_1 = \boldsymbol{\tau}\, W(E+E^T-2H)\,\mathbf{v}^T + p\,\boldsymbol{\tau}\, W\big[(E+E^T)\,\mathbf{y}^T - 2\mathbf{s}^T\big].$$

We add

$$\begin{aligned}& \tau_1(L_1-2L_2+L_3) + \tau_2(L_3-2L_4+L_5) - \tau_3 L_1 - (\tau_4+\tau_5)L_2 \\ & - (\tau_6+\tau_7+\tau_8+\tau_9)L_3 - (\tau_{10}+\tau_{11})L_4 - \tau_{12}L_5 - \tau_{16}(L_3+L_4)/4 - \tau_{17}(L_2+L_3)/4\end{aligned}$$

to Δ_1, and write the result $\widetilde{\Delta}_1$ in the form $\boldsymbol{\tau}\,\mathbf{d}^T$. The components of $\mathbf{d}$ are given below.

$$
\begin{aligned}
d_1 &= v_1 + v_{12} & d_4 &= v_1 + v_4 + v_8 + v_{11} + 2v_{12} + v_{14} + v_{15}\\
d_2 &= v_2 + v_{11} & d_5 &= v_1 + v_8 + v_9 + v_{11} + 2v_{12} + v_{14} + v_{15}\\
d_3 &= v_8 + v_{11} + v_{12} + v_{14} + v_{15} & d_6 &= v_8 + v_9 + 2v_{11} + 2v_{12} + 2v_{14} + v_{15}\\
& & d_7 &= v_4 + v_8 + 2v_{11} + 2v_{12} + 2v_{14} + v_{15}\\
d_{12} &= v_{10} + v_{11} + v_{12} + v_{13} + v_{14} & d_8 &= v_4 + v_{10} + 2v_{11} + 2v_{12} + v_{13} + 2v_{14}\\
d_{13} &= -v_8 + v_{13} & d_9 &= v_9 + v_{10} + 2v_{11} + 2v_{12} + v_{13} + 2v_{14}\\
d_{14} &= -v_9 + v_{14} & d_{10} &= v_2 + v_9 + v_{10} + 2v_{11} + v_{12} + v_{13} + v_{14}\\
d_{15} &= -v_{10} + v_{15} & d_{11} &= v_2 + v_4 + v_{10} + 2v_{11} + v_{12} + v_{13} + v_{14}
\end{aligned}
$$

$$
\begin{aligned}
d_{16} &= \frac{v_2}{4} + \frac{v_4}{2} + \frac{v_5}{4} - \frac{v_6}{2} + \frac{v_7}{2} + \frac{v_8}{2} - \frac{v_9}{2} + \frac{v_{10}}{4} + \frac{3v_{11}}{2} + \frac{v_{12}}{4} + \frac{5v_{14}}{4}\\
d_{17} &= \frac{v_1}{4} + \frac{v_3}{4} + \frac{v_4}{2} + \frac{v_6}{2} - \frac{v_7}{2} + \frac{v_8}{4} - \frac{v_9}{2} + \frac{v_{10}}{2} + \frac{v_{11}}{4} + \frac{3v_{12}}{2} + \frac{5v_{14}}{4}
\end{aligned}
$$

We thus have

$$
\begin{aligned}
\widetilde{\Delta}_1 \geq{}& \left(\frac{\tau_6 + \tau_7 + \tau_{16}}{2} - \tau_{13}\right) v_8 - \left(\frac{\tau_{16} + \tau_{17}}{2} + \tau_{14}\right) v_9 + \left(\frac{\tau_8 + \tau_9 + \tau_{17}}{2} - \tau_{15}\right) v_{10}\\
&- \frac{\tau_{16} - \tau_{17}}{2}\left(v_6 - v_7 + \frac{v_3 - v_5 + v_8 - v_{10}}{2}\right).
\end{aligned}
$$

