Jordan Triples and Operads

Allahtan Victor Gnedbaye and Marc Wambst

Abstract. The Jordan triple of any Jordan algebra gives rise to a ternary Jordan algebra. We study these ternary algebras in terms of operads. We give the description of the operad of ternary Jordan algebras as a quadratic operad and prove that the quadratic dual of this operad is the operad of partially antisymmetric, partially associative ternary algebras.

Résumé. Le triple de Jordan munit toute algèbre de Jordan d'une structure d'algèbre de Jordan ternaire. Nous étudions ces algèbres ternaires en termes d'opérades. Nous donnons une description de l'opérade quadratique des algèbres de Jordan ternaires et montrons que son dual quadratique est l'opérade des algèbres ternaires partiellement associatives, partiellement antisymétriques.

Mathematics Subject Classifications (1991): 17A30, 17A40, 17B60, 17C05, 17C10, 17C45, 17C50, 17D99, 18G40, 18G50, 18G60, 18G99

Key words: Jordan algebras, Jordan triple system, n-ary algebra, Koszul duality, operad.

Introduction.

Jordan algebras were introduced by P. Jordan in the 1930's [Jo] in order to axiomatize the algebraic relations of operators arising in quantum mechanics. The algebras and their generalizations were also studied by N. Jacobson, K. MacCrimmon, K. Meyberg, E. Neher ([Ja2, McC, Mey, Neh]).

This article should be seen as a first step in the construction of a proper homological theory which seems lacking for Jordan algebras, as remarked by M. Dubois-Violette [D-V]. Actually, N. Glassman ([Gla]) defined a cohomology theory for Jordan algebras but his definition does not provide a cochain complex that could be used for computations. In order to fill this gap, we use the theory of operads which is a natural framework to study classes of algebras. Indeed, a homology theory given by a chain complex arises naturally in the context of Koszul duality of quadratic operads. For example, in the case of associative, commutative or Lie algebras, one recovers in this way the Hochschild homology, Harrison homology or Chevalley-Eilenberg homology, respectively.

Jordan algebras are not quadratic in the standard sense of operads. Nevertheless, as pointed out by J.-L. Loday, *the triple Jordan product* which can be associated with any Jordan algebra satisfies a quadratic relation. It is then natural to consider ternary algebras, *the ternary Jordan algebras*, which are quadratic in the sense of operads. In this article, we describe the quadratic operad of ternary Jordan algebras and prove that its dual is the quadratic operad of partially associative and partially anti-symmetric ternary algebras. The latter had already been considered by the first author in [Gn1, Gn2]. They are natural in the context of n-ary algebras.

The article is organized in three sections and an appendix. In the first section, we define the ternary Jordan algebras which are the main object of our study. We also briefly recall the bestiary of ternary algebras. In particular, we define the partially associative and partially antisymmetric algebras. In Section 2, we construct the operad of ternary Jordan algebras and the operad of partially associative and partially antisymmetric algebras. We describe a basis in low degrees for each of them and state our main result stating that the two operads are dual. The third section is devoted to the proofs of the theorems of Section 2. We end with an appendix in which we prove the equivalence of the structure of Jordan algebras and that of ternary Jordan algebras in the presence of units.

Throughout the article, k will be a field of characteristic different from 2 and 3. We denote the symmetric group acting on the set of $\{1, \dots, n\}$ by S_n and by $\varepsilon(\sigma)$ the sign of $\sigma \in S_n$.

1. Ternary Jordan algebras and other ternary algebras

In this section, we define the main object of our article, namely the *ternary Jordan algebras.* We also recall other related ternary algebra structures.

Definition 1.1. — *A ternary Jordan algebra is a vector space A over k equipped with a linear map $\{\text{-}\,\text{-}\,\text{-}\} : A \otimes A \otimes A \to A$, the triple product, which satisfies the partial symmetry condition*

$$\{x\,y\,z\} = \{z\,y\,x\} \tag{1.1}$$

and the four-term relation

$$\{x\,y\,\{z\,u\,v\}\} + \{z\,\{y\,x\,u\}\,v\} = \{\{x\,y\,z\}\,u\,v\} + \{z\,u\,\{x\,y\,v\}\} \tag{1.2}$$

for all $x, y, z, u, v \in A$.

This structure is also known under the name of *Jordan triple systems*; it plays a fundamental role in the theory of Jordan algebras and in the study of symmetric spaces (see [Neh]).

Let us give two examples of ternary Jordan algebras.

Exemple 1.2. — Recall that a *Jordan algebra* is a vector space A over k together with a linear map $-\star- : A \otimes A \to A$ such that for all $a, b \in A$, we have $a \star b = b \star a$ and $a^2 \star (a \star b) = a \star (a^2 \star b)$, where we set $a^2 = a \star a$. Any Jordan algebra A is endowed with a ternary Jordan algebra structure whose *triple product* is defined by

$$\{a\,b\,c\} = a\star(b\star c) - b\star(c\star a) + c\star(a\star b) \tag{1.3}$$

for all $a, b, c \in A$.

In the appendix, we prove that the two structures are equivalent in the case of unital algebras. Moreover, to any associative algebra A one associates a Jordan algebra A^+ by setting $a\star b = \frac{1}{2}(ab + ba)$ for all $a, b \in A$. A Jordan algebra whose product is derived from an associative algebra in this way is called a *special Jordan algebra.* For a special Jordan algebra, the triple product (1.3) is simply given by $\{a\,b\,c\} = \frac{1}{2}(abc + cba)$.

Exemple 1.3. — A *triple Jordan system* is a k-module V equipped with a quadratic map $P : V \to \mathrm{End}_k(V)$ satisfying

$$L(x,y)P(x) = P(x)L(y,x)\,, \qquad L(P(x)y,y) = L(x,P(y)x)\,,$$
$$\text{and} \quad P(P(x)y) = P(x)P(y)P(x)$$

for all $x, y \in V$, where $L(x,y) \in \mathrm{End}_k(V)$ is defined by $L(x,y) = P(x+z)y - P(x)y - P(z)y$ for any $x, y, z \in V$. These systems have been studied by Meyberg [Mey] and Jacobson [Ja2]. In [Loos], it is proved that the triple product defined by $\{x\,y\,z\} = L(x,y)z$ for all $x, y, z \in V$ satisfies Relations (1.1) and (1.2). Consequently, to any triple Jordan system (V, P) corresponds a ternary Jordan algebra structure on V.

Other generalizations of Jordan algebras, such as *quadratic Jordan algebras* and *Jordan pairs* (see [Ja2, Loos, McC, Mey]), also give rise to a triple product satisfying (1.1) and (1.2). More examples of ternary Jordan algebras are provided by Proposition 1.4 below.

We now define some other ternary algebra structures and emphasize their relations with the Jordan ternary algebras. The general definitions for the corresponding n-ary algebras can be found in [H-W] and [Gn2].

Let A be a vector space over k. A linear map $(\text{-}\,\text{-}\,\text{-}) : A \otimes A \otimes A \to A$ endows A with the structure of a *totally associative algebra* if it satisfies the relations

$$((a\,b\,c)\,d\,e) = (a\,(b\,c\,d)\,e) = (a\,b\,(c\,d\,e)) \tag{1.4}$$

for all $a, b, c, d, e \in A$. The map equips A with the structure of a *partially associative algebra* if it satisfies

$$((a\,b\,c)\,d\,e) + (a\,(b\,c\,d)\,e) + (a\,b\,(c\,d\,e)) = 0 \tag{1.5}$$

for all $a, b, c, d, e \in A$. Formulas (1.4) and (1.5) give two different generalizations of the usual (binary) associative algebras. We will consider the following two subclasses of these algebras. A ternary totally associative and totally symmetric algebra (*tats algebra* for short) is a totally associative algebra $(A, (\text{-}\,\text{-}\,\text{-}))$ such that

$$(a_1\,a_2\,a_3) = (a_{\sigma(1)}\,a_{\sigma(2)}\,a_{\sigma(3)}) \tag{1.6}$$

for all $a_1, a_2, a_3 \in A$ and $\sigma \in S_3$. A partially associative and partially antisymmetric algebra (*papas algebra* for short) is a partially associative algebra $(A, (\text{-}\,\text{-}\,\text{-}))$ such that

$$(a_1\,a_2\,a_3) = -(a_3\,a_2\,a_1) \tag{1.7}$$

for all $a_1, a_2, a_3 \in A$.

Intuitively, tats algebras should be seen as ternary generalizations of commutative associative (binary) algebras and papas algebras as generalizations of anticommutative associative (binary) algebras.

There is also a notion of ternary Lie algebra. Let L be a vector space over k equipped with a linear map $[\text{-},\text{-},\text{-}] : L\otimes L\otimes L \to L$. We say that L is a *ternary Lie algebra* if the triple bracket $[\text{-},\text{-},\text{-}]$ satisfies the condition of total antisymmetry

$$[a_1, a_2, a_3] = \varepsilon(\sigma)[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}]$$

for all $a_1, a_2, a_3 \in L$ and all $\sigma \in S_3$ and the ternary variant of the Jacobi identity

$$\sum_{\sigma\in S_5} \varepsilon(\sigma)[[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}], a_{\sigma(4)}, a_{\sigma(5)}] = 0$$

for all $a_1, \dots, a_5 \in L$.

For binary algebras the anticommutator defines a functor from the category of associative algebras to that of Lie algebras. This can be generalized to ternary algebras. It was proven in [Gn2] that, if $(A, (\text{-}\text{-}\text{-}))$ is a partially associative algebra, then the bracket $[\text{-},\text{-},\text{-}] : A\otimes A\otimes A \to A$ defined by

$$[a_1, a_2, a_3] = \sum_{\sigma\in S_3} \varepsilon(\sigma)(a_{\sigma(1)}\, a_{\sigma(2)}\, a_{\sigma(3)})$$

for all $a_1, a_2, a_3 \in A$ endows $(A, [\text{-},\text{-},\text{-}])$ with a ternary Lie algebra structure. We denote this algebra by A_Σ.

Similarly, the next proposition defines a functor from the category of totally associative algebras to that of ternary Jordan algebras.

Proposition 1.4. — *Let $(A, (\text{-}\text{-}\text{-}))$ be a totally associative algebra. The triple product $\{\text{-}\text{-}\text{-}\} : A\otimes A\otimes A \to A$ defined for all $a_1, a_2, a_3 \in A$ by*

$$\{a_1\, a_2\, a_3\} = \frac{(a_1\, a_2\, a_3) + (a_3\, a_2\, a_1)}{2}$$

equips $(A, \{\text{-}\text{-}\text{-}\})$ with a ternary Jordan algebra structure.

We denote the ternary Jordan algebra of Proposition 1.4 by A^+. The proof of the proposition is a straightforward computation.

The above considerations can be summarized in the following two sequences of functors:

$$\{\,\text{papas algebras}\,\} \hookrightarrow \{\,\text{partially associative algebras}\,\} \xrightarrow{\Sigma} \{\,\text{ternary Lie algebras}\,\}$$

$$\{\,\text{tats algebras}\,\} \hookrightarrow \{\,\text{totally associative algebras}\,\} \xrightarrow{+} \{\,\text{ternary Jordan algebras}\,\}$$

We will see in Section 2 that these two sequences of functors are dual of each other for some appropriate duality.

2. The operad of ternary Jordan algebras

In this section, we construct the operad of ternary Jordan algebras and the operad of papas algebras. Both will be defined as ternary quadratic operads. We also state our main result, namely that these operads are dual of each other.

First, we briefly recall some facts about operads. For a general theory on operads, we refer to [G-K], [L] and [L-S-V], but since we are dealing with non-necessarily unitary operads, we follow the notation of [L]. Recall that an operad $\mathcal{P}$ is a familly $\{\mathcal{P}(n)\}_{n\in\mathbf{N}}$ of vector spaces over k, each $\mathcal{P}(n)$ being a S_n-module, together with linear maps

$$\mathcal{P}(n)\otimes\mathcal{P}(\ell_1)\otimes\mathcal{P}(\ell_2)\otimes\cdots\otimes\mathcal{P}(\ell_n)\to\mathcal{P}(\ell_1+\ell_2+\cdots+\ell_n) \tag{2.1}$$

for all $n,\ell_1,\ldots,\ell_n\in\mathbf{N}$. These maps are compatible with the action of the symmetric groups and their compositions satisfy the axioms of May [May].

Given an operad $\mathcal{P}$, a $\mathcal{P}$-*algebra* is a vector space A over k endowed with linear maps

$$\mathcal{P}(n)\otimes A^{\otimes n}\to A \tag{2.2}$$

for all $n\in\mathbf{N}$ which are S_n-equivariant and compatible with the maps (2.1).

The module $\mathcal{P}(n)$ is the vector space of all possible expressions in n variables for a given type of algebras. The map (2.2) is the evaluation of such an expression in n elements of A. The description of the operads of associative algebras, of commutative algebras and of Lie algebras can be found, for example, in [G-K] or [L].