Last, $\Delta_2 = \boldsymbol{\tau}\, W\, E\, W^T\, \boldsymbol{\tau}^T$ is the quadratic form given by

$$
\begin{aligned}
\Delta_2 ={}& \tau_1^2 + \tau_2^2 + \tau_3^2 + \tau_4^2 + \tau_5^2 + \tau_6^2 + \tau_7^2 + \tau_8^2 + \tau_9^2 + \tau_{10}^2 + \tau_{11}^2 + \tau_{12}^2 + \tau_{13}^2 + \tau_{14}^2 + \tau_{15}^2\\
&+ \tau_{16}^2 + \tau_{17}^2 + \tau_1\tau_3 + \tau_1\tau_4 + \tau_1\tau_5 + \tau_1\tau_6 + \tau_1\tau_7 + \tau_1\tau_{14} + \tau_1\tau_{15} + \tau_1\tau_{16} + \tau_2\tau_8\\
&+ \tau_2\tau_9 + \tau_2\tau_{10} + \tau_2\tau_{11} + \tau_2\tau_{12} + \tau_2\tau_{13} + \tau_2\tau_{14} + \tau_2\tau_{17} + \tau_3\tau_4 + \tau_3\tau_5 + \tau_3\tau_{14}\\
&+ \tau_3\tau_{16} + \tau_4\tau_5 + \tau_4\tau_7 - \tau_4\tau_9 - \tau_4\tau_{10} + \tau_4\tau_{15} + \tau_5\tau_6 - \tau_5\tau_8 - \tau_5\tau_{11} + \tau_5\tau_{14}\\
&+ \tau_5\tau_{15} + \tau_6\tau_7 - \tau_6\tau_8 - \tau_6\tau_{11} - \tau_6\tau_{13} + \tau_6\tau_{14} + \tau_6\tau_{15} + \tau_6\tau_{16} - \tau_7\tau_9 - \tau_7\tau_{10}\\
&- \tau_7\tau_{13} + \tau_7\tau_{15} + \tau_7\tau_{16} + \tau_8\tau_9 + \tau_8\tau_{11} + \tau_8\tau_{13} - \tau_8\tau_{15} + \tau_8\tau_{17} + \tau_9\tau_{10} + \tau_9\tau_{13}\\
&+ \tau_9\tau_{14} - \tau_9\tau_{15} + \tau_9\tau_{17} + \tau_{10}\tau_{11} + \tau_{10}\tau_{12} + \tau_{10}\tau_{13} + \tau_{10}\tau_{14} + \tau_{11}\tau_{12} + \tau_{11}\tau_{13}\\
&+ \tau_{12}\tau_{14} + \tau_{12}\tau_{17} - \tau_{13}\tau_{16} + \tau_{14}\tau_{16} + \tau_{14}\tau_{17} - \tau_{15}\tau_{17},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\Delta_2 \geq{}& \tau_1^2 + \tau_2^2 + \tau_3^2 + \tau_4^2 + \tau_5^2 + \tau_6^2 + \tau_7^2 + \tau_8^2 + \tau_9^2 + \tau_{10}^2 + \tau_{11}^2 + \tau_{12}^2 + \tau_{13}^2 + \tau_{14}^2 + \tau_{15}^2\\
&+ \tau_{16}^2 + \tau_{17}^2 - \tau_{13}(\tau_6 + \tau_7 + \tau_{16}) - \tau_{15}(\tau_8 + \tau_9 + \tau_{17}) - (\tau_4 + \tau_7)(\tau_9 + \tau_{10})\\
&- (\tau_5 + \tau_6)(\tau_8 + \tau_{11}) + \tau_{16}\left(\tau_{14} + \frac{\tau_6 + \tau_7}{2}\right) + \tau_{17}\left(\tau_{14} + \frac{\tau_8 + \tau_9}{2}\right)\\
&+ (\tau_4\tau_5 + \tau_5\tau_6 + \tau_6\tau_7 + \tau_4\tau_7) + (\tau_8\tau_9 + \tau_9\tau_{10} + \tau_{10}\tau_{11} + \tau_8\tau_{11}).
\end{aligned}
$$

Modulo the relations L_i, Δ is congruent to $\widetilde{\Delta} = \widetilde{\Delta}_0 + \widetilde{\Delta}_1 + \Delta_2$, and we have

$$\begin{aligned}
\widetilde{\Delta} \geq\ & \left(\frac{v_{12}-v_1}{2}\right)^2 + \left(\frac{v_{11}-v_2}{2}\right)^2 + \frac{1}{2}\left(v_6 - v_7 + \frac{v_3 - v_5 + v_8 - v_{10}}{2} - \frac{\tau_{16}-\tau_{17}}{2}\right)^2 \\
& + \frac{1}{2}\left(\frac{v_3 - v_8}{2} - \frac{v_5 - v_{10}}{2}\right)^2 + \left(\frac{v_3 - v_8}{2} - \frac{v_4 - v_9}{2} + \frac{v_5 - v_{10}}{2}\right)^2 \\
& + \frac{v_3^2 + v_4^2 + v_5^2}{4} + \frac{v_3 v_8 + v_4 v_9 + v_5 v_{10}}{2} + \left(\frac{v_8}{2} + \frac{\tau_6 + \tau_7 + \tau_{16}}{2} - \tau_{13}\right)^2 \\
& + \left(\frac{v_9}{2} - \frac{\tau_{16} + \tau_{17}}{2} - \tau_{14}\right)^2 + \left(\frac{v_{10}}{2} + \frac{\tau_8 + \tau_9 + \tau_{17}}{2} - \tau_{15}\right)^2 \\
& + \frac{(\tau_{16} - \tau_{17})^2}{8} + \frac{\tau_{16}^2 + \tau_{17}^2}{4} + \tau_1^2 + \tau_2^2 + \tau_3^2 + \tau_{12}^2 + \left(\frac{\tau_6 + \tau_7}{2}\right)^2 + \left(\frac{\tau_8 + \tau_9}{2}\right)^2 \\
& + \frac{1}{2}\big((\tau_4 + \tau_5)^2 + (\tau_{10} + \tau_{11})^2 + (\tau_4 + \tau_7 - \tau_9 - \tau_{10})^2 + (\tau_5 + \tau_6 - \tau_8 - \tau_{11})^2\big) \\
\geq\ & 0.
\end{aligned}$$

This concludes the proof of equation (8).

To finish the proof of Proposition 5.6, it remains to study the case of equality. From the minoration above, one easily sees that $\widetilde{\Delta} = 0$ is possible only if all the τ_k vanish, except perhaps τ_{13}, τ_{14} and τ_{15}, and if

$$\begin{gathered}
v_1 = v_{12}, \quad v_2 = v_{11}, \quad v_3 = v_4 = v_5 = 0, \quad v_6 = v_7, \\
v_8 = v_9/2 = v_{10}, \quad v_8 = 2\tau_{13}, \quad v_9 = 2\tau_{14}, \quad v_{10} = 2\tau_{15}.
\end{gathered}$$

Substituting into $L_i = 0$, we then get

$$v_1 = v_2 = v_{11} = v_{12}, \quad v_8 = v_9 = v_{10} = v_{13} = v_{14} = v_{15} = 0, \quad p = v_1 + v_6, \quad \boldsymbol{\tau} = 0.$$

Therefore $\mathbf{v} = (p - 2v_6)\mathbf{y} + v_6\mathbf{z}$ and $\mathbf{u} = \mathbf{v} + p\mathbf{y} = 2(p - v_6)\mathbf{y} + v_6\mathbf{z}$, as desired.

The proof of the converse (that is, if $\mathbf{u}$ and $\mathbf{v}$ have the required form, then $\widetilde{\mathscr{F}}^{\mathbf{u},\mathbf{v}} \neq \varnothing$ and (8) in an equality) is easy and left to the reader.

6 The canonical basis and the quantum Frobenius morphism

In this section, we study the compatibility of the quantum Frobenius morphism and its splitting with bases of canonical type. These morphisms require the use of the quantum group, so from now on, $\mathbf{f}$ denotes the quantum deformation of the algebra defined in Section 2.2. In addition, we also need a basis that lifts to the quantum group and can be specialized to a quantum root of unity. This invites us to restrict our attention to the canonical basis.

6.1 Background on the quantum Frobenius morphism

We follow the notation set up in Lusztig's book [38], and in particular assume that the conditions (a) and (b) in Section 35.1.2 of that book hold true.

Let (d_i) be a family of positive integers such that the matrix $(d_i a_{i,j})$ is symmetric. Let v be an indeterminate. For $n \in \mathbb{N}$ and $i \in I$, we define the Gaussian number $[n]_i$ and the Gaussian factorial $[n]_i!$ as in Section 3.1. From these data, we define $\mathbf{f}$ as the $\mathbb{Q}(v)$-algebra generated by elements θ_i, for $i \in I$, submitted to the relations (1), in which the ordinary factorials $p!$ and $q!$ are replaced by their Gaussian counterparts $[p]_i!$ and $[q]_i!$.