In [G-K], V. Ginzburg and M. Kapranov developed the notion of a *quadratic operad* for binary algebras. We adapt their definition to ternary algebras. For more details about quadratic operads of multinary algebras, see [Gn1, Gn2].

The forgetful functor from the category of operads into the category of families of S_n-modules admits a left ajoint functor $\mathcal{T}$. If E is a family $\{E(n)\}_{n\in\mathbf{N}}$ of S_n-modules, the operad $\mathcal{T}(E)$ will be called the *free operad* over E.

We will consider the case where E is concentrated in degree 3, *i.e.*, $E(k)=0$ for $k\neq 3$. In this case, one has $\mathcal{T}(E)(n)=0$ when n is even. Let R be a subspace of $\mathcal{T}(E)(5)$ that is stable under the action of S_5 and let (R) be the ideal (in the sense of operads) generated by R. We define the *ternary quadratic operad* $\mathcal{P}(E,R)$ generated by E and R as the quotient $\mathcal{T}(E)/(R)$.

For any $k[S_n]$-module V we denote by $V^\vee$ its linear dual $\mathrm{Hom}(V,k)$ endowed with its canonical S_n-action tensored by the sign representation of S_n. The *dual ternary quadratic operad* $\mathcal{P}^!$ of $\mathcal{P}=\mathcal{P}(E,R)$ is defined as the quadratic ternary operad $\mathcal{P}(E^\vee,R^\perp)$, where $R^\perp$ is the orthogonal of R in $\mathcal{T}(E^\vee)(5)\cong\mathcal{T}(E)(5)^\vee$.

Examples of ternary quadratic operads are described in [Gn2], where the first author gave a construction of the operads **ta**, **pa**, **tats** and $\mathbf{lie_3}$ which are the operads of totally associative, partially associative, totally associative and totally symmetric and Lie ternary algebras respectively. It is also proved in [Gn2] that $\mathbf{ta}^!\cong\mathbf{pa}$ and $\mathbf{tats}^!\cong\mathbf{lie_3}$.

We now define the operads **jord$_\mathbf{3}$** and **papas** for ternary Jordan algebras and papas algebras. We adopt the following convention. Let us write elements of the symmetric group as product of disjoint cyclic permutations. A cyclic permutation of the elements $i_1, \ldots, i_\ell \in \{1, \ldots, n\}$ will be denoted by $\tau_{i_1 \ldots i_\ell}$. Moreover, we equip the set of the expressions containing the symbols $a_1, a_2 \ldots$ or $b_1, b_2 \ldots$ with a left action of the symmetric group by setting

$$\sigma{\cdot} f(a_1, a_2 \ldots, b_1, b_2, \ldots) = f(a_{\sigma(1)}, a_{\sigma(2)} \ldots, b_{\sigma(1)}, b_{\sigma(2)}, \ldots)$$

where f is any expression containing $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ This action can naturally be extended by k-linearity to the group algebra $k[S_n]$ for n sufficiently large. For example, one has

$$(\tau_{12} - 3\tau_{123}){\cdot}\{a_1\, a_3\, a_2\} = \{a_2\, a_3\, a_1\} - 3\{a_2\, a_1\, a_3\}\,.$$

Consider the symbols

$$X_1 = \{a_2\, a_1\, a_3\}\,, \quad X_2 = \{a_1\, a_2\, a_3\} \quad \text{and} \quad X_3 = \{a_1\, a_3\, a_2\}$$

and

$$Y_1 = (b_2\, b_1\, b_3)\,, \quad Y_2 = (b_3\, b_2\, b_1) \quad \text{and} \quad Y_3 = (b_1\, b_3\, b_2)\,.$$

We denote by E the vector space spanned over k by X_1, X_2 and X_3, and by F the one spanned by Y_1, Y_2 and Y_3. They are both endowed with an action of S_3 given by $\sigma{\cdot}X_i = X_{\sigma(i)}$ and $\sigma{\cdot}Y_i = \varepsilon(\sigma)Y_{\sigma(i)}$ for all $i = 1, 2, 3$ and $\sigma \in S_3$. Note that the symbols X_i (resp. the symbols Y_i) represent the only three different ways to "multiply" three elements in a ternary Jordan algebra (resp. in a papas algebra) and that the action of S_3 is compatible with the partial symmetry (1.1) (resp. antisymmetry (1.7)) of the product.

The tensor product $E{\otimes}E$ (resp. the product $F{\otimes}F$) is endowed with the action of $S_3{\times}S_2$ where S_3 acts on the first term and S_2 on the second by exchanging 2 and 3. It was proved in [Gn2], that the module $\mathcal{T}(E)(5)$ is the induced module $\mathrm{Ind}_{S_3\times S_2}^{S_5}(E{\otimes}E)$ and is isomorphic to the direct sum of 10 copies of $E{\otimes}E$. A basis of the space $\mathcal{T}(E)(5)$ is given by the set

$$\big\{\sigma{\cdot}A \,\big|\, \sigma \in S_5\,, \sigma(1) < \sigma(3)\big\} \cup \big\{\sigma{\cdot}B \,\big|\, \sigma \in S_5\,, \sigma(1) < \sigma(5)\,, \sigma(2) < \sigma(4)\big\}\,, \tag{2.3}$$

where $A = \{\{a_1\, a_2\, a_3\}\, a_4\, a_5\}$ and $B = \{a_1\, \{a_2\, a_3\, a_4\}\, a_5\}$. Similarly, a basis of $\mathcal{T}(F)(E)$ is given by the set

$$\big\{\sigma{\cdot}A' \,\big|\, \sigma \in S_5\,, \sigma(1) < \sigma(3)\big\} \cup \big\{\sigma{\cdot}B' \,\big|\, \sigma \in S_5\,, \sigma(1) < \sigma(5)\,, \sigma(2) < \sigma(4)\big\}\,, \tag{2.4}$$

where $A' = ((b_1\, b_2\, b_3)\, b_4\, b_5)$ and $B' = (b_1\, (b_2\, b_3\, b_4)\, b_5)$.

In our conventions the two sets of symbols are endowed with an action of S_5 taking into account the partial symmetry of the product $\{{-}{-}{-}\}$ and the partial antisymmetry of $({-}{-}{-})$.