We set $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. It is known that the $\mathcal{A}$-subalgebra of $\mathbf{f}$ generated by the divided powers $\theta_i^{(n)} = \theta_i^n/[n]_i!$ is an $\mathcal{A}$-form ${}_{\mathcal{A}}\mathbf{f}$ in $\mathbf{f}$. We can then specialize the parameter v to any invertible element in a commutative ring R by a base change $R \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathbf{f}$ ([38], Section 31.1). Thus for instance, the algebra $\mathbf{f}$ from Section 2.2 is the specialization over $\mathbb{Q}$ at the value $v = 1$. The canonical basis is in fact an $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathbf{f}$, so it induces an R-basis in each specialization ${}_R\mathbf{f}$.

Let ℓ be a positive integer. For $i \in I$, let ℓ_i be the smallest positive integer such that $\ell_i d_i \in \ell\mathbb{Z}$. We define a new symmetrizable Cartan matrix $A^* = (a^*_{i,j})$ by $a^*_{i,j} = a_{i,j}\ell_j/\ell_i$; setting $d^*_i = d_i\ell_i^2$, the matrix $(d^*_i a^*_{i,j})$ is symmetric. We then have a $\mathbb{Q}(v)$-algebra $\mathbf{f}^*$ and an $\mathcal{A}$-form ${}_{\mathcal{A}}\mathbf{f}^*$. The simple roots and coroots for the starred Cartan datum are chosen to be $\alpha^*_i = \ell_i\alpha_i$ and $(\alpha^*_i)^\vee = \alpha_i^\vee/\ell_i$.

Let $\Phi_{2\ell}$ be the 2ℓ-th cyclotomic polynomial, and let $R = \mathbb{Q}[\zeta]/(\Phi_{2\ell}(\zeta))$. Let ${}_R\mathbf{f}$ and ${}_R\mathbf{f}^*$ be the specializations of $\mathbf{f}$ and $\mathbf{f}^*$ over R at the value $v = \zeta$. There is then an algebra homomorphism $Fr_\ell : {}_R\mathbf{f} \to {}_R\mathbf{f}^*$ that maps the generator $\theta_i^{(n)}$ to $\theta_i^{(n/\ell_i)}$ if n is a multiple of ℓ_i and to 0 otherwise; this morphism is called the quantum Frobenius map. In the other direction, there is an algebra homomorphism $Fr'_\ell : {}_R\mathbf{f}^* \to {}_R\mathbf{f}$ that maps $\theta_i^{(n)}$ to $\theta_i^{(n\ell_i)}$; this map is called the quantum Frobenius splitting.

Let $B(-\infty)^*$ be the analogue of the crystal $B(-\infty)$ for the Cartan matrix A^* and the algebra $\mathbf{f}^*$. By construction, the Frobenius splitting Fr'_ℓ has some kind of compatibility with the conditions (i)–(iii) in the definition of a basis of canonical type. One may thus expect the existence of a map $S_\ell : B(-\infty)^* \to B(-\infty)$ that reflects the action of Fr'_ℓ at the level of the crystals. Such a map S_ℓ has been constructed by Kashiwara ([27], Theorems 3.2 and 5.1); it satisfies

$$\operatorname{wt}(S_\ell(b)) = \operatorname{wt}(b), \qquad \varepsilon_i(S_\ell(b)) = \ell_i\varepsilon_i(b), \qquad \varphi_i(S_\ell(b)) = \ell_i\varphi_i(b),$$
$$S_\ell(\tilde{e}_i b) = \tilde{e}_i^{\ell_i} S_\ell(b), \qquad S_\ell\big(\tilde{f}_i b\big) = \tilde{f}_i^{\ell_i} S_\ell(b).$$

These equations take into account the convention that the simple roots and coroots for the starred root datum are given by $\alpha^*_i = \ell_i\alpha_i$ and $(\alpha^*_i)^\vee = \alpha_i^\vee/\ell_i$.

6.2 Compatibility up to a filtration

As promised, we now study the compatibility of Fr_ℓ and Fr'_ℓ with the canonical bases of ${}_R\mathbf{f}$ and ${}_R\mathbf{f}^*$. The best compatibility we could hope for would be

$$Fr_\ell(G(b')) = \begin{cases} G(b'') & \text{if } b' = S_\ell(b'') \\ 0 & \text{if } b' \notin \operatorname{im} S_\ell \end{cases} \qquad \text{and} \qquad Fr'_\ell(G(b'')) = G(S_\ell(b'')), \tag{9}$$

where $G(b')$ and $G(b'')$ denote the elements in the canonical bases of ${}_R\mathbf{f}$ and ${}_R\mathbf{f}^*$ that correspond to $b' \in B(-\infty)$ and $b'' \in B(-\infty)^*$.

This property holds true in types A_1, A_2, A_3 and B_2. I owe this nice observation to Littelmann, who checked it using the explicit formulas for the canonical basis given by Lusztig ([35], Section 3.4) and Xi [49, 50]. Alas, (9) fails in types A_5 and D_4, as we will see in Section 6.3.

One can however hope to restore the compatibility by working with filtrations, so as to be able to neglect undesired terms in the expansion of $Fr_\ell(G(b'))$ and $Fr'_\ell(G(b''))$. To this aim, we must study how these terms compare with the expected one.

The next proposition is a crude result, whose proof relies solely on the property that the canonical basis is of canonical type.

Proposition 6.1 *Let $\ell \geq 1$ and let $(b', b'') \in (B(-\infty)^*)^2$.*

(i) In order that $G(b'')$ actually occurs in the expansion of $Fr_\ell(G(S_\ell(b')))$ on the canonical basis of ${}_R\mathbf{f}^*$, it is necessary that $b' \leq_{\text{str}} b''$. Moreover, $G(b')$ occurs with coefficient 1 in the expansion of $Fr_\ell(G(S_\ell(b')))$.

(ii) In order that $G(S_\ell(b'))$ actually occurs in the expansion of $Fr'_\ell(G(b''))$ on the canonical basis of ${}_R\mathbf{f}$, it is necessary that $b'' \leq_{\text{str}} b'$. Moreover, $G(S_\ell(b''))$ occurs with coefficient 1 in the expansion of $Fr_\ell(G(b''))$.

Proof. To avoid confusion with the star in the notation $\mathbf{f}^*$, it will be convenient to denote duality with a superscript $\vee$. Thus $({}_R\mathbf{f})^\vee = \operatorname{Hom}_R({}_R\mathbf{f}, R)$ and $({}_R\mathbf{f}^*)^\vee = \operatorname{Hom}_R({}_R\mathbf{f}^*, R)$. The elements in the dual canonical bases of these algebras are denoted by $G(b)^\vee$, where b is in $B(-\infty)$ or in $B(-\infty)^*$, respectively.

Let $\ell \geq 1$ and let $(b, b'') \in B(-\infty) \times B(-\infty)^*$. Choose $i \in I$ and set $k = \lfloor \varphi_i(b)/\ell_i \rfloor$, the largest integer smaller than or equal to $\varphi_i(b)/\ell_i$. By Theorem 14.3.2 in [38], there are elements $x_n \in {}_R\mathbf{f}$ such that

$$\theta_i^{(\varphi_i(b))}\, G\big(\tilde f_i^{\max} b\big) = G(b) + \sum_{n > \varphi_i(b)} \theta_i^{(n)} x_n.$$

Applying Fr_ℓ to this equation, we obtain that modulo $\theta_i^{k+1}\,{}_R\mathbf{f}^*$,

$$Fr_\ell(G(b)) \equiv \begin{cases} \theta_i^{(k)} Fr_\ell\big(G\big(\tilde f_i^{\max} b\big)\big) & \text{if } \varphi_i(b) = k\ell_i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $G(b'')$ actually occurs in the expansion of $Fr_\ell(G(b))$ on the canonical basis of ${}_R\mathbf{f}^*$, then either $\varphi_i(b'') \geq k+1$, or $\varphi_i(b) = k\ell_i$ and $\varphi_i(b'') = k$; in any case, $\varphi_i(b'') \geq \varphi_i(b)/\ell_i$. Moreover, when $\varphi_i(b'') = \varphi_i(b)/\ell_i$,

$$\big\langle G(b'')^\vee, Fr_\ell(G(b))\big\rangle = \big\langle G(b'')^\vee, \theta_i^{(k)} Fr_\ell\big(G\big(\tilde{f}_i^{\max} b\big)\big)\big\rangle = \big\langle G\big(\tilde{f}_i^{\max} b''\big)^\vee, Fr_\ell\big(G\big(\tilde{f}_i^{\max} b\big)\big)\big\rangle.$$