Let R be the subspace of $T(E)(5)$ spanned by the elements

$$\sigma\cdot\big(A+\tau_{35}\cdot A-\tau_{15}\tau_{24}\cdot A-\tau_{13}\cdot B\big)\,, \tag{2.5}$$

where σ runs over S_5. Similarly, let S be the subspace of $T(F)(5)$ spanned by the elements

$$\sigma\cdot\big(A'+\tau_{15}\tau_{24}\cdot A'+B'\big)\,, \tag{2.6}$$

where σ runs over S_5. Observe that the anihilation of the elements (2.5) is equivalent to the four-term relation (1.2) and the anihilation of the elements (2.6) to Relation (1.5).

Finally, we define the operad of ternary Jordan algebras and the operad of papas algebras respectively by

$$\mathbf{jord_3}=\mathcal{P}(E,R) \qquad \text{and} \qquad \mathbf{papas}=\mathcal{P}(F,S)\,.$$

Theorem 2.1. — *The ternary quadratic operads* **jord₃** *and* **papas** *are dual of each other :*

$$\mathbf{jord_3}^! \cong \mathbf{papas}\,.$$

The proof of Theorem 2.1 is based on the study of the S_5 modules $\mathbf{jord_3}(5)$ and $\mathbf{papas}(5)$ for which we have following result.

Theorem 2.2. — *The subset of* S_5

$$\big\{\sigma\cdot A \,\big|\, \sigma\in S_5\,, \sigma(1)<\sigma(3), \sigma(1)<\sigma(5) \text{ or } \sigma(2)<\sigma(4)\big\}$$

has 50 *elements and is a basis of the space* $\mathbf{jord_3}(5)$*. The subset of* S_5

$$\big\{\sigma\cdot A' \,\big|\, \sigma\in S_5\,, \sigma(1)<\sigma(3), \sigma(3)<\sigma(5) \text{ or } \sigma(2)<\sigma(4)\big\}$$

has 40 *elements and is a basis of the space* $\mathbf{papas}(5)$*.*

The proofs for Theorems 2.1 and 2.2 will be given in Section 3. We can state the following general result for ternary quadratic operads.

Theorem 2.3. — *Let* $\mathcal{P}=\mathcal{P}(E,R)$ *be a ternary quadratic operad. Let* $\{\mu_i\}_{1\le i\le n}$ *be a basis of* E *and* $\{\mu_i^\vee\}_{1\le i\le n}$ *the corresponding dual basis in* $E^\vee$*. If* A *is a* $\mathcal{P}$*-algebra and* B *a* $\mathcal{P}^!$*-algebra, then* $A\otimes B$ *is a ternary Lie algebra, whose bracket is defined by*

$$[a_1\otimes b_1, a_2\otimes b_2, a_3\otimes b_3]=\frac{1}{6}\sum_{\sigma\in S_3}\sum_{1\le i\le n}\mu_i(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)})\otimes\mu_i^\vee(b_{\sigma(1)},b_{\sigma(2)},b_{\sigma(3)})$$

for all elements $a_1,a_2,a_3\in A$ *and* $b_1,b_2,b_3\in B$*, and where* $\mu_i(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)})$ *and* $\mu_i^\vee(b_{\sigma(1)},b_{\sigma(2)},b_{\sigma(3)})$ *denote the images by the map (2.2) of* $\mu_i\otimes a_{\sigma(1)}\otimes a_{\sigma(2)}\otimes a_{\sigma(3)}$ *and* $\mu_i^\vee\otimes b_{\sigma(1)}\otimes b_{\sigma(2)}\otimes b_{\sigma(3)}$ *respectively.*

Proof. — We refer to the proof given by Ginzburg and Kapranov ([G-K]) for quadratic binary operads. The crucial point which allows to generalize their proof to ternary operads lays in the duality $\mathbf{tats}^!=\mathbf{lie_3}$ and in $\mathbf{tats}(n)=k$ for all even integer n (proven in [Gn2]). □

Theorem 2.1 implies following statement.

COROLLARY 2.4. — *Let $(A, \{---\})$ and $(B, (---))$ be a ternary Jordan algebra and a papas algebra respectively. The tensor product $A \otimes B$ is endowed with a ternary Lie algebra structure whose bracket is defined by*

$$[a_1 \otimes b_1, a_2 \otimes b_2, a_3 \otimes b_3] = \{a_2\, a_1\, a_3\} \otimes (b_2\, b_1\, b_3) + \{a_1\, a_2\, a_3\} \otimes (b_3\, b_2\, b_1) + \{a_1\, a_3\, a_2\} \otimes (b_1\, b_3\, b_2)\,.$$

for all $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$.

We end this section with a few words on possible further developments. In the theory of Koszul duality for a binary quadratic operad $\mathcal{P}$ ([G-K]), any $\mathcal{P}$-algebra A gives canonically rise to a chain complex

$$C_n^{\mathcal{P}}(A) = A^{\otimes n} \otimes_{S_n} \mathcal{P}^!(n)^\vee \tag{2.7}$$

whose differential is defined in terms of binary trees. In particular, when $\mathcal{P}$ is the operad of associative, commutative or Lie algebras, this complex is the standard Hochschild complex, the Harrison complex or the Chevalley-Eilenberg complex respectively. We plan to extend this to ternary operads and construct a chain complex for ternary Jordan algebras in this way. Namely, we conjecture that the complex (2.7) can be generalized to ternary quadratic operads. If $(A, \{---\})$ is a $\mathbf{jord_3}$-algebra, one has

$$A^{\otimes 3} \otimes_{S_3} \mathbf{papas}(3)^\vee \cong A^{\otimes 3}/I_3 \quad \text{and} \quad A^{\otimes 5} \otimes_{S_5} \mathbf{papas}(5)^\vee \cong A^{\otimes 5}/I_5\,,$$

where I_3 is the subspace of $A^{\otimes 3}$ spanned by the elements $a \otimes b \otimes c - c \otimes b \otimes a \in A^{\otimes 3}$, and I_5 is the subspace of $A^{\otimes 5}$ spanned by all elements $a \otimes b \otimes (c \otimes d \otimes e - e \otimes d \otimes c) \in A^{\otimes 5}$ and

$$a \otimes b \otimes c \otimes d \otimes e - c \otimes b \otimes a \otimes d \otimes e - a \otimes d \otimes c \otimes b \otimes e + c \otimes d \otimes a \otimes b \otimes e \in A^{\otimes 5}\,.$$

Consider the maps $d_3 : A^{\otimes 3} \to A$ and $d_5 : A^{\otimes 5} \to A^{\otimes 3}$ defined by $d_3(a \otimes b \otimes c) = \{a\, b\, c\}$ for all homogeneous elements $a \otimes b \otimes c \in A^{\otimes 3}$, and

$$d_5(x) = \{a\, b\, c\} \otimes d \otimes e - c \otimes \{b\, a\, d\} \otimes e + c \otimes d \otimes \{a\, b\, e\} - a \otimes b \otimes \{c\, d\, e\}$$

for all homogeneous elements $x = a \otimes b \otimes c \otimes d \otimes e \in A^{\otimes 5}$. They satisfy $d_3 \circ d_5 = 0$ and pass to the quotient in maps

$$A^{\otimes 5}/I_5 \xrightarrow{d_5} A^{\otimes 3}/I_3 \xrightarrow{d_3} A\,. \tag{2.8}$$

We expect that, in low degrees, the conjectured chain complex for ternary Jordan algebras coincides with Sequence (2.8).