Now let $b' \in B(-\infty)^*$. Applying the previous reasoning to $b = S_\ell(b')$ and using that σ commutes with Fr_ℓ and with S_ℓ, we eventually obtain:

- If $\big\langle G(b'')^\vee, Fr_\ell(G(S_\ell(b')))\big\rangle \neq 0$, then $\varphi_i(b'') \geq \varphi_i(b')$ for each $i \in I$.
- If $(c', c'') \in (B(-\infty)^*)^2$ is such that $(b', b'') \approx (c', c'')$, then

$$\big\langle G(b'')^\vee, Fr_\ell(G(S_\ell(b')))\big\rangle = \big\langle G(c'')^\vee, Fr_\ell(G(S_\ell(c')))\big\rangle.$$

This proves (i).

The proof of (ii) is completely analogous. □

Suppose now that the Cartan matrix A is of finite type and adopt the notation of Section 3.1. Observing that the Weyl group for the Cartan matrices A and A^* are the same, we can use the same $\mathbf{i} \in \mathscr{X}$ to construct a PBW basis in $\mathbf{f}$ and in $\mathbf{f}^*$. Abusing slightly the notation, we denote the elements in these bases by $E_{\mathbf{i}}^{(\mathbf{n})}$ and ${}^*E_{\mathbf{i}}^{(\mathbf{n})}$, for $\mathbf{n} \in \mathbb{N}^N$ — the abuse is that in Section 3.1, $E_{\mathbf{i}}^{(\mathbf{n})}$ was a basis of $U_q(\mathfrak{n}_+)$, which we now transport to $\mathbf{f}$. By Section 7.1 in [37], these bases are in fact bases of ${}_\mathcal{A}\mathbf{f}$ and ${}_\mathcal{A}\mathbf{f}^*$, so they can be specialized to ${}_R\mathbf{f}$ and ${}_R\mathbf{f}^*$. Lastly, we define the map $S_{\mathbf{i},\ell} : (n_1, \ldots, n_N) \mapsto (\ell_{i_1} n_1, \ldots, \ell_{i_N} n_N)$ from $\mathbb{N}^N$ to itself, and we denote by $\leq_{\mathbf{i}}$ and $\leq_{\mathbf{i}}^*$ the orders on $\mathbb{N}^N$ relative to the Cartan matrices A and A^*, respectively.

Lemma 6.2 *Let $\ell \geq 1$ and let $\mathbf{i} \in \mathscr{X}$.*

(i) The diagram

$$\begin{array}{ccc} B(-\infty)^* & \xrightarrow{\;S_\ell\;} & B(-\infty) \\ {\scriptstyle n_{\mathbf{i}}}\downarrow & & \downarrow{\scriptstyle n_{\mathbf{i}}} \\ \mathbb{N}^N & \xrightarrow{\;S_{\mathbf{i},\ell}\;} & \mathbb{N}^N \end{array}$$

commutes.

(ii) Let $\mathbf{m}$ and $\mathbf{n}$ in $\mathbb{N}^N$. Then $\mathbf{n} \leq_{\mathbf{i}}^* \mathbf{m}$ if and only if $S_{\mathbf{i},\ell}(\mathbf{n}) \leq_{\mathbf{i}} S_{\mathbf{i},\ell}(\mathbf{m})$.

(iii) Let $\mathbf{n} \in \mathbb{N}^N$. If $\mathbf{n}$ does not belong to the image of $S_{\mathbf{i},\ell}$, then $Fr_\ell\big(E_{\mathbf{i}}^{(\mathbf{n})}\big) = 0$. Otherwise, $Fr_\ell\big(E_{\mathbf{i}}^{(\mathbf{n})}\big) = {}^*E_{\mathbf{i}}^{(\mathbf{m})}$, where $\mathbf{m} = S_{\mathbf{i},\ell}^{-1}(\mathbf{n})$.

(iv) Let $\mathbf{m}$ and $\mathbf{n}$ in $\mathbb{N}^N$. If $E_{\mathbf{i}}^{(\mathbf{m})}$ actually occurs in the expansion of $Fr'_\ell\big({}^*E_{\mathbf{i}}^{(\mathbf{n})}\big)$ on the PBW basis of ${}_R\mathbf{f}$, then $\mathbf{m} \geq_{\mathbf{i}} S_{\mathbf{i},\ell}(\mathbf{n})$.

Proof. Given $(\mathbf{i},\mathbf{j}) \in \mathscr{X}^2$, let $R_{\mathbf{i}}^{\mathbf{j}}$ be the composition $\mathbb{N}^N \xrightarrow{\mathbf{b_i}} B(-\infty) \xrightarrow{\mathbf{n_j}} \mathbb{N}^N$ and let ${}^*R_{\mathbf{i}}^{\mathbf{j}}$ be the composition $\mathbb{N}^N \xrightarrow{\mathbf{b_i}} B(-\infty)^* \xrightarrow{\mathbf{n_j}} \mathbb{N}^N$. Using the explicit formulas for these piecewise linear bijections $R_{\mathbf{i},\mathbf{j}}$ and ${}^*R_{\mathbf{i},\mathbf{j}}$ (see Section 12.6 in [37] and Theorem 5.2 and Proposition 7.1 in [9]), one checks that the diagram

$$\begin{array}{ccc} \mathbb{N}^N & \xrightarrow{S_{\mathbf{i},\ell}} & \mathbb{N}^N \\ {\scriptstyle {}^*R_{\mathbf{i}}^{\mathbf{j}}}\downarrow & & \downarrow{\scriptstyle R_{\mathbf{i}}^{\mathbf{j}}} \\ \mathbb{N}^N & \xrightarrow{S_{\mathbf{j},\ell}} & \mathbb{N}^N \end{array}$$

commutes. From there, one shows assertion (i) by induction on the weight, using the same arguments as those used in [9], proof of Theorem 5.7 (see in particular p. 112, l. 5–12).

Assertion (ii) follows from the definitions by a straightforward computation.

Assertion (iii) comes from the fact that the quantum Frobenius morphism Fr_ℓ is compatible with Lusztig symmetries $T'_{i,-1}$ ([38], Section 41.1.9), which are the main ingredient in the construction of the PBW bases.

To prove (iv), one begins with the particular case where all entries of $\mathbf{n}$ but one vanish. Certainly then $S_\ell(\mathbf{n})$ has the same property, so every $\mathbf{m} \in \mathbb{N}^N$ such that $|\mathbf{m}| = |S_{\mathbf{i},\ell}(\mathbf{n})|$ satisfies $\mathbf{m} \geq_{\mathbf{i}} S_{\mathbf{i},\ell}(\mathbf{n})$. This obvioulsy implies the desired property. The general case then follows by induction on the number of nonzero entries in $\mathbf{n}$, using Lemma 3.4 and the fact that Fr'_ℓ is a morphism of algebras. □

The next proposition states that the compatibility condition (9) can be restored by filtering $\mathbf{f}$ and $\mathbf{f}^*$ with the help of $\leq_{\text{pol}}$. It thus tells us that S_ℓ is the crystal version of Fr'_ℓ.

Proposition 6.3 *Let $\ell \geq 1$ and let $(b', b'') \in B(-\infty) \times B(-\infty)^*$.*

(i) If $G(b'')$ actually occurs in the expansion of $Fr_\ell(G(b'))$ on the canonical basis of ${}_R\mathbf{f}^*$, then $b' \leq_{\text{pol}} S_\ell(b'')$.

(ii) If $G(b')$ actually occurs in the expansion of $Fr'_\ell(G(b''))$ on the canonical basis of ${}_R\mathbf{f}$, then $S(b'') \leq_{\text{pol}} b'$.