3. Proofs of Theorems 2.1 and 2.2

This section is entirely devoted to the proofs of Theorems 2.1 and 2.2. We begin with a technical lemma.

Lemma 3.1. — *The S_5-module*

$$k[S_5]/\Big(k[S_5](1-\tau_{13})+k[S_5](1-\tau_{24})(1+\tau_{15}+\tau_{35})\Big)$$

has dimension 50 over k whereas the S_5-module

$$k[S_5]/\Big(k[S_5](1+\tau_{13})+k[S_5](1+\tau_{24})(1+\tau_{35})\Big)$$

has dimension 40.

Proof. — We will first prove that the first S_5-module of the lemma is isomorphic to

$$k[S_5]\big(1+\tau_{24}\big)\big(1+\tau_{13}\big)\big(1+\tau_{13}+\tau_{35}\big)\big(1+\tau_{13}\big)\oplus k[S_5]\big(1+\tau_{13}\big)\big(2-\tau_{13}-\tau_{35}\big)\big(1+\tau_{13}\big)\,.$$

We use the following basic result of linear algebra. Let A be a k-algebra. If one is provided with n quasi-idempotents, *i.e.* elements $c_1,\ldots,c_n\in A$ such that $c_i^2=\lambda_i c_i$ where $\lambda_i\in k\backslash\{0\}$ and $c_ic_j=0$ for $i\neq j$, then

$$A(c_1+\cdots+c_n)\cong Ac_1\oplus\cdots\oplus Ac_n\,.$$

We set $A=k[S_5]$, $X=1-\tau_{13}$, $Z=1-\tau_{24}$ and $T=1+\tau_{15}+\tau_{35}$. The quotient $A/(AX+AZT)$ is isomorphic to $\big(A/AX\big)/\big((AX+AZT)/AX\big)$ which is also isomorphic to $\big(A/AX\big)/\big(A/AX\big)\overline{ZT}$ where $\overline{ZT}$ denotes the projection of $ZT\in A$ in A/AX.

We also set $X'=1+\tau_{13}$. So we get

$$XX'=X'X=0\,,\quad X^2=2X\,,\quad {X'}^2=2X'\quad\text{and}\quad X+X'=2\,.$$

One immediately deduces that $A=AX\oplus AX'$ and $A/AX\cong AX'$. The projection of ZT in AX' is ZTX'. Consequently, we get $A/(AX+AZT)\cong AX'/AX'ZTX'$.

Now we set $Z'=1+\tau_{24}$. One checks that the relations

$$Z^2=2Z\,,\quad {Z'}^2=2Z'\,,\quad TX'=X'T\,,\quad Z'Z=ZZ'=0\quad\text{and}\quad T(3-T)=(T-1)X$$

are satisfied and that Z and Z' commute with X, X' and T. These last relations involve

$$\begin{aligned}R_i^2&=24R_i\quad\text{for } i=1,2,3,\\ R_iR_j&=0\quad\text{for } i\neq j,\text{ and}\\ R_1+R_2+R_3&=12X'\,,\end{aligned}$$

where

$$R_1=X'ZTX'\,,\quad R_2=Z'X'TX'\quad\text{and}\quad R_3=2X'(3-T)X'\,.$$

This family of quasi-idempotents gives rise to the decomposition $AX'/AR_1 \cong AR_2 \oplus AR_3$ which proves the announced isomorphism

$$A/(AX + AZ'T) \cong AX'/AR_1 \cong AR_2 \oplus AR_3 \,.$$

To determine the dimensions of the module $AR_2 \oplus AR_3$, we calculate the rank of the right multiplication by R_2 and R_3. For any integer ℓ, we denote the matrix of dimension ℓ whose entries are all equal to 1 by J_ℓ, and the identity matrix of dimension ℓ by I_ℓ. One has

$$R_2 = 2(1+\tau_{24})(1+\tau_{13}+\tau_{15}+\tau_{35}+\tau_{135}+\tau_{153}) \,.$$

It is easy to see that, in the basis of $k[S_5]$ formed by the elements of S_5 in an appropriate order, the matrix of $V \mapsto VR_2$ is the Kronecker product of matrices $2I_{10} \otimes J_2 \otimes J_6$. The rank of this matrix is 10. On the other hand, we have

$$R_3 = 6(1+\tau_{13}) - 2(1+\tau_{13}+\tau_{15}+\tau_{35}+\tau_{135}+\tau_{153}) \,.$$

In an appropriate basis of $k[S_5]$, the matrix of $V \mapsto VR_3$ is given by

$$I_{20} \otimes \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \otimes J_2 \,,$$

whose rank is 40.

The dimension of the second module in the lemma is computed in the same way. We will not give all details here. One first proves that the module is isomorphic to

$$k[S_5](1-\tau_{13})(1+\tau_{35})(1-\tau_{13})(1-\tau_{24}) \oplus k[S_5](1-\tau_{35}-\tau_{15})(1-\tau_{13}) \,.$$

This part of the proof uses the quasi-idempotents $Y_1 = XUZ'X$, $Y_2 = XUZX$ and $Y_3 = 2(2-T)X$, where $U = 1+\tau_{15}$. To prove that the dimension of the module $AY_2 \oplus AY_3$ is 40, one computes the rank of the right multiplication by Y_2 and Y_3. □

We now prove Theorem 2.2.

Proof of theorem 2.2. — By definition, the module $\mathbf{jord_3}(5)$ is the quotient of the module $\mathcal{T}(E)(5)$ by the subspace R. They were both defined in Section 2. The space $\mathcal{T}(E)(5)$ is spanned by the symbols $\sigma{\cdot}A = \sigma{\cdot}\{\{a_1\, a_2\, a_3\}\, a_4\, a_5\}$ and $\sigma{\cdot}B = \sigma{\cdot}\{a_1\, \{a_2\, a_3\, a_4\}\, a_5\}$, where σ runs over S_5 and where we take in account Relation (1.1). Namely, we have $\tau_{13}{\cdot}A = A$ and $\tau_{15}{\cdot}B = \tau_{24}{\cdot}B = B$. By definition of R, the relation

$$\sigma\tau_{13}{\cdot}B = \sigma{\cdot}\big(A - \tau_{24}\tau_{15}{\cdot}A + \tau_{35}{\cdot}A\big) \tag{3.1}$$

holds in the quotient. So, we may restrict the generators of $\mathbf{jord_3}(5)$ to the set $\{\sigma{\cdot}A \mid \sigma \in S_5\}$.

There are exactly two ways to express $\sigma\tau_{13}\cdot B$ in terms of A. The one given above and also

$$\sigma\tau_{13}\cdot B = \sigma\tau_{13}\tau_{24}\cdot B = \sigma\cdot\big(\tau_{24}\cdot A - \tau_{15}\cdot A + \tau_{24}\tau_{35}\cdot A\big)\,. \tag{3.2}$$

Now, we can determine all the relations holding in $\mathcal{T}(E)(5)/R$. They are

$$\sigma(1-\tau_{13})\cdot A = 0 \qquad \text{and} \qquad \sigma(1-\tau_{24})(1+\tau_{15}+\tau_{35})\cdot A = 0\,, \tag{3.3}$$

where the later is the difference of Relations (3.1) and (3.2) and where σ still runs over S_5. These considerations imply that $\mathbf{jord_3}(5)$ is isomorphic to the left S_5-module

$$k[S_5]/\Big(k[S_5](1-\tau_{13}) + k[S_5](1-\tau_{24})(1+\tau_{15}+\tau_{35})\Big)\,,$$

whose dimension is 50 by Lemma 3.1.

Note that the set

$$K = \big\{\sigma \in S_5\,,\big|\,\sigma(1)<\sigma(3), \sigma(1)<\sigma(5) \text{ or } \sigma(2)<\sigma(4)\big\}$$

has cardinal 50. It is sufficient to show that the set $\{\sigma\cdot A\,|\,\sigma\in K\}$ spans $\mathbf{jord_3}(5)$. To this end, we introduce the following subsets of S_5. We set

$$\begin{aligned}
K_0 &= \{\sigma\in S_5\,|\,\sigma(1)<\sigma(3)\}\,,\\
K_1 &= \{\sigma\in K_0\,|\,\sigma(1)<\sigma(3)<\sigma(5) \text{ and } \sigma(2)<\sigma(4)\}\,,\\
K_2 &= \{\sigma\in K_0\,|\,\sigma(1)<\sigma(3)<\sigma(5) \text{ and } \sigma(2)>\sigma(4)\}\,,\\
K_3 &= \{\sigma\in K_0\,|\,\sigma(1)<\sigma(5)<\sigma(3) \text{ and } \sigma(2)<\sigma(4)\}\,,\\
K_4 &= \{\sigma\in K_0\,|\,\sigma(1)<\sigma(5)<\sigma(3) \text{ and } \sigma(2)>\sigma(4)\}\,,\\
K_5 &= \{\sigma\in K_0\,|\,\sigma(5)<\sigma(1)<\sigma(3) \text{ and } \sigma(2)<\sigma(4)\}\,,\\
K_6 &= \{\sigma\in K_0\,|\,\sigma(5)<\sigma(1)<\sigma(3) \text{ and } \sigma(2)>\sigma(4)\}\,.
\end{aligned}$$

The sets $K_1,\ldots,K_6$ form a partition of K_0 and we have

$$K = K_1\cup K_2\cup K_3\cup K_4\cup K_5\,.$$

Moreover, $\mathrm{Card}(K_i)=10$ for all $i=1,\cdots,6$ and one has

$$K_6\tau_{24}\tau_{35}\tau_{13} = K_1\,,\ K_6\tau_{35}\tau_{13} = K_2\,,\ K_6\tau_{24}\tau_{15} = K_3\,,\ K_6\tau_{15} = K_4\,,\ K_6\tau_{24} = K_5\,. \tag{3.4}$$

Note that as $\tau_{13}\cdot A = A$, we can restrict the set of generators to $\{\sigma\cdot A\,|\,\sigma\in K_0\}$. After the second of Relations (3.3), for any $\sigma\in S_5$, one has

$$\sigma\cdot A = \sigma(\tau_{24}+\tau_{24}\tau_{15}+\tau_{24}\tau_{35}-\tau_{15}-\tau_{35})\cdot A$$

in $\mathbf{jord_3}(5)$. Consequently, according to Relations (3.4), if an element $\sigma\in K_0$ is in K_6, then $\sigma\cdot A\in\mathbf{jord_3}(5)$ can be written as a linear combination of elements $\tau_i\cdot A\in\mathbf{jord_3}(5)$, where $\tau_i\in K_i$ for $i=1,\ldots,5$. This proves that K spans $\mathbf{jord_3}(5)$ and ends the proof of the Theorem for the operad $\mathbf{jord_3}$.

The proof for the S_5-module $\mathbf{papas}(5)$ is similar. One first proves that it is isomorphic to

$$k[S_5]/\Big(k[S_5](1+\tau_{13}) + k[S_5](1+\tau_{24})(1+\tau_{35})\Big)\,,$$

whose dimension is 40 by Lemma 3.1. Then, in a same way as above, one proves that the set $\{\sigma\cdot A'\,|\,\sigma\in K'\}$ where $K' = K_1\cup K_2\cup K_3\cup K_5$ is a family of 40 generators. $\square$

We can now prove Theorem 2.1.

Proof of Theorem 2.1. — In order to show that $\mathcal{P}(E,R)^! = \mathcal{P}(F,S)$, we have to prove that $E^\vee \cong F$ and that R and S are orthogonal. The isomorphism $E \cong F^\vee$ is obvious by definition of E and F. To establish the orthogonality of the spaces R and S, we first express the duality of $\mathcal{T}(E)(5)$ and $\mathcal{T}(F)(5)$ in a proper way.

In view of the form of the basis (2.3) of $\mathcal{T}(E)(5)$, we can state the S_5-module isomorphism

$$\mathcal{T}(E)(5) \cong \Big(k[S_5]/(1-\tau_{13})\Big)A \oplus \Big(k[S_5]/(1-\tau_{15}, 1-\tau_{24})\Big)B\,.$$

We put symbols A and B here to distinguish the two summands of the direct sum. From this isomorphism, we deduce that

$$\mathcal{T}(E)(5) \cong k[S_5](1+\tau_{13})A \oplus k[S_5](1+\tau_{15})(1+\tau_{24})B\,. \tag{3.5}$$

In the same way the form of the basis (2.4) of $\mathcal{T}(F)(5)$ yields the isomorphism

$$\mathcal{T}(F)(5) \cong \Big(k[S_5]/(1+\tau_{13})\Big)A' \oplus \Big(k[S_5]/(1+\tau_{15}, 1+\tau_{24})\Big)B'$$

and so

$$\mathcal{T}(F)(5) \cong k[S_5](1-\tau_{13})A' \oplus k[S_5](1-\tau_{15})(1-\tau_{24})B'\,. \tag{3.6}$$

Now, we consider the isomorphism of S_5-modules $\mathcal{T}(F)(5) \to \mathcal{T}(E)(5)^\vee$ defined between the spaces (3.5) and (3.6) by $\sigma A + \tau B \mapsto \varepsilon(\sigma)\sigma A' + \varepsilon(\tau)\tau B'$ for all $\sigma, \tau \in S_5$. Thus, we can define a non-degenerate bilinear form $\langle - \,|\, - \rangle : \mathcal{T}(E)(5) \times \mathcal{T}(F)(5) \to k$ by setting

$$\langle \sigma A + \tau B | \sigma' A' + \tau' B' \rangle = \frac{1}{2}\varepsilon(\sigma)\delta_{\sigma,\sigma'} + \frac{1}{4}\varepsilon(\tau)\delta_{\tau,\tau'}$$

for all $\sigma, \sigma'\tau, \tau' \in S_5$.

We will prove that $\langle R \,|\, S\rangle = \{0\}$. If σ is an element of S_5, then

$$x = \sigma(1+\tau_{35}-\tau_{15}\tau_{24})(1+\tau_{13})A - \sigma\tau_{13}(1+\tau_{15})(1+\tau_{24})B$$

is a generator of R (see (2.5)) via the isomorphism (3.5). For $\sigma' \in S_5$ consider the generator (see (2.6))

$$y = \sigma'(1+\tau_{15}\tau_{24})(1-\tau_{13})A' + \sigma'(1-\tau_{15})(1-\tau_{24})B'$$

of the space S via the isomorphism (3.6). We calculate

$$\begin{aligned}
\langle x \,|\, y\rangle =& \big\langle \sigma(1+\tau_{35}-\tau_{15}\tau_{24})(1+\tau_{13})A \,|\, \sigma'(1+\tau_{15}\tau_{24})(1-\tau_{13})A'\big\rangle \\
& - \big\langle \sigma\tau_{13}(1+\tau_{15})(1+\tau_{24})B \,|\, \sigma'(1-\tau_{15})(1-\tau_{24})B'\big\rangle \\
=& \big\langle \sigma(1+\tau_{35}-\tau_{15}\tau_{24})(1+\tau_{13})(1+\tau_{13})(1+\tau_{15}\tau_{24})A \,|\, \sigma'A'\big\rangle \\
& - \big\langle \sigma\tau_{13}(1+\tau_{15})(1+\tau_{24})(1+\tau_{24})(1+\tau_{15})B \,|\, \sigma'B'\big\rangle \\
=& \, 2\big\langle \sigma(1+\tau_{35}-\tau_{15}\tau_{24})(1+\tau_{13})(1+\tau_{15}\tau_{24})A \,|\, \sigma'A'\big\rangle \\
& - 4\big\langle \sigma\tau_{13}(1+\tau_{15})(1+\tau_{24})B, \sigma'B'\big\rangle\,.
\end{aligned}$$

The first equality is given by the S_5-invariance of the scalar product and the second by a simple computation in $k[S_5]$. Now, since $\langle A \mid A' \rangle = 2\langle B \mid B' \rangle$, we have $\langle x \mid y \rangle = 2\langle \sigma z A \mid \sigma' A \rangle$, where

$$z = (1 + \tau_{35} - \tau_{15}\tau_{24})(1 + \tau_{13})(1 + \tau_{15}\tau_{24}) - \tau_{13}(1 + \tau_{15})(1 + \tau_{24})\,.$$

Another computation in $k[S_5]$ gives $z = 0$. We proved that $\langle R, S \rangle = 0$. By Theorem 3.2, as the dimension of $T(E)(5)$ is 90, the dimension of R is 40 and that of S is 50. Consequently, we have $R = S^{\perp}$. □

Appendix

In this appendix, we prove the equivalence of the notions of Jordan algebras and ternary Jordan algebras in the unitary case. The results presented here may be well known to the experts, but we did not find complete proofs of them in the literature. We give them here for the sake of completeness.

Theorem A.1. — *If $(J, \star)$ is a Jordan algebra, then the ternary product*

$$\{a\,b\,c\} = (a\star b)\star c + a\star(b\star c) - (a\star c)\star b$$

endows J with a ternary Jordan algebra structure.

We say that a ternary Jordan algebra $(J, \{---\})$ is *unital* if there exists $1 \in J$ such that $\{1\,a\,1\} = a$ for all $a \in J$. Remark that, if one replaces x, z, u and v by 1 in Relation (1.2) and uses (1.1), one gets $\{1\,1\,y\} = \{y\,1\,1\} = \{1\,y\,1\} = y$ for all $y \in J$. When the ternary Jordan algebra is unital, we have the converse theorem of Theorem A.1.

Theorem A.2. — *If $(J, \{---\})$ is a unital ternary Jordan algebra of unit 1, then the product $a\star b = \{a\,b\,1\}$ endows J with a Jordan algebra structure.*

Proof of Theorem A.1. — Relation (1.1) is an obvious consequence of the commutativity of $(J, \star)$. Let us check that the map $\{---\}$ defined in the theorem satisfies Relation (1.2). We claim that the relations

$$\{x\,y\,\{x\,z\,x\}\} = \{x\,\{z\,x\,y\}\,x\} \qquad (A.1)$$

and

$$\{\{x\,y\,x\}\,y\,z\} = \{x\,\{y\,x\,y\}\,z\} \qquad (A.2)$$

hold for all x, y, z in any Jordan algebra. Indeed, these are relations in three variables of degree 1 in z. By the theorems of MacDonald and Shirshov (see [Ja1, ch.I.9.]), it is sufficient to check them for any special algebra. Recall that in this case, we have $\{x\,y\,z\} = \frac{1}{2}(xyz + zxy)$ for all x, y, z. Indeed we have

$$\{x\,y\,\{x\,z\,x\}\} = \frac{1}{2}(xyxzx + xzxyx) = \{x\,\{z\,x\,y\}\,x\}$$

and

$$\{\{x\,y\,x\}\,y\,z\} = \frac{1}{2}(xyxyz + zyxyx) = \{x\,\{y\,x\,y\}\,z\}\,.$$

We follow [Loos] to complete the proof. Linearizing $(A.1)$, we obtain

$$2\{x\,y\,\{x\,z\,t\}\} + \{t\,y\,\{x\,z\,x\}\} = 2\{x\,\{z\,x\,y\}\,t\} + \{x\,\{z\,t\,y\}\,x\}\,. \qquad (A.3)$$

We can also linearize $(A.2)$ and obtain the relation

$$\{\{x\,y\,x\}\,z\,t\} + \{\{x\,z\,x\}\,y\,t\} = \{x\,\{y\,x\,z\}\,t\} + \{x\,\{z\,x\,y\}\,t\} = 2\{x\,\{y\,x\,z\}\,t\}\,,$$

which is equivalent to

$$\{\{x\,y\,x\}\,z\,t\} + \{t\,y\,\{x\,z\,x\}\} == 2\{x\,\{z\,x\,y\}\,t\}\,, \qquad (A.4)$$

The difference between Relations (A.3) and (A.4) is

$$2\{x\,y\,\{x\,z\,t\}\} = \{x\,\{z\,t\,y\}\,x\} + \{\{x\,y\,x\}\,z\,t\}\,,$$

which, after linearization, yields Relation (1.2). □

Proof of Theorem A.2. — First we note that for any $a, b \in J$, the partial symmetry of $\{---\}$ implies $\{a\,b\,1\} = \{1\,b\,a\}$ and that, from (1.2), we get

$$\{a\,b\,\{c\,d\,c\}\} + \{c\,\{b\,a\,d\}\,c\} = 2\{\{a\,b\,c\}\,d\,c\} \qquad (A.5)$$

for all $a, b, c, d \in J$.

Replacing a, b and c by 1 in (A.5) gives $d = d\star 1$ and replacing d and c by 1 gives $a\star b = b\star a$. We proved that $\star$ is an unitary and commutative law.

Replacing a and c by 1 in (A.5) gives us $\{b\,1\,d\} = b\star d$. So, for all $a, b \in J$, we have

$$a\star b = \{1\,a\,b\} = \{a\,1\,b\} = \{a\,b\,1\}\,.$$

Now, if we replace z by a, x by b, v by c and y and u by 1 in Relation (1.2), we get

$$\{a\,b\,c\} = (a\star b)\star c + a\star(b\star c) - (a\star c)\star b\,. \qquad (A.6)$$

Note that $\{a\,a\,b\} = a^2\star b$ for any $a, b \in J$.

We compute

$$\begin{aligned}(a^2\star b)\star a &= \{1\,a\,\{a\,a\,b\}\}\\ &= \{\{1\,a\,a\}\,a\,b\} - \{a\,\{a\,1\,a\}\,b\} + \{a\,a\,\{1\,a\,b\}\}\\ &= \{a^2\,a\,b\} - \{a\,a^2\,b\} + \{a\,a\,a\star b\}\\ &= 3a^2\star(a\star b) - 2(a^2\star b)\star a\,.\end{aligned}$$

The second equality is a consequence of the four-term relation and the last one of (A.6). For any $a, b \in J$, we get $a^2\star(a\star b) = (a^2\star b)\star a$, which proves that $(J, \star)$ is a Jordan algebra. □

Acknowledgements. The authors would like to thank Jean-Louis Loday who suggested the operadic approach to Jordan algebras. We also thank Christian Kassel for his careful reading of our work and his useful and enlightening remarks. The first author is grateful to UNESCO and the Abdus Salam ICTP (Trieste, Italy) for support and hospitality.

References

[D-V] M. Dubois-Violette. — $d^N = 0$: *generalized homology*, K-Theory, t. **14**, 1998, p. 371–404.

[G-K] V. Ginzburg, M. Kapranov. — *Koszul duality for operads*, Duke Math. J., t. **76**, 1994, p. 203–272.

[Gla] N. Glassman. — *Cohomology of Jordan algebras*, J. Algebra, t. **15**, 1970, p. 167–194.

[Gn1] A. V. Gnedbaye. — *Les algèbres k-aires et leurs opérades*, C. R. Acad. Sci. Paris, Série I, t. **321**, 1995, p. 147–152.

[Gn2] A. V. Gnedbaye. — *Opérades des algèbres $(k+1)$-aires*, Contemp. Math., Amer. Math. Soc., Providence, RI, t. **202**, 1997, p. 83–113.

[H-W] Ph. Hanlon, M. Wachs. — *On Lie k-algebras*, Advances in Math., t. **113**, 1995, p. 206–236.

[Ja1] N. Jacobson. — *Structure and representations of Jordan algebras*, Amer. Math. Soc. Colloq. Publ., Providence RI, t. **39**, 1969.

[Ja2] N. Jacobson. — *Lie and Jordan Triples Systems*, Amer. J. Math., t. **49**, 1971, p. 149–170.

[Jo] P. Jordan. — *Über Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik*, Nachr. Ges. Wiss. Göttingen, 1933, p. 149–170.

[L] J.-L. Loday. — *La renaissance des opérades*, Séminaire Bourbaki (Exp. n° 792), Astérisque, Soc. Math. France, t. **237**, 1996, p. 47–74.

[L-S-V] J.-L. Loday, J. Stasheff, A. Voronov (editors). — *Operads: Proceedings of Renaissance Conferences*, Contemp. Math., Amer. Math. Soc., Providence, RI, t. **202**, 1997.

[Loos] O. Loos. — *Jordan Pairs*, Lect. Notes in Math., Springer-Verlag, t. **460**, 1975.

[May] J. P. May. — *The Geometry of Iterated Loop Spaces*, Lecture Notes in Math., Springer-Verlag, t. **271**, 1972.

[McC] K. McCrimmon. — *A General Theory of Jordan Rings*, Proc. Nat. Acad. Sci. U.S.A., t. **56**, 1960, p. 1072–1079.

[Mey] K. Meyberg. — *Lectures on Algebras and Triple Systems*, Lecture Notes, Univ. of Virginia (Charlottesville), 1972.

[Neh] E. Neher. — *On the Classification of Lie and Jordan Triples*, Commun. Alg., t. **13**, 1985, p. 2615–2667.

A. V. Gnedbaye
B.P. 102, N'Djaména (Tchad)

M. Wambst
Institut de Recherche Mathématique Avancée
Université Louis Pasteur et CNRS
7, rue René-Descartes
F-67084 Strasbourg Cedex (France)
E-mail: wambst@math.u-strasbg.fr