Proof. This follows from Proposition 3.1, Corollary 3.6, and Lemma 6.2 by routine arguments. □

Remark 6.4. In the Hall algebra model for quantum groups, the natural basis of the Hall algebra corresponds to a PBW basis (see [45] for a survey). The compatibility of the quantum Frobenius morphism with the PBW bases (Lemma 6.2 (iii)) then leads to an interpretation of Fr_ℓ within the framework of Hall algebras, which can be used as an alternate definition of Fr_ℓ [40]. Conversely, one can adopt this Hall algebra approach to show the compatibility of Fr_ℓ with the automorphisms $T'_{i,-1}$, using Theorem 6 in [45] or Theorem 13.1 in [48].

6.3 Counterexamples in type A_5 and D_4

As mentioned at the beginning of Section 6.2, counterexamples to (9) do exist in type A_5 and D_4.

Let us take $d_i = 1$ for each i, whence $\ell_i = \ell$ and $A^* = A$, and let us adopt the notation of Section 5.1. Since

$$Fr_\ell(\xi_p) = \begin{cases} \xi_{p/\ell} & \text{if } \ell \text{ divides } p, \\ 0 & \text{otherwise,} \end{cases} \qquad Fr_\ell(\eta_p) = \begin{cases} \eta_{p/\ell} & \text{if } \ell \text{ divides } p, \\ 0 & \text{otherwise,} \end{cases}$$

$$Fr'_\ell(\xi_p) = \xi_{\ell p}, \qquad Fr'_\ell(\eta_p) = \eta_{\ell p},$$

Proposition 5.1 and Theorem 5.2 (ii) lead to

$$Fr_\ell(G(b_{p,0})) = \begin{cases} G(b_{p/\ell,0}) & \text{if } \ell \text{ divides } p, \\ 0 & \text{otherwise,} \end{cases}$$

$$Fr_2(G(b_{0,1})) + G(b_{1,0}) = 0,$$
$$Fr_2(G(b_{0,2})) + Fr_2(G(b_{2,1})) = G(b_{0,1}),$$
$$Fr_4(G(b_{0,2})) + Fr_4(G(b_{2,1})) + G(b_{1,0}) = 0,$$

and to

$$Fr'_\ell(G(b_{p,0})) = G(b_{\ell p,0}),$$
$$Fr'_\ell(G(b_{0,1})) = G(b_{0,\ell}) + G(b_{2,\ell-1}) + G(b_{4,\ell-2}) + \cdots + G(b_{2\ell-2,1}).$$

In addition, in the case (III), calculations made by a computer running GAP and its package QuaGroup [15, 13] lead to further relations in ${}_{\mathcal{A}}\mathbf{f}$. Since all $d_i = 1$, we may drop the subscript i in the notation for the Gaussian numbers. Let us introduce a linear operator $R_{i,\ell} : {}_{\mathcal{A}}\mathbf{f} \to {}_{\mathcal{A}}\mathbf{f}$ by

$$R_{i,\ell}(x) = \theta_i^{(\ell-1)} x \theta_i - [\ell - 2]\, \theta_i^{(\ell)} x,$$

where $i \in \{1, 3, 4\}$ and $\ell \in \mathbb{N}$. Then

$$\theta_2^{(2)} \Big(R_{1,3} \circ R_{3,3} \circ R_{4,3}\big(\theta_2^{(2)}\big)\Big)\, \theta_2^{(2)} = G(b_{1,1}) + [2]^2\, G(b_{3,0}),$$
$$\theta_2^{(3)} \Big(R_{1,4} \circ R_{3,4} \circ R_{4,4}\big(\theta_2^{(2)}\big)\Big)\, \theta_2^{(3)} = G(b_{2,1}) + [3]^2\, G(b_{4,0}).$$

Using the congruences $[3]^2 \equiv 1$ modulo Φ_4 or Φ_8 and $[2]^2 \equiv 1 \pmod{\Phi_6}$, we deduce

$$Fr_2(G(b_{2,1})) + G(b_{2,0}) = 0,$$
$$Fr_3(G(b_{1,1})) + G(b_{1,0}) = 0,$$
$$Fr_4(G(b_{2,1})) + G(b_{1,0}) = 0,$$

whence

$$Fr_2(G(b_{0,2})) = G(b_{0,1}) + G(b_{2,0}).$$

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Pierre Baumann

Institut de Recherche Mathématique Avancée, Université de Strasbourg et CNRS, 7 rue René Descartes, 67084 Strasbourg Cedex, France

p.baumann@unistra.fr

